Mathematical Physics, Analysis and Geometry (2024) 27:12
https://doi.org/10.1007/s11040-024-09484-x

Nonrelativistic Limit of Generalized MIT Bag Models and Spectral Inequalities

Jussi Behrndt[1] · Dale Frymark[1] · Markus Holzmann[1] · Christian Stelzer-Landauer[1]

Received: 8 March 2024 / Accepted: 25 June 2024 / Published online: 22 July 2024

Abstract
For a family of self-adjoint Dirac operators $-ic(\alpha \cdot \nabla) + \frac{c^2}{2}$ subject to generalized MIT bag boundary conditions on domains in $\mathbb{R}^3$, it is shown that the nonrelativistic limit in the norm resolvent sense is the Dirichlet Laplacian. This allows to transfer spectral geometry results for Dirichlet Laplacians to Dirac operators for large c.

Keywords Dirac operator · Generalized MIT bag boundary conditions · Nonrelativistic limit · Spectral inequalities

Mathematics Subject Classification Primary: 81Q10, 58J50 · Secondary: 35Q40

1 Introduction

The MIT bag operator and more general types of self-adjoint Dirac operators on domains $\Omega \subset \mathbb{R}^3$ have attracted a lot of attention in the last years. The MIT bag model itself originates from the investigation of quarks in hadrons from the 1970s [22, 26, 28, 34] and has been studied from a more mathematical perspective in [3–5, 16, 40, 42, 44, 47, 48]. The present paper is inspired by the recent contribution [7], where

✉ Markus Holzmann
holzmann@math.tugraz.at

Jussi Behrndt
behrndt@tugraz.at

Dale Frymark
dfrymark2@gmail.com

Christian Stelzer-Landauer
christian.stelzer@tugraz.at

1 Institut für Angewandte Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria

spectral properties of the family H_κ^Ω, $\kappa \in \mathbb{R}$, of self-adjoint Dirac operators

$$\begin{aligned} H_\kappa^\Omega f &= -ic(\alpha \cdot \nabla) f + \frac{c^2}{2}\beta f, \\ \operatorname{dom} H_\kappa^\Omega &= \left\{ f \in H^1(\Omega; \mathbb{C}^4) : f = i(\sinh(\kappa) I_4 - \cosh(\kappa)\beta)(\alpha \cdot \nu) f \text{ on } \partial\Omega \right\}, \end{aligned} \tag{1.1}$$

in $L^2(\Omega; \mathbb{C}^4)$ were studied. Here $\alpha \cdot \nabla = \alpha_1 \partial_1 + \alpha_2 \partial_2 + \alpha_3 \partial_3$ with the usual Dirac matrices $\alpha_1, \alpha_2, \alpha_3, \beta \in \mathbb{C}^{4 \times 4}$ (see (1.5) and (1.7) below), $c > 0$ is the speed of light, Ω is a C^2-domain with unit normal vector ν, and $H^1(\Omega; \mathbb{C}^4)$ is the first order L^2-based Sobolev space. The operators H_κ^Ω model the propagation of a relativistic spin $\frac{1}{2}$ particle with mass $m = \frac{1}{2}$ subject to the boundary conditions in (1.1), which are a three-dimensional counterpart of the quantum dot boundary conditions; cf. [19, 20], the introduction in [7] for more references in dimension two, and Sect. 2.2 for a further motivation of these boundary conditions. In particular, for $\kappa = 0$ the standard MIT bag boundary conditions are recovered. If Ω is bounded, then the spectrum of H_κ^Ω is purely discrete and consists of eigenvalues

$$\cdots \leq \lambda_2^-(H_\kappa^\Omega) \leq \lambda_1^-(H_\kappa^\Omega) \leq -\frac{c^2}{2} < \frac{c^2}{2} \leq \lambda_1^+(H_\kappa^\Omega) \leq \lambda_2^+(H_\kappa^\Omega) \leq \dots, \tag{1.2}$$

that accumulate at $\pm\infty$. The main objective in [7] is the analysis of the eigenvalue curves $\kappa \mapsto \lambda_j^\pm(H_\kappa^\Omega)$ and their asymptotic behaviour, which then leads to spectral geometry results for H_κ^Ω with κ sufficiently large. The most remarkable result therein is a variant of the Faber–Krahn inequality for κ sufficiently large minimizing the first positive eigenvalue when Ω is a ball. For related spectral geometry results for two-dimensional Dirac operators with infinite mass boundary conditions we refer to [2, 20, 23, 39, 51].

In this paper we propose a different approach to obtain spectral inequalities and spectral geometry results for the Dirac operators H_κ^Ω, which is based on the analysis of a nonrelativistic limit. This allows us to conclude for all sufficiently large c and *all* $\kappa \in \mathbb{R}$, e.g., the Faber–Krahn inequality for the first two positive eigenvalues $\lambda_1^+(H_\kappa^\Omega)$, $\lambda_2^+(H_\kappa^\Omega)$, the Hong–Krahn–Szegö inequality minimizing the second two positive eigenvalues $\lambda_3^+(H_\kappa^\Omega)$, $\lambda_4^+(H_\kappa^\Omega)$, or the Payne–Pólya–Weinberger inequality for the ratios $\lambda_j^+(H_\kappa^\Omega)/\lambda_l^+(H_\kappa^\Omega)$, $j = 1, 2, l = 3, 4$, of the first two and the second two positive eigenvalues, relying on classical counterparts for the Dirichlet Laplacian [8, 29, 33, 36, 37, 45]; here the spectral inequalities come for pairs of eigenvalues, as all eigenvalues of H_κ^Ω have even multiplicity, and remain valid in an analogous form also for the first two pairs of negative eigenvalues, see Remark 3.7. In the same spirit other results from spectral geometry can be transferred from Laplacians to Dirac operators; we refer the reader to the monographs [31, 38, 46] for an introduction to and overview of this topic, but limit ourselves to the above-mentioned three examples.

The nonrelativistic limit provides a connection of the generalized MIT bag models with their nonrelativistic counterparts, i.e. Schrödinger operators, and is of independent interest, as it gives a physical interpretation of H_κ^Ω. To find it one has to subtract the energy of the resting particle $\frac{c^2}{2}$ and compute the limit of the resolvent of $H_\kappa^\Omega - \frac{c^2}{2}$

as $c \to \infty$. Nonrelativistic limits of Dirac operators have been computed in many different settings. More information on three dimensional Dirac operators with regular potentials, for example, can be found in [50, Chapter 6] and the references therein. In [27] it was shown that the nonrelativistic limit of a family of one-dimensional Dirac operators with boundary conditions containing the counterpart of H_κ^Ω is a Dirichlet or a Neumann Laplacian. Moreover, the nonrelativistic limit of one-dimensional Dirac operators with singular interactions supported on points, which are closely related to one-dimensional Dirac operators with boundary conditions, was studied extensively in [24, 25, 30, 32]. In higher dimensions, the nonrelativistic limit of Dirac operators with singular potentials supported on curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$ was computed in various situations in [10, 11, 13, 18]. We also point out the paper [3], where it is shown that for bounded Ω the discrete eigenvalues of the MIT bag model, i.e. of H_κ^Ω in (1.1) for $\kappa = 0$, converge in the nonrelativistic limit to the eigenvalues of the Dirichlet Laplacian. However, in [3] only the convergence of the eigenvalues and not of the operator itself was studied.

In order to state our main result on the nonrelativistic limit of the operators H_κ^Ω we make the following assumption on Ω, where we use the definition of a C^2-domain as, e.g., in [41].

Hypothesis 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a (bounded or unbounded) C^2-domain, not necessarily connected, with a compact boundary and unit normal vector field ν pointing outwards of Ω. The bounded element in $\{\Omega, \mathbb{R}^3 \setminus \overline{\Omega}\}$ is denoted by Ω_+, the unbounded element in $\{\Omega, \mathbb{R}^3 \setminus \overline{\Omega}\}$ is denoted by Ω_-, and ν_+ is the unit normal vector field pointing outwards of Ω_+, so that $\nu = \nu_+$ if $\Omega = \Omega_+$ and $\nu = -\nu_+$ if $\Omega = \Omega_-$. For the common boundary we write $\Sigma := \partial\Omega = \partial\Omega_+ = \partial\Omega_-$.*

Then, the main result of the present paper reads as follows:

Theorem 1.2 *Let $\kappa \in \mathbb{R}$, $\Omega \subset \mathbb{R}^3$ be as in Hypothesis 1.1, and $z \in \mathbb{C} \setminus [0, \infty)$. Then, there exists a constant $K(z)$ such that for all c sufficiently large $z + \frac{c^2}{2} \in \rho(H_\kappa^\Omega) \cap \rho(-\Delta_D^\Omega)$ and*

$$\left\| \left(H_\kappa^\Omega - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta_D^\Omega - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\|_{L^2(\Omega;\mathbb{C}^4) \to L^2(\Omega;\mathbb{C}^4)} \leq \frac{K(z)}{\sqrt{c}}, \tag{1.3}$$

where $-\Delta_D^\Omega$ denotes the self-adjoint Dirichlet Laplacian in $L^2(\Omega; \mathbb{C})$.

The strategy to prove Theorem 1.2 is to consider the self-adjoint orthogonal sum $H_\kappa^{\Omega_+} \oplus H_\kappa^{\Omega_-}$ in $L^2(\Omega_+; \mathbb{C}^4) \oplus L^2(\Omega_-; \mathbb{C}^4)$, which can be identified with a self-adjoint Dirac operator A_κ^Σ in $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with a δ-shell potential supported on Σ, see [6, 11, 16, 21]. Such types of Dirac operators with singular interactions are well-studied, see the review article [17] and the references therein. We collect some properties of A_κ^Σ in Sect. 2.2 and provide a Krein type formula in Proposition 2.2 for its resolvent, which is the key tool for the analysis of the nonrelativistic limit. Each of the terms appearing in the resolvent formula will be examined separately and the main technical difficulty is the limit behavior of the inverse of

$$\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c, \tag{1.4}$$

involving a strongly singular boundary integral operator $\mathcal{C}_{z+c^2/2}$ on Σ, a coefficient matrix ϑ_c modelling the boundary condition in dom H_κ^Ω, and a scaling matrix $\mathcal{M}_c$ (see (2.5), (2.9), and (2.10) for details). In fact, it turns out that the operator in (1.4) does not converge to a boundedly invertible operator in one Sobolev space on Σ, but instead it is necessary to study the convergence of the inverse of (1.4) as an operator acting between different fractional order Sobolev spaces on Σ. Here we argue via the Schur complement and rely on an advanced and deep analysis of various boundary integral operators appearing in this context. Eventually, it turns out that the limit of A_κ^Σ in the norm resolvent sense is an orthogonal sum of Dirichlet Laplacians and compressing the resolvents onto the original domain leads to (1.3).

It is well-known that the operator norm convergence in (1.3) implies the convergence of the corresponding spectra (see, e.g. [35, 49, 52]) and, in particular, if Ω is bounded, the spectrum of H_κ^Ω is discrete and we conclude convergence of eigenvalues. This leads to spectral inequalities for the positive eigenvalues of the Dirac operators H_κ^Ω, $\kappa \in \mathbb{R}$, for $c > 0$ sufficiently large; cf. Remark 3.7 for analogous results for the negative eigenvalues.

Corollary 1.3 *Let $\kappa \in \mathbb{R}$, $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain, $B \subset \mathbb{R}^3$ be a ball such that $|B| = |\Omega|$ and $B_1, B_2 \subset \mathbb{R}^3$ be identical and disjoint balls such that $|B_1|+|B_2| = |\Omega|$. Then, the following assertions hold for $c > 0$ sufficiently large:*

(i) $\lambda_j^+(H_\kappa^B) \leq \lambda_j^+(H_\kappa^\Omega)$ for $j \in \{1,2\}$ and equality holds if and only if Ω is a ball.

(ii) $\lambda_j^+(H_\kappa^{B_1\cup B_2}) \leq \lambda_j^+(H_\kappa^\Omega)$ for $j \in \{3,4\}$ and equality holds if and only if Ω is the union of two identical disjoint balls.

(iii) If, in addition, Ω is connected, then

$$\frac{\lambda_j^+(H_\kappa^B)}{\lambda_l^+(H_\kappa^B)} \leq \frac{\lambda_j^+(H_\kappa^\Omega)}{\lambda_l^+(H_\kappa^\Omega)}, \qquad j \in \{1,2\},\ l \in \{3,4\},$$

and equality holds if and only if Ω is a ball.

The article is organized as follows. In Sect. 2 we introduce the free Dirac operator in $\mathbb{R}^3$ and some associated integral operators, show the connection of H_κ^Ω and Dirac operators A_κ^Σ with singular interactions supported on $\Sigma = \partial\Omega$, and recall some properties of the Dirichlet Laplacian. In Sect. 3 we compute the nonrelativistic limit of A_κ^Σ, which allows us to prove Theorem 1.2 and Corollary 1.3.

Notations

The Dirac matrices are denoted by

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k \in \{1,2,3\}, \qquad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{1.5}$$

where I_n is the $n \times n$ identity matrix, $n \in \mathbb{N}$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices. The Dirac matrices satisfy

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \quad \alpha_j \beta + \beta \alpha_j = 0, \quad j, k \in \{1, 2, 3\}, \tag{1.6}$$

where δ_{jk} is the Kronecker symbol. Moreover, the notations

$$\alpha \cdot \nabla = \sum_{j=1}^{3} \alpha_j \partial_j \quad \text{and} \quad \alpha \cdot x = \sum_{j=1}^{3} \alpha_j x_j, \quad x = (x_1, x_2, x_3) \in \mathbb{C}^3, \tag{1.7}$$

will often be used.

If $M \subset \mathbb{R}^3$ and $k, l \in \mathbb{N}$, then the set of all continuous and k times continuously differentiable functions $f : M \to \mathbb{C}^l$ is denoted by $C(M; \mathbb{C}^l)$ and $C^k(M; \mathbb{C}^l)$, respectively. Next, denote by $\mathcal{F}$ the Fourier transform on the space $\mathcal{S}'(\mathbb{R}^3; \mathbb{C})$ of tempered distributions. For the Sobolev spaces $H^s(\mathbb{R}^3; \mathbb{C})$, $s \in \mathbb{R}$, we shall use the definition

$$H^s(\mathbb{R}^3; \mathbb{C}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3; \mathbb{C}) \, : \, \int_{\mathbb{R}^3} (1 + |x|^2)^s |\mathcal{F} f(x)|^2 dx < \infty \right\}, \tag{1.8}$$

with Hilbert space norm

$$\| f \|^2_{H^s(\mathbb{R}^3; \mathbb{C})} := \int_{\mathbb{R}^3} (1 + |x|^2)^s |\mathcal{F} f(x)|^2 dx, \quad f \in H^s(\mathbb{R}^3; \mathbb{C}). \tag{1.9}$$

For Ω as in Hypothesis 1.1 the Sobolev spaces $H^s(\Omega; \mathbb{C})$, $s > 0$, are defined via restrictions of functions from $H^s(\mathbb{R}^3; \mathbb{C})$ onto Ω, and the spaces $H^t(\Sigma; \mathbb{C})$, $t \in [-2, 2]$, on the boundary Σ of Ω are defined by using an open cover of Σ and a corresponding partition of unity, reducing it to Sobolev spaces on hypographs; see, e.g., [41, Chapter 3] for more details. We denote by $\gamma_D : H^1(\Omega; \mathbb{C}) \to H^{1/2}(\Sigma; \mathbb{C})$ the bounded Dirichlet trace operator and we shall use the same symbol for the trace operator $\gamma_D : H^1(\mathbb{R}^3; \mathbb{C}) \to H^{1/2}(\Sigma; \mathbb{C})$. Sobolev spaces of vector valued functions are defined component-wise and in this context the action of the Dirichlet trace operator is also understood component-wise.

If A is a linear operator acting between two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, then its domain, range, and kernel are denoted by dom A, ran A, and ker A, respectively. Whenever A is bounded and everywhere defined, then $\|A\|_{\mathcal{H} \to \mathcal{G}}$ is the operator norm of A. If A is self-adjoint in $\mathcal{H}$, then the symbols $\rho(A)$, $\sigma(A)$, $\sigma_{\text{ess}}(A)$, and $\sigma_{\text{disc}}(A)$ are used for the resolvent set, spectrum, essential spectrum, and discrete spectrum of A, respectively.

2 Preliminaries

In this preliminary section we first collect several results about the free Dirac operator in $\mathbb{R}^3$ and associated integral operators. Afterwards, we show how the operators H_κ^Ω in (1.1) are related to Dirac operators with δ-shell potentials, and we recall some useful properties of the single layer potential, single layer boundary integral operator, and the Dirichlet Laplacian that are needed to prove Theorem 1.2. Throughout this section we assume that Ω, $\Omega_\pm$, and Σ are as in Hypothesis 1.1.

2.1 The Free Dirac Operator and Associated Integral Operators

It is well-known that the free Dirac operator

$$A_0 f = -ic(\alpha \cdot \nabla) f + \frac{c^2}{2}\beta f, \qquad \operatorname{dom} A_0 = H^1(\mathbb{R}^3; \mathbb{C}^4), \tag{2.1}$$

in $\mathbb{R}^3$ is self-adjoint in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and its spectrum is $\sigma(A_0) = (-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)$. For $z \in \rho(A_0) = \mathbb{C} \setminus ((-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty))$ and $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$, the resolvent of A_0 is given by

$$(A_0 - z)^{-1} f(x) = \int_{\mathbb{R}^3} G_z(x-y) f(y) dy, \qquad x \in \mathbb{R}^3,$$

where the function $G_z : \mathbb{R}^3 \setminus \{0\} \to \mathbb{C}^{4\times 4}$ is defined by

$$G_z(x) = \left(\frac{z}{c^2} I_4 + \frac{1}{2}\beta + \left(1 - i\sqrt{\frac{z^2}{c^2} - \frac{c^2}{4}}|x|\right) \frac{i(\alpha \cdot x)}{c|x|^2} \right) \frac{e^{i\sqrt{z^2/c^2 - c^2/4}|x|}}{4\pi|x|} \tag{2.2}$$

and the square root is chosen such that $\operatorname{Im} \sqrt{z^2/c^2 - c^2/4} > 0$; cf. [50, Section 1.E].

Next, we introduce several integral operators and summarize some of their properties that are necessary to prove Theorem 1.2; we refer to [11, 15, 17] for more details. In the following $\gamma_D : H^1(\mathbb{R}^3;\mathbb{C}^4) \to H^{1/2}(\Sigma;\mathbb{C}^4)$ denotes the Dirichlet trace operator. For $z \in \rho(A_0)$ the map

$$\Phi_z^* := \gamma_D (A_0 - \overline{z})^{-1} : L^2(\mathbb{R}^3;\mathbb{C}^4) \to H^{1/2}(\Sigma;\mathbb{C}^4) \tag{2.3}$$

is well-defined and bounded. It is not difficult to see that Φ_z^* acts on $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$ as

$$\Phi_z^* f(x) = \int_{\mathbb{R}^3} G_{\overline{z}}(x-y) f(y) dy, \quad x \in \Sigma.$$

The definition of Φ_z^* in (2.3) allows to define the bounded anti-dual map

$$\Phi_z := (\Phi_z^*)' : H^{-1/2}(\Sigma;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4). \tag{2.4}$$

With the help of Fubini's theorem and $(G_{\overline{z}}(x))^* = G_z(-x)$ one shows that Φ_z acts on $\varphi \in L^2(\Sigma; \mathbb{C}^4)$ as

$$\Phi_z\varphi(x) := \int_\Sigma G_z(x-y)\varphi(y)d\sigma(y), \qquad x \in \mathbb{R}^3 \setminus \Sigma,$$

where $d\sigma$ denotes the surface measure on Σ. We will also make use of the strongly singular boundary integral operator $\mathcal{C}_z : L^2(\Sigma; \mathbb{C}^4) \to L^2(\Sigma; \mathbb{C}^4)$, $z \in \rho(A_0)$, acting via

$$\mathcal{C}_z\varphi(x) := \lim_{\varepsilon\to 0+} \int_{\Sigma\setminus B(x,\varepsilon)} G_z(x-y)\varphi(y)d\sigma(y), \qquad x \in \Sigma,\ \varphi \in L^2(\Sigma; \mathbb{C}^4), \tag{2.5}$$

where $B(x, \varepsilon)$ is the ball of radius ε centered at x. For $s \in [0, \frac{1}{2}]$ the map $\mathcal{C}_z$ gives rise to a bounded operator

$$\mathcal{C}_z : H^s(\Sigma; \mathbb{C}^4) \to H^s(\Sigma; \mathbb{C}^4). \tag{2.6}$$

The adjoint of the realization of $\mathcal{C}_z$ in $L^2(\Sigma; \mathbb{C}^4)$ satisfies $\mathcal{C}_z^* = \mathcal{C}_{\overline{z}}$ and it follows from (2.6) that $\mathcal{C}_z$ admits a bounded extension to $H^s(\Sigma; \mathbb{C}^4)$, $s \in [-\frac{1}{2}, 0]$, such that

$$\mathcal{C}_z = (\mathcal{C}_{\overline{z}})' : H^s(\Sigma; \mathbb{C}^4) \to H^s(\Sigma; \mathbb{C}^4), \quad s \in \big[-\tfrac{1}{2}, 0\big], \tag{2.7}$$

where $(\mathcal{C}_{\overline{z}})'$ denotes the anti-dual of $\mathcal{C}_{\overline{z}}$.

2.2 H_κ^Ω and Dirac Operators with δ-Shell Potentials

In this subsection we show how the operators H_κ^Ω defined in (1.1) are related to Dirac operators A_κ^Σ with δ-shell potentials supported on Σ; the latter operators are well-studied, see, e.g., [6, 11, 17] and the references therein. Recall the notation $\Omega_\pm$ and the unit outward normal vector field ν_+ from Hypothesis 1.1. For a function $f : \mathbb{R}^3 \to \mathbb{C}^4$ we write $f_\pm := f \upharpoonright \Omega_\pm$. Define the operator

$$\begin{aligned} A_\kappa^\Sigma &= \big(-ic(\alpha\cdot\nabla) + \tfrac{c^2}{2}\beta\big)f_+ \oplus \big(-ic(\alpha\cdot\nabla) + \tfrac{c^2}{2}\beta\big)f_-, \\ \operatorname{dom} A_\kappa^\Sigma &= \big\{f = f_+ \oplus f_- \in H^1(\Omega_+; \mathbb{C}^4) \oplus H^1(\Omega_-; \mathbb{C}^4) : \\ &\quad -i(\alpha\cdot\nu_+)(\gamma_D f_+ - \gamma_D f_-) \\ &\qquad = (\sinh(\kappa)I_4 + \cosh(\kappa)\beta)(\gamma_D f_+ + \gamma_D f_-)\big\}, \end{aligned} \tag{2.8}$$

in $L^2(\mathbb{R}^3; \mathbb{C}^4)$. We note that A_κ^Σ is the rigorously defined operator associated with the formal differential expression $-ic(\alpha\cdot\nabla) + \frac{c^2}{2}\beta + 2c(\sinh(\kappa)I_4 + \cosh(\kappa)\beta)\delta_\Sigma$.

Our first observation is an immediate consequence from [11, Lemma 3.1 (ii)], which says that the operator formally given by $-ic(\alpha\cdot\nabla) + \frac{c^2}{2}\beta + (\eta I_4 + \tau\beta)\delta_\Sigma$ decouples to the orthogonal sum of two Dirac operators with boundary conditions acting on functions in $\Omega_\pm$ if and only if $\eta^2 - \tau^2 = -4c^2$; in the present setting the strength η of the electrostatic interaction in [11] is $2c\sinh(\kappa)$, the strength τ of the Lorentz scalar interaction is $2c\cosh(\kappa)$, and the normal vector in the definition of $H_\kappa^{\Omega_-}$ in (1.1) is $-\nu_+$. Note that this choice of η and τ is a natural parametrization of the arm of the hyperbola $\eta^2 - \tau^2 = -4c^2$ that contains the MIT bag boundary conditions. We also refer the reader to [6, Section 5], [16, Section 5.3], or [17, Section 5.2] for similar statements.

Lemma 2.1 *The equality* $A_\kappa^\Sigma = H_\kappa^{\Omega_+} \oplus H_\kappa^{\Omega_-}$ *holds.*

In the next proposition we summarize some properties of the operator A_κ^Σ that will be particularly useful for our analysis. Recall that A_0 is the free Dirac operator defined in (2.1) and that Φ_z and $\mathcal{C}_z$ are the operators defined in (2.4) and (2.5), respectively. Moreover, define the two numbers

$$a_+ := \frac{1}{2}(\cosh(\kappa) - \sinh(\kappa)) > 0, \quad a_- := -\frac{1}{2}(\cosh(\kappa) + \sinh(\kappa)) < 0, \tag{2.9}$$

and for $c > 0$ the coefficient matrix ϑ_c and the scaling matrix $\mathcal{M}_c$

$$\vartheta_c := \begin{pmatrix} \frac{1}{c}a_+ I_2 & 0 \\ 0 & a_- I_2 \end{pmatrix} \in \mathbb{C}^{4\times 4}, \qquad \mathcal{M}_c := \begin{pmatrix} I_2 & 0 \\ 0 & \sqrt{c} I_2 \end{pmatrix} \in \mathbb{C}^{4\times 4}. \tag{2.10}$$

Proposition 2.2 *Let* $\kappa \in \mathbb{R}$ *and* $c > 0$. *Then, the operator* A_κ^Σ *in* (2.8) *is self-adjoint in* $L^2(\mathbb{R}^3; \mathbb{C}^4)$, $\sigma(A_\kappa^\Sigma) = (-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)$, *for* $z \in \rho(A_\kappa^\Sigma)$ *the linear operator* $\vartheta_c + \mathcal{M}_c \mathcal{C}_z \mathcal{M}_c$ *admits a bounded inverse in* $H^{1/2}(\Sigma; \mathbb{C}^4)$, *and the formula*

$$(A_\kappa^\Sigma - z)^{-1} = (A_0 - z)^{-1} - \Phi_z \mathcal{M}_c \big(\vartheta_c + \mathcal{M}_c \mathcal{C}_z \mathcal{M}_c\big)^{-1} \mathcal{M}_c \Phi_{\bar{z}}^*$$

holds.

Proof It follows from [11, Lemma 3.3 and Theorems 3.4 & 4.1] or [17, Theorem 5.6] (in the case $c = 1$) that A_κ^Σ is self-adjoint in $L^2(\mathbb{R}^3; \mathbb{C}^4)$, that $\sigma_{\mathrm{ess}}(A_\kappa^\Sigma) = (-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)$, that $I_4 + 2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\mathcal{C}_z$ is bijective in $H^{1/2}(\Sigma; \mathbb{C}^4)$ for $z \in \rho(A_\kappa^\Sigma) \cap \rho(A_0)$ and that the resolvent formula

$$\begin{aligned}(A_\kappa^\Sigma - z)^{-1} = (A_0 - z)^{-1} - \Phi_z \big(I_4 \; &+ 2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\mathcal{C}_z\big)^{-1} \\ &\cdot 2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\Phi_{\bar{z}}^*\end{aligned}$$

holds. Note that the matrix $2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)$ is invertible with inverse

$$\big(2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\big)^{-1} = \frac{1}{2c}(-\sinh(\kappa) I_4 + \cosh(\kappa)\beta) = \mathcal{M}_c^{-1} \vartheta_c \mathcal{M}_c^{-1}.$$

Hence, also $\vartheta_c + \mathcal{M}_c \mathcal{C}_z \mathcal{M}_c = \mathcal{M}_c(\frac{1}{2c}(-\sinh(\kappa) I_4 + \cosh(\kappa)\beta) + \mathcal{C}_z)\mathcal{M}_c$ is bijective in $H^{1/2}(\Sigma; \mathbb{C}^4)$ and the claimed resolvent formula is true.

It remains to show that $(-\frac{c^2}{2}, \frac{c^2}{2}) \cap \sigma(A_\kappa^\Sigma) = (-\frac{c^2}{2}, \frac{c^2}{2}) \cap \sigma_p(A_\kappa^\Sigma) = \emptyset$. For this, we use the Birman–Schwinger principle for A_κ^Σ from [11, Lemma 3.3], which states that

$$z \in \left(-\frac{c^2}{2}, \frac{c^2}{2}\right) \cap \sigma_p(A_\kappa^\Sigma) \quad \text{if and only if} \quad 0 \in \sigma_p(I_4 + 2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\mathcal{C}_z). \tag{2.11}$$

Let $z \in (-\frac{c^2}{2}, \frac{c^2}{2})$ and assume that $\varphi \in \ker(I_4 + 2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\mathcal{C}_z)$. Then,

$$\begin{aligned} 0 &= \big((I_4 + 2c\mathcal{C}_z(-\sinh(\kappa) I_4 + \cosh(\kappa)\beta)) \\ &\quad \cdot (I_4 + 2c(\sinh(\kappa) I_4 + \cosh(\kappa)\beta)\mathcal{C}_z)\varphi, \varphi\big)_{L^2(\Sigma;\mathbb{C}^4)} \\ &= \Big((I_4 + 4c^2\mathcal{C}_z^2 + 2c\cosh(\kappa)(\mathcal{C}_z\beta + \beta\mathcal{C}_z))\varphi, \varphi\Big)_{L^2(\Sigma;\mathbb{C}^4)}. \end{aligned} \tag{2.12}$$

With (1.6) and (2.2) one finds that

$$\mathcal{C}_z\beta + \beta\mathcal{C}_z = 2\left(\frac{1}{2} I_4 + \frac{z}{c^2}\beta\right)\mathcal{S}_{z^2/c^2 - c^2/4},$$

where $\mathcal{S}_{z^2/c^2-c^2/4}$ is the single layer boundary integral operator defined below in (2.14). In the present situation we have $z^2/c^2 - c^2/4 < 0$ and hence it follows that $\mathcal{S}_{z^2/c^2-c^2/4}$ is a non-negative operator in $L^2(\Sigma; \mathbb{C})$; cf. the text below (2.15) in the next subsection. Therefore, $I_4 + 4c^2\mathcal{C}_z^2 + 2c\cosh(\kappa)(\mathcal{C}_z\beta + \beta\mathcal{C}_z)$ is a strictly positive operator in $L^2(\Sigma; \mathbb{C}^4)$ and we obtain $\varphi = 0$ from (2.12). Therefore, by (2.11) we have $z \notin \sigma_p(A_\kappa^\Sigma)$. □

From the properties of A_κ^Σ one can now easily deduce the properties of H_κ^Ω stated in the following corollary, when Ω coincides either with Ω_+ or Ω_-. The claims follow immediately from Lemma 2.1 and Proposition 2.2; for (i) one additionally uses that $\operatorname{dom} H_\kappa^\Omega \subset H^1(\Omega; \mathbb{C}^4)$ is compactly embedded in $L^2(\Omega; \mathbb{C}^4)$ if Ω is bounded, see also [7, Lemma 1.2], and that H_κ^Ω commutes with the anti-linear time reversal operator $\mathcal{T} f = -i\left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)\alpha_2 \overline{f}$, see the proof of [11, Proposition 4.2 (ii)] for details.

Corollary 2.3 *Let $\kappa \in \mathbb{R}$ and $c > 0$. Then, the operator H_κ^Ω in (1.1) is self-adjoint in $L^2(\Omega; \mathbb{C}^4)$ and the following holds:*

(i) If Ω is bounded, then $\sigma(H_\kappa^\Omega) = \sigma_{\mathrm{disc}}(H_\kappa^\Omega) \subset (-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)$ and all eigenvalues of H_κ^Ω have even multiplicity.

(ii) If Ω is unbounded, then $\sigma(H_\kappa^\Omega) = (-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)$.

Moreover, for $z \in \mathbb{C} \setminus ((-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty))$ the resolvent formula

$$(H_\kappa^\Omega - z)^{-1} = P_\Omega (A_0 - z)^{-1} P_\Omega^* - P_\Omega \Phi_z \mathcal{M}_c \big(\vartheta_c + \mathcal{M}_c \mathcal{C}_z \mathcal{M}_c\big)^{-1} \mathcal{M}_c \Phi_{\overline{z}}^* P_\Omega^*$$

holds, where $P_\Omega : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\Omega; \mathbb{C}^4)$ is the projection operator acting as $f \mapsto f \upharpoonright \Omega$ and its adjoint $P_\Omega^* : L^2(\Omega; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4)$ is the embedding operator which extends a function $g \in L^2(\Omega; \mathbb{C}^4)$ by zero.

2.3 The Dirichlet Laplacian and Associated Integral Operators

We begin by briefly recalling some properties of the single layer potential and single layer boundary integral operator associated with $-\Delta-\mu$, where $-\Delta$ is the self-adjoint Laplacian in $L^2(\mathbb{R}^3;\mathbb{C})$ defined on $H^2(\mathbb{R}^3;\mathbb{C})$ and $\mu\in\rho(-\Delta)=\mathbb{C}\setminus[0,\infty)$.

For $\varphi\in L^2(\Sigma;\mathbb{C})$ the single layer potential SL_μ is the formal integral operator that acts as

$$SL_\mu\varphi(x)=\int_\Sigma \frac{e^{i\sqrt{\mu}|x-y|}}{4\pi|x-y|}\varphi(y)d\sigma(y),\qquad x\in\mathbb{R}^3\setminus\Sigma, \tag{2.13}$$

and the single layer boundary integral operator $\mathcal{S}_\mu$ is the mapping defined by

$$\mathcal{S}_\mu\varphi(x)=\int_\Sigma \frac{e^{i\sqrt{\mu}|x-y|}}{4\pi|x-y|}\varphi(y)d\sigma(y),\qquad x\in\Sigma, \tag{2.14}$$

where $\sqrt{\mu}$ is again the complex square root satisfying $\operatorname{Im}\sqrt{\mu}>0$ for $\mu\in\mathbb{C}\setminus[0,\infty)$. It is well-known that for any $s\in[-\frac{1}{2},\frac{1}{2}]$ the map $\mathcal{S}_\mu$ gives rise to a bounded and bijective operator

$$\mathcal{S}_\mu: H^s(\Sigma;\mathbb{C})\to H^{s+1}(\Sigma;\mathbb{C}). \tag{2.15}$$

Moreover, we will use that for $\mu<0$ the realization of $\mathcal{S}_\mu$ in $L^2(\Sigma;\mathbb{C})$ is self-adjoint and non-negative. These claims can be shown in the same way as in [12, Lemma 2.6], where the two-dimensional case and $\mu=-1$ is treated. Furthermore, by [41, Corollary 6.14] the mapping SL_μ gives for any $s\in(-\frac{1}{2},1]$ rise to a bounded operator

$$SL_\mu: H^{s-1/2}(\Sigma;\mathbb{C})\to H^{s+1}(\Omega_+;\mathbb{C})\oplus H^{s+1}(\Omega_-;\mathbb{C}).$$

Moreover, the representations

$$SL_\mu=(-\Delta-\mu)^{-1}\gamma_D': H^{-1/2}(\Sigma;\mathbb{C})\to H^1(\mathbb{R}^3;\mathbb{C})$$

and

$$\mathcal{S}_\mu=\gamma_D(-\Delta-\mu)^{-1}\gamma_D': H^{-1/2}(\Sigma;\mathbb{C})\to H^{1/2}(\Sigma;\mathbb{C}) \tag{2.16}$$

hold, where $\gamma_D: H^1(\mathbb{R}^3;\mathbb{C})\to H^{1/2}(\Sigma;\mathbb{C})$ is the bounded Dirichlet trace operator and $\gamma_D': H^{-1/2}(\Sigma;\mathbb{C})\to H^{-1}(\mathbb{R}^3;\mathbb{C})$ its anti-dual map. We will also use that the L^2-adjoint of SL_μ is given by

$$SL_\mu^*=\gamma_D(-\Delta-\overline{\mu})^{-1}: L^2(\mathbb{R}^3;\mathbb{C})\to H^{3/2}(\Sigma;\mathbb{C}), \tag{2.17}$$

which is bounded, as the restriction $\gamma_D: H^2(\mathbb{R}^3;\mathbb{C})\to H^{3/2}(\Sigma;\mathbb{C})$ is bounded. Next, we state a useful continuity property of the map $\mu\mapsto\mathcal{S}_\mu$.

Lemma 2.4 *Let $M\subset\mathbb{C}\setminus[0,\infty)$ be compact. Then, for all $\mu_1,\mu_2\in M$ the operator $\mathcal{S}_{\mu_1}-\mathcal{S}_{\mu_2}$ has a bounded extension from $H^{-3/2}(\Sigma;\mathbb{C})$ to $H^{3/2}(\Sigma;\mathbb{C})$ and there exists* *a constant $K(M) > 0$ such that the estimate*

$$\left\| \mathcal{S}_{\mu_1} - \mathcal{S}_{\mu_2} \right\|_{H^{-3/2}(\Sigma;\mathbb{C}) \to H^{3/2}(\Sigma;\mathbb{C})} \leq K(M)|\mu_1 - \mu_2| \tag{2.18}$$

holds. In particular, for any $s \in [-\frac{3}{2}, \frac{3}{2}]$ the operator $\mathcal{S}_\mu : H^s(\Sigma;\mathbb{C}) \to H^s(\Sigma;\mathbb{C})$ is uniformly bounded in $\mu \in M$.

Proof It suffices to show that

$$(-\Delta - \mu_1)^{-1} - (-\Delta - \mu_2)^{-1} = (\mu_1 - \mu_2)(-\Delta - \mu_1)^{-1}(-\Delta - \mu_2)^{-1}$$

gives rise to a bounded operator from $H^{-2}(\mathbb{R}^2;\mathbb{C})$ to $H^2(\mathbb{R}^2;\mathbb{C})$ that satisfies

$$\left\| (-\Delta - \mu_1)^{-1} - (-\Delta - \mu_2)^{-1} \right\|_{H^{-2}(\mathbb{R}^3;\mathbb{C}) \to H^2(\mathbb{R}^3;\mathbb{C})} \leq K(M)|\mu_1 - \mu_2|, \tag{2.19}$$

as then (2.18) follows from (2.16) and the fact that γ_D has a continuous restriction $\gamma_D : H^2(\mathbb{R}^3;\mathbb{C}) \to H^{3/2}(\Sigma;\mathbb{C})$ and γ_D' a continuous extension $\gamma_D' : H^{-3/2}(\Sigma;\mathbb{C}) \to H^{-2}(\mathbb{R}^3;\mathbb{C})$. To show (2.19), we compute for $f \in H^{-2}(\mathbb{R}^3;\mathbb{C})$, taking (1.9) into account,

$$\begin{aligned}
&\left\| \big((-\Delta - \mu_1)^{-1} - (-\Delta - \mu_2)^{-1}\big) f \right\|^2_{H^2(\mathbb{R}^3;\mathbb{C})} \\
&\qquad = \int_{\mathbb{R}^3} (1 + |x|^2)^2 \left| \mathcal{F}\big((-\Delta - \mu_1)^{-1} - (-\Delta - \mu_2)^{-1}\big) f(x) \right|^2 dx \\
&\qquad = \int_{\mathbb{R}^3} (1 + |x|^2)^2 \left| \frac{1}{|x|^2 - \mu_1} - \frac{1}{|x|^2 - \mu_2} \right|^2 |\mathcal{F} f(x)|^2 dx \\
&\qquad = \int_{\mathbb{R}^3} \frac{(1 + |x|^2)^4 |\mu_1 - \mu_2|^2}{|(|x|^2 - \mu_1)(|x|^2 - \mu_2)|^2} \frac{|\mathcal{F} f(x)|^2}{(1 + |x|^2)^2} dx \\
&\qquad \leq \sup_{x \in \mathbb{R}^3} \frac{(1 + |x|^2)^4 |\mu_1 - \mu_2|^2}{|(|x|^2 - \mu_1)(|x|^2 - \mu_2)|^2} \cdot \| f \|^2_{H^{-2}(\mathbb{R}^3;\mathbb{C})}.
\end{aligned}$$

This shows (2.19) with $K(M) := \sup_{x \in \mathbb{R}^3, \mu_1, \mu_2 \in M} \frac{(1+|x|^2)^2}{|(|x|^2 - \mu_1)(|x|^2 - \mu_2)|}$.

Eventually, it follows from (2.18) that $\mathcal{S}_\mu : H^{-3/2}(\Sigma;\mathbb{C}) \to H^{3/2}(\Sigma;\mathbb{C})$ is uniformly bounded in $\mu \in M$. Since $H^{s_1}(\Sigma;\mathbb{C})$ is continuously embedded in $H^{s_2}(\Sigma;\mathbb{C})$ for $s_1 > s_2$, we conclude that $\mathcal{S}_\mu : H^s(\Sigma;\mathbb{C}) \to H^s(\Sigma;\mathbb{C})$ is also uniformly bounded in $\mu \in M$ for any $s \in [-\frac{3}{2}, \frac{3}{2}]$. □

Let again Ω be a C^2-domain as in Hypothesis 1.1. In the next lemma we express the resolvent of the self-adjoint Dirichlet Laplacian

$$-\Delta_D^\Omega f = -\Delta f, \qquad \operatorname{dom}(-\Delta_D^\Omega) = \big\{ f \in H^2(\Omega;\mathbb{C}) : \gamma_D f = 0 \big\}, \tag{2.20}$$

in $L^2(\Omega;\mathbb{C})$ as the compression of the resolvent of the self-adjoint Laplacian $-\Delta$ in $L^2(\mathbb{R}^3;\mathbb{C})$ and a perturbation term. The statement follows from, e.g., [1, Theorem 4.4], [9, Theorem 3.2] or [14, Theorem 8.6.3], where instead of the single layer potential (2.13) and the single layer boundary integral operator (2.14) the terminology of γ-fields, Weyl functions or Q-functions, and Dirichlet-to-Neumann maps is used.

Lemma 2.5 *Let Ω and $\Omega_\pm$ be as in Hypothesis 1.1 and $-\Delta_D^\Omega$ and $-\Delta_D^{\Omega_\pm}$ be the corresponding Dirichlet Laplacians defined as in (2.20). Then, for the orthogonal sum $-\Delta_D := (-\Delta_D^{\Omega_+}) \oplus (-\Delta_D^{\Omega_-})$ and any $z \in \rho(-\Delta_D) = \mathbb{C} \setminus [0, \infty)$ the resolvent formula*

$$(-\Delta_D - z)^{-1} = (-\Delta - z)^{-1} - SL_z \mathcal{S}_z^{-1} SL_{\bar{z}}^*$$

holds. In particular, one has

$$(-\Delta_D^\Omega - z)^{-1} = P_\Omega(-\Delta - z)^{-1} P_\Omega^* - P_\Omega SL_z \mathcal{S}_z^{-1} SL_{\bar{z}}^* P_\Omega^*$$

with the projection and embedding operators P_Ω and P_Ω^* from Corollary 2.3.

3 The Nonrelativistic Limit

In this section we compute the nonrelativistic limit of the operator A_κ^Σ defined in (2.8) and use this to show Theorem 1.2 and Corollary 1.3. Again, we will always assume that $\Omega_\pm$ is as in Hypothesis 1.1 and $\Sigma = \partial\Omega_\pm$. Furthermore, we will often assume that $z \in \mathbb{C} \setminus [0, \infty)$ and $c > \sqrt{|z|}$, as then $z + \frac{c^2}{2} \in \rho(A_0) = \rho(A_\kappa^\Sigma)$; cf. Proposition 2.2. In the following, the Krein type resolvent formula

$$\begin{aligned}\left(A_\kappa^\Sigma - \left(z + \frac{c^2}{2}\right)\right)^{-1} &= \left(A_0 - \left(z + \frac{c^2}{2}\right)\right)^{-1} \\ &\quad - \Phi_{z+c^2/2}\mathcal{M}_c\big(\vartheta_c + \mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c\big)^{-1}\mathcal{M}_c\Phi_{\bar{z}+c^2/2}^* \end{aligned} \tag{3.1}$$

from Proposition 2.2 will play an important role. We will compute the limit of each of the terms on the right hand side separately. The convergence of $(A_0 - (z + c^2/2))^{-1}$, $\Phi_{z+c^2/2}\mathcal{M}_c$, and $\mathcal{M}_c\Phi_{\bar{z}+c^2/2}^*$ is investigated in Sect. 3.1, the convergence of the map $(\vartheta_c + \mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c)^{-1}$ is treated in Sect. 3.2, and the nonrelativistic limit of A_κ^Σ is computed in Sect. 3.3.

3.1 Convergence of $(A_0 - (z + c^2/2))^{-1}$, $\Phi_{z+c^2/2}\mathcal{M}_c$, and $\mathcal{M}_c\Phi_{\bar{z}+c^2/2}^*$

First, the nonrelativistic limit of the free Dirac operator A_0 defined in (2.1) is discussed. This result is well-known, it follows, e.g., as a special case of the results in [50, Section 6]. However, since the result and the topology, in which the convergence takes place, are of importance in the analysis of $\Phi_{z+c^2/2}\mathcal{M}_c$ and $\mathcal{M}_c\Phi_{\bar{z}+c^2/2}^*$, we give a direct simple proof here to keep the presentation self-contained.

Proposition 3.1 *Let $z \in \mathbb{C} \setminus [0, \infty)$ and $c > \sqrt{|z|}$. Then, there exists a constant $K(z)$ such that*

$$\left\| \left(A_0 - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4) \to H^1(\mathbb{R}^3;\mathbb{C}^4)} \leq \frac{K(z)}{c}.$$

Proof We shall use (1.9) and compute for $f \in L^2(\mathbb{R}^3; \mathbb{C}^4)$ and $c > \sqrt{|z|}$

$$\begin{aligned} &\int_{\mathbb{R}^3} (1 + |x|^2) \left| \mathcal{F} \left[\left(A_0 - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right] f(x) \right|^2 dx \\ &\quad = \int_{\mathbb{R}^3} (1 + |x|^2) \left| \left[\frac{\alpha \cdot x + \frac{c}{2}\beta + \left(\frac{z}{c} + \frac{c}{2} \right) I_4}{c(|x|^2 - (\frac{z^2}{c^2} + z))} - \frac{1}{|x|^2 - z} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right] \mathcal{F} f(x) \right|^2 dx. \end{aligned}$$

Next, we decompose the part of the integrand that does not depend on $\mathcal{F} f$ in the last line of the equation. Using $\frac{1}{2}(\beta + I_4) = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$ we find

$$\begin{aligned} &\sup_{x \in \mathbb{R}^3} (1 + |x|^2) \left| \frac{\alpha \cdot x + \frac{z}{c} I_4}{c\left(|x|^2 - \frac{z^2}{c^2} - z\right)} + \frac{\frac{1}{2}(\beta + I_4)}{|x|^2 - \frac{z^2}{c^2} - z} - \frac{1}{|x|^2 - z} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right|^2 \\ &\quad = \sup_{x \in \mathbb{R}^3} \frac{1 + |x|^2}{c^2} \left| \frac{\alpha \cdot x + \frac{z}{c} I_4}{|x|^2 - \frac{z^2}{c^2} - z} + \frac{z^2}{c(|x|^2 - \frac{z^2}{c^2} - z)(|x|^2 - z)} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right|^2 \leq \frac{K(z)^2}{c^2} \end{aligned}$$

for some constant $K(z)$ that does not depend on c, since the assumptions $z \in \mathbb{C} \setminus [0, \infty)$ and $c > \sqrt{|z|}$ ensure that there is no singularity in the last x-dependent expression. As $\|\mathcal{F} f\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \|f\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)}$ we conclude

$$\begin{aligned} &\int_{\mathbb{R}^3} (1 + |x|^2) \left| \mathcal{F} \left[\left(A_0 - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right] f(x) \right|^2 dx \\ &\quad \leq \frac{K(z)^2}{c^2} \|f\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^4)}, \end{aligned}$$

which shows the desired result. □

By using the convergence result from Proposition 3.1 and the definition (2.3) of Φ_z^*, it is not difficult to obtain the convergence of $\Phi_{z+c^2/2}$ and $\Phi^*_{z+c^2/2}$. Recall that SL_μ, $\mu \in \mathbb{C} \setminus [0, \infty)$, is the single layer potential defined in (2.13) and that $\mathcal{M}_c$ is the scaling matrix given by (2.10). Since there is a multiplication by $\sqrt{c}$ involved, the rate of convergence in the following proposition reduces to $\mathcal{O}(c^{-1/2})$. This is the main reason why we get this rate of convergence in Theorem 1.2.

Proposition 3.2 *Let $z \in \mathbb{C} \setminus [0, \infty)$ and $c > \sqrt{|z|}$. Then, there exists a constant $K(z)$ such that*

$$\left\| \Phi_{z+c^2/2} \mathcal{M}_c - SL_z \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\|_{H^{-1/2}(\Sigma;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)} \leq \frac{K(z)}{\sqrt{c}} \tag{3.2}$$

and

$$\left\| \mathcal{M}_c \Phi^*_{z+c^2/2} - SL^*_z \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} \leq \frac{K(z)}{\sqrt{c}}. \tag{3.3}$$

In particular, the operators $\Phi_{z+c^2/2}\mathcal{M}_c : H^{-1/2}(\Sigma;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)$ *and the mappings* $\mathcal{M}_c\Phi^*_{z+c^2/2} : L^2(\mathbb{R}^3;\mathbb{C}^4) \to H^{1/2}(\Sigma;\mathbb{C}^4)$ *are uniformly bounded in* c.

Proof Recall from (2.3) and (2.17) that

$$\Phi^*_{z+c^2/2} = \gamma_D \left(A_0 - \left(\overline{z} + \frac{c^2}{2} \right) \right)^{-1} \quad \text{and} \quad SL^*_z = \gamma_D(-\Delta - \overline{z})^{-1}.$$

Hence, (3.3) follows from Proposition 3.1 and the mapping properties of the trace operator; the stated rates of convergence are obtained by accounting for the matrix terms in equation (3.3). The claim in (3.2) follows from (3.3) by duality. The uniform boundedness of $\Phi_{z+c^2/2}\mathcal{M}_c$ and $\mathcal{M}_c\Phi^*_{z+c^2/2}$ is clear as these operators converge. □

3.2 Convergence of $(\vartheta_c + \mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c)^{-1}$

The more difficult part in the analysis of (3.1) is $(\vartheta_c + \mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c)^{-1}$. To handle it in the computation of the nonrelativistic limit, first a more detailed consideration of $\mathcal{C}_{z+c^2/2}$ is provided. Define for $z \in \rho(A_0)$ the auxiliary operator $\mathcal{T}_z$ that formally acts on a sufficiently smooth function $\varphi : \Sigma \to \mathbb{C}^2$ via

$$\mathcal{T}_z\varphi(x) := \lim_{\varepsilon\to 0^+} \int_{\Sigma\setminus B(x,\varepsilon)} t_z(x-y)\varphi(y)d\sigma(y), \qquad x \in \Sigma,$$

with

$$t_z(x) := \left(1 - i\sqrt{\frac{z^2}{c^2} - \frac{c^2}{4}}|x| \right) \frac{i(\sigma\cdot x)}{4\pi|x|^3} e^{i\sqrt{z^2/c^2 - c^2/4}|x|}, \quad x \neq 0.$$

Next, the definition of G_z in (2.2) implies

$$G_{z+c^2/2}(x) = \left(\frac{z}{c^2} I_4 + \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} + \left(1 - i\sqrt{z + \frac{z^2}{c^2}}|x| \right) \frac{i(\alpha\cdot x)}{c|x|^2} \right) \frac{e^{i\sqrt{z+z^2/c^2}|x|}}{4\pi|x|}.$$

This and the definitions of $\mathcal{C}_z$ and $\mathcal{S}_z$ in Eqs. (2.5) and (2.14) lead to

$$\mathcal{C}_{z+c^2/2} = \begin{pmatrix} (\frac{z}{c^2}+1)\mathcal{S}_{z+z^2/c^2} I_2 & \frac{1}{c}\mathcal{T}_{z+c^2/2} \\ \frac{1}{c}\mathcal{T}_{z+c^2/2} & \frac{z}{c^2}\mathcal{S}_{z+z^2/c^2} I_2 \end{pmatrix}. \tag{3.4}$$

It follows from the latter representation and (2.6)–(2.7) that $\mathcal{T}_{z+c^2/2}$ gives rise to a bounded operator

$$\mathcal{T}_{z+c^2/2} : H^s(\Sigma; \mathbb{C}^2) \to H^s(\Sigma; \mathbb{C}^2), \quad s \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right], \tag{3.5}$$

and that the anti-dual of $\mathcal{T}_{z+c^2/2}$ satisfies $\mathcal{T}'_{z+c^2/2} = \mathcal{T}_{\overline{z}+c^2/2}$. In the next proposition we show that these operators are even uniformly bounded in c.

Proposition 3.3 *Let $z \in \mathbb{C} \setminus [0, \infty)$ and $c > \sqrt{|z|}$. Then, for any $s \in [-\frac{1}{2}, \frac{1}{2}]$ the operators $\mathcal{T}_{z+c^2/2} : H^s(\Sigma; \mathbb{C}^2) \to H^s(\Sigma; \mathbb{C}^2)$ are uniformly bounded in c.*

Proof The proof of this proposition is split into three steps. In *Step 1* the integral kernel $t_{z+c^2/2}$ of $\mathcal{T}_{z+c^2/2}$ is decomposed into a singular part d, which is independent of c, and a remainder term $\widetilde{t}_{z,c}$ which is easier to analyze. In *Step 2* it is shown that the integral operator with kernel $\widetilde{t}_{z,c}$ gives rise to a bounded operator from $L^2(\Sigma; \mathbb{C}^2)$ to $H^1(\Omega_+; \mathbb{C}^2)$ that is uniformly bounded in c. By combining the results from *Step 1 & 2*, the proof of the proposition is completed in *Step 3*.

Step 1 Rewriting the exponential in the kernel

$$t_{z+c^2/2}(x) = \left(1 - i\sqrt{z + \frac{z^2}{c^2}}|x|\right) \frac{i(\sigma \cdot x)}{4\pi |x|^3} e^{i\sqrt{z+z^2/c^2}|x|}, \quad x \neq 0,$$

as a power series shows that the terms with $|x|^{-2}$ cancel out. After combining and rearranging the coefficients of the remaining terms we obtain

$$t_{z+c^2/2}(x) = d(x) + \widetilde{t}_{z,c}(x), \tag{3.6}$$

where

$$d(x) = \frac{i(\sigma \cdot x)}{4\pi |x|^3}, \quad x \neq 0, \tag{3.7}$$

and

$$\widetilde{t}_{z,c}(x) = \sum_{k=0}^{\infty} \left(\frac{i^{k+3}}{(k+2)!} + \frac{i^{k+1}}{(k+1)!}\right) \left(\sqrt{z + \frac{z^2}{c^2}}\right)^{k+2} |x|^{k-1} \frac{\sigma \cdot x}{4\pi}, \quad x \neq 0.$$

Step 2 Now we consider $\widetilde{t}_{z,c}(x - y)$ for $x \in \Omega_+$ and $y \in \Sigma$ and define the integral operator $\widetilde{T}_{z,c}$ for sufficiently smooth functions $\varphi : \Sigma \to \mathbb{C}^2$ as

$$\widetilde{T}_{z,c}\varphi(x) := \int_{\Sigma} \widetilde{t}_{z,c}(x - y)\varphi(y) d\sigma(y), \quad x \in \Omega_+. \tag{3.8}$$

We will show that

$$\widetilde{T}_{z,c} : L^2(\Sigma; \mathbb{C}^2) \to H^1(\Omega_+; \mathbb{C}^2) \quad \text{is uniformly bounded in } c. \tag{3.9}$$

For this, we first establish some simple bounds on $\widetilde{t}_{z,c}$ and its first order derivatives that are independent of c. Observe that for a constant $K_1 = K_1(z)$ one has for all $x \in \Omega_+$ and $y \in \Sigma$

$$\left|\widetilde{t}_{z,c}(x-y)\right| \leq \sum_{k=0}^{\infty} \frac{2}{k!} \frac{\left(\sqrt{2|z|}\right)^{k+2}|x-y|^k}{4\pi} \leq K_1, \tag{3.10}$$

as the latter series is absolutely converging and defines a continuous function on the compact set $\overline{\Omega_+} \times \Sigma$. Likewise, there exists a constant $K_2 = K_2(z)$ such that for the partial derivatives of $\widetilde{t}_{z,c}$ and all $x \in \Omega_+$ and $y \in \Sigma$ one has

$$\begin{aligned} \left|\partial_{x_j}\widetilde{t}_{z,c}(x-y)\right| &\leq \sum_{k=0}^{\infty} \frac{2}{k!} \frac{\left(\sqrt{2|z|}\right)^{k+2}}{4\pi} \left|\partial_{x_j}\left((\sigma\cdot(x-y))|x-y|^{k-1}\right)\right| \\ &= \sum_{k=0}^{\infty} \frac{\left(\sqrt{2|z|}\right)^{k+2}}{2\pi k!} \left|\sigma_j + \frac{(k-1)(x_j-y_j)(\sigma\cdot(x-y))}{|x-y|^2}\right| |x-y|^{k-1} \\ &\leq \frac{K_2}{|x-y|}. \end{aligned} \tag{3.11}$$

Since Ω_+ is bounded, (3.10) and (3.11) imply

$$\begin{aligned} (x,y) &\mapsto \widetilde{t}_{z,c}(x-y) \in L^2(\Omega_+ \times \Sigma; \mathbb{C}^{2\times 2}), \\ (x,y) &\mapsto \partial_{x_j}\widetilde{t}_{z,c}(x-y) \in L^2(\Omega_+ \times \Sigma; \mathbb{C}^{2\times 2}) \end{aligned}$$

and there exists a constant $K_3 = K_3(z)$ such that

$$\int_{\Omega_+}\int_{\Sigma} \left|\widetilde{t}_{z,c}(x-y)\right|^2 d\sigma(y)dx \leq K_3, \qquad \int_{\Omega_+}\int_{\Sigma} \left|\partial_{x_j}\widetilde{t}_{z,c}(x-y)\right|^2 d\sigma(y)dx \leq K_3. \tag{3.12}$$

Furthermore, using that for any $y \in \Sigma$ one has $x \mapsto \widetilde{t}_{z,c}(x-y) \in C^\infty(\Omega_+; \mathbb{C}^{2\times 2})$, (3.11), and the dominated convergence theorem, it is not difficult to see that for any $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ one has $\widetilde{T}_{z,c}\varphi \in C^1(\Omega_+; \mathbb{C}^2)$ and

$$\partial_{x_j}\widetilde{T}_{z,c}\varphi(x) = \int_{\Sigma} \partial_{x_j}\widetilde{t}_{z,c}(x-y)\varphi(y)d\sigma(y), \quad x \in \Omega_+. \tag{3.13}$$

Combining (3.8) and (3.13) with (3.12) shows that $\widetilde{T}_{z,c}, \partial_{x_j}\widetilde{T}_{z,c} : L^2(\Sigma; \mathbb{C}^2) \to L^2(\Omega_+; \mathbb{C}^2)$ are Hilbert–Schmidt operators that are uniformly bounded in c, and hence (3.9) is true.

Step 3 We verify that $\mathcal{T}_{z+c^2/2} : H^s(\Sigma; \mathbb{C}^2) \to H^s(\Sigma; \mathbb{C}^2)$ is uniformly bounded in c for any $s \in [-\frac{1}{2}, \frac{1}{2}]$. First, we do this for $s = \frac{1}{2}$. For that purpose, consider the operator $\gamma_D\widetilde{T}_{z,c} : L^2(\Sigma; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2)$, which is uniformly bounded in c by the results in *Step 2*, and hence also the restriction

$$\gamma_D \widetilde{T}_{z,c} : H^{1/2}(\Sigma; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2) \tag{3.14}$$

is uniformly bounded in c. Furthermore, we shall use that

$$\gamma_D \widetilde{T}_{z,c}\varphi(x) = \int_\Sigma \widetilde{t}_{z,c}(x-y)\varphi(y)d\sigma(y), \quad x \in \Sigma, \tag{3.15}$$

holds for all $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ (and, in particular, for all $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^2)$). In fact, the estimate (3.10) extends to $\overline{\Omega_+} \times \Sigma$ and this implies that the function $\widetilde{T}_{z,c}\varphi : \Omega_+ \to \mathbb{C}^2$ admits a continuous extension onto $\overline{\Omega_+}$, which shows (3.15).

Next, recall that the function d is defined by (3.7). For $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^2)$ consider the integral operator

$$\mathcal{D}\varphi(x) := \lim_{\varepsilon \to 0^+} \int_{\Sigma \setminus B(x,\varepsilon)} d(x-y)\varphi(y)d\sigma(y), \quad x \in \Sigma,$$

which is bounded in $H^{1/2}(\Sigma; \mathbb{C}^2)$ by [43, Theorem 4.3.1] as d is a homogeneous kernel of order 0 in the sense of [43, Section 4.3.2], see also [43, Example 4.2] (the boundedness of $\mathcal{D}$ would also follow from (3.16) and the reasoning below, as $\mathcal{T}_{z+c^2/2}$ is bounded in $H^{1/2}(\Sigma; \mathbb{C}^2)$ by (3.5)). From (3.6) and (3.15) we obtain

$$\mathcal{T}_{z+c^2/2}\varphi = \mathcal{D}\varphi + \gamma_D \widetilde{T}_{z,c}\varphi, \quad \varphi \in H^{1/2}(\Sigma; \mathbb{C}^2), \tag{3.16}$$

and now it follows from the uniform boundedness of the operator $\gamma_D \widetilde{T}_{z,c}$ in (3.14) that also $\mathcal{T}_{z+c^2/2} : H^{1/2}(\Sigma; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2)$ is uniformly bounded in c.

To show the claim for $s = -\frac{1}{2}$, recall that $\mathcal{T}_{z+c^2/2}$ has a bounded extension in $H^{-1/2}(\Sigma; \mathbb{C}^2)$ given by $\mathcal{T}_{z+c^2/2} = (\mathcal{T}_{\overline{z}+c^2/2})'$. Hence, by the already shown uniform boundedness in c of $\mathcal{T}_{\overline{z}+c^2/2}$ in $H^{1/2}(\Sigma; \mathbb{C}^2)$ also

$$(\mathcal{T}_{\overline{z}+c^2/2})' = \mathcal{T}_{z+c^2/2} : H^{-1/2}(\Sigma; \mathbb{C}^2) \to H^{-1/2}(\Sigma; \mathbb{C}^2)$$

is uniformly bounded in c. Finally, as $\mathcal{T}_{z+c^2/2}$ is uniformly bounded in $H^{-1/2}(\Sigma; \mathbb{C}^2)$ and $H^{1/2}(\Sigma; \mathbb{C}^2)$ in c, it follows with an interpolation argument using [41, Theorems B.2 and B.11] that $\mathcal{T}_{z+c^2/2} : H^s(\Sigma; \mathbb{C}^2) \to H^s(\Sigma; \mathbb{C}^2)$ is also uniformly bounded in c for any $s \in (-\frac{1}{2}, \frac{1}{2})$. This finishes the proof. □

Next, the convergence of $(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2}\mathcal{M}_c)^{-1}$ is analyzed. Recall that $a_\pm$ is defined by (2.9). By (3.4) one has the block structure

$$\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2}\mathcal{M}_c = \begin{pmatrix} \left(\frac{1}{c}a_+ + \left(\frac{z}{c^2}+1\right)\mathcal{S}_{z+z^2/c^2}\right)I_2 & \frac{1}{\sqrt{c}}\mathcal{T}_{z+c^2/2} \\ \frac{1}{\sqrt{c}}\mathcal{T}_{z+c^2/2} & \left(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}\right)I_2 \end{pmatrix}.$$

To proceed, note that for $z \in \mathbb{C} \setminus [0, \infty)$ and $c > 0$ sufficiently large the operator $a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}$ is boundedly invertible in $H^s(\Sigma; \mathbb{C})$, $s \in [-\frac{1}{2}, \frac{1}{2}]$, with inverse given by

$$\Big(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}\Big)^{-1} = \frac{1}{a_-}\sum_{n=0}^{\infty}\Big(-\frac{z}{a_- c}\mathcal{S}_{z+z^2/c^2}\Big)^n, \tag{3.17}$$

as $a_- < 0$ and by Lemma 2.4 the operator $\mathcal{S}_{z+z^2/c^2}$ is (uniformly) bounded in $H^s(\Sigma;\mathbb{C})$ in c. Thus, one can write

$$\begin{aligned}\vartheta_c + \mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c = &\begin{pmatrix} I_2 & \frac{1}{\sqrt{c}}\mathcal{T}_{z+c^2/2}\big(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}\big)^{-1} \\ 0 & I_2\end{pmatrix}\\ &\cdot\begin{pmatrix}\widetilde{\mathcal{S}}_{z,c} & 0 \\ 0 & \big(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}\big)I_2\end{pmatrix}\begin{pmatrix} I_2 & 0 \\ \frac{1}{\sqrt{c}}\big(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}\big)^{-1}\mathcal{T}_{z+c^2/2} & I_2\end{pmatrix},\end{aligned} \tag{3.18}$$

where the Schur complement $\widetilde{\mathcal{S}}_{z,c}$ is given by

$$\widetilde{\mathcal{S}}_{z,c} = \frac{1}{c}a_+I_2 + \Big(\frac{z}{c^2}+1\Big)\mathcal{S}_{z+z^2/c^2}I_2 - \frac{1}{c}\mathcal{T}_{z+c^2/2}\Big(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}\Big)^{-1}\mathcal{T}_{z+c^2/2}. \tag{3.19}$$

The first and the third factor in (3.18) are bijective in $H^{1/2}(\Sigma;\mathbb{C}^4)$. Since the map $\vartheta_c + \mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c$ has this property as well by Proposition 2.2, we conclude that also $\widetilde{\mathcal{S}}_{z,c}$ is bijective in $H^{1/2}(\Sigma;\mathbb{C}^2)$. In the following proposition, the convergence of $\widetilde{\mathcal{S}}_{z,c}^{-1}$ is analyzed.

Proposition 3.4 *Let $z < 0$ and $c > \sqrt{|z|}$. Then, there exists a constant $K(z)$ such that for all c sufficiently large*

$$\big\|\widetilde{\mathcal{S}}_{z,c}^{-1} - \mathcal{S}_z^{-1}I_2\big\|_{H^{3/2}(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)} \le \frac{K(z)}{c}. \tag{3.20}$$

Moreover, $\widetilde{\mathcal{S}}_{z,c}^{-1} : H^{1/2}(\Sigma;\mathbb{C}^2) \to H^{-1/2}(\Sigma;\mathbb{C}^2)$ is uniformly bounded in c.

Proof The proof of this proposition is split into four steps. In *Step 1* we show that for $s \in [-\frac{1}{2}, \frac{1}{2}]$ there exists a constant $K_1 = K_1(z,s)$ such that

$$\big\|\widetilde{\mathcal{S}}_{z,c} - \mathcal{S}_zI_2\big\|_{H^s(\Sigma;\mathbb{C}^2)\to H^s(\Sigma;\mathbb{C}^2)} \le \frac{K_1}{c} \tag{3.21}$$

for $c > 0$ sufficiently large. In *Step 2* we verify that the realization of $\widetilde{\mathcal{S}}_{z,c}$ in $L^2(\Sigma;\mathbb{C}^2)$ is bijective and there exists a constant K_2 such that for $c > 0$ sufficiently large

$$\big\|\widetilde{\mathcal{S}}_{z,c}^{-1}\big\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} \le K_2 c. \tag{3.22}$$

Using this, we show in *Step 3* our claim that $\widetilde{\mathcal{S}}_{z,c}^{-1} : H^{1/2}(\Sigma;\mathbb{C}^2) \to H^{-1/2}(\Sigma;\mathbb{C}^2)$ is uniformly bounded in c, while in *Step 4* we prove (3.20).

Step 1 For $s \in [-\frac{1}{2}, \frac{1}{2}]$ fixed and $c > 0$ sufficiently large we obtain the estimate

$$\begin{aligned} &\| \widetilde{\mathcal{S}}_{z,c} - \mathcal{S}_z I_2 \|_{H^s(\Sigma;\mathbb{C}^2)\to H^s(\Sigma;\mathbb{C}^2)} \\ &\quad\le \left\| (\mathcal{S}_{z+z^2/c^2} - \mathcal{S}_z) I_2 \right\|_{H^s(\Sigma;\mathbb{C}^2)\to H^s(\Sigma;\mathbb{C}^2)} + \frac{1}{c} \left\| a_+ I_2 + \frac{z}{c}\mathcal{S}_{z+z^2/c^2} I_2 \right. \\ &\qquad \left. - \mathcal{T}_{z+c^2/2} \left(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2} \right)^{-1} \mathcal{T}_{z+c^2/2} \right\|_{H^s(\Sigma;\mathbb{C}^2)\to H^s(\Sigma;\mathbb{C}^2)} \end{aligned} \tag{3.23}$$

from (3.19). For the first term on the right-hand side of (3.23) one has by Lemma 2.4

$$\begin{aligned} &\| (\mathcal{S}_{z+z^2/c^2} - \mathcal{S}_z) I_2 \|_{H^s(\Sigma;\mathbb{C}^2)\to H^s(\Sigma;\mathbb{C}^2)} \\ &\quad \le \| (\mathcal{S}_{z+z^2/c^2} - \mathcal{S}_z) I_2 \|_{H^{-3/2}(\Sigma;\mathbb{C}^2)\to H^{3/2}(\Sigma;\mathbb{C}^2)} \le K_1' \frac{z^2}{c^2} \end{aligned}$$

with some constant $K_1' = K_1'(z)$. Note also that $\mathcal{S}_{z+z^2/c^2}$ is uniformly bounded in $H^s(\Sigma;\mathbb{C})$ for $c > 0$ sufficiently large by Lemma 2.4. Therefore, since $a_- < 0$ we conclude from (3.17) and the estimate

$$\begin{aligned} &\left\| \left(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2} \right)^{-1} \right\|_{H^s(\Sigma;\mathbb{C})\to H^s(\Sigma;\mathbb{C})} \\ &\quad \le \frac{1}{-a_-} \left(1 - \frac{z}{a_- c} \| \mathcal{S}_{z+z^2/c^2} \|_{H^s(\Sigma;\mathbb{C})\to H^s(\Sigma;\mathbb{C})} \right)^{-1} \end{aligned}$$

that $(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2})^{-1}$ is also uniformly bounded in $H^s(\Sigma;\mathbb{C})$ for $c > 0$ sufficiently large. Combining this with Proposition 3.3 it follows that the second term on the right-hand side of (3.23) is bounded by $\frac{K_1''}{c}$ with some constant $K_1'' = K_1''(z, s)$; thus we conclude (3.21).

Step 2 For $z < 0$ and $c > 0$ sufficiently large we shall now consider the operator

$$\widetilde{\mathcal{S}}_{z,c} = \frac{1}{c} a_+ I_2 + \left(\frac{z}{c^2} + 1 \right) \mathcal{S}_{z+z^2/c^2} I_2 - \frac{1}{c} \mathcal{T}_{z+c^2/2} \left(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2} \right)^{-1} \mathcal{T}_{z+c^2/2}$$

in $L^2(\Sigma;\mathbb{C}^2)$. Note that for $c > 0$ sufficiently large $\mathcal{S}_{z+z^2/c^2}$ is bounded, self-adjoint and nonnegative in $L^2(\Sigma;\mathbb{C}^2)$ (see the discussion after (2.15)) and hence the same holds for the operator $(\frac{z}{c^2} + 1)\mathcal{S}_{z+z^2/c^2}$. Furthermore, for $c > 0$ sufficiently large $\mathcal{T}_{z+c^2/2}$ is bounded and self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$ (see (3.5)), and together with (3.17) we conclude that also $\widetilde{\mathcal{S}}_{z,c}$ is bounded and self-adjoint. As $a_- < 0$ and $\mathcal{S}_{z+z^2/c^2}$ is uniformly bounded in c it is clear that $a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2}$ is a negative operator in

$L^2(\Sigma;\mathbb{C}^2)$ for $c>0$ sufficiently large, and the same is true for its inverse. Therefore,

$$-\frac{1}{c}\mathcal{T}_{z+c^2/2}\left(a_-+\frac{z}{c}\mathcal{S}_{z+z^2/c^2}\right)^{-1}\mathcal{T}_{z+c^2/2}$$

is a nonnegative operator in $L^2(\Sigma;\mathbb{C}^2)$ for $c>0$ sufficiently large. This implies $\widetilde{\mathcal{S}}_{z,c}\geq\frac{a_+}{c}$ for $c>0$ sufficiently large, which in turn yields (3.22) with $K_2=a_+^{-1}$.

Step 3 We claim that $\widetilde{\mathcal{S}}_{z,c}^{-1}:H^{1/2}(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)$ is uniformly bounded in c. For this it suffices to prove that for $c>0$ sufficiently large there exists a constant $K_3=K_3(z)$ such that

$$\|\widetilde{\mathcal{S}}_{z,c}^{-1}\|_{H^1(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\leq K_3, \tag{3.24}$$

as then by duality and formal symmetry one also has

$$\|\widetilde{\mathcal{S}}_{z,c}^{-1}\|_{L^2(\Sigma;\mathbb{C}^2)\to H^{-1}(\Sigma;\mathbb{C}^2)}\leq K_3,$$

and an interpolation argument (see [41, Theorems B.2 and B.11]) leads to the assertion.

To show (3.24), we use

$$\widetilde{\mathcal{S}}_{z,c}^{-1}=\mathcal{S}_z^{-1}I_2-\widetilde{\mathcal{S}}_{z,c}^{-1}\left(\widetilde{\mathcal{S}}_{z,c}-\mathcal{S}_zI_2\right)\mathcal{S}_z^{-1}I_2 \tag{3.25}$$

and the fact that $\mathcal{S}_z^{-1}:H^1(\Sigma;\mathbb{C})\to L^2(\Sigma;\mathbb{C})$ is bounded; cf. (2.15). Using (3.21) for $s=0$ with $K_1=K_1(z,0)$ and (3.22) we obtain

$$\begin{aligned}
&\|\widetilde{\mathcal{S}}_{z,c}^{-1}\|_{H^1(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\\
&\quad\leq\|\mathcal{S}_z^{-1}\|_{H^1(\Sigma;\mathbb{C})\to L^2(\Sigma;\mathbb{C})}\\
&\qquad+\|\widetilde{\mathcal{S}}_{z,c}^{-1}\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\|\mathcal{S}_zI_2-\widetilde{\mathcal{S}}_{z,c}\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\\
&\qquad\qquad\|\mathcal{S}_z^{-1}\|_{H^1(\Sigma;\mathbb{C})\to L^2(\Sigma;\mathbb{C})}\\
&\quad\leq\|\mathcal{S}_z^{-1}\|_{H^1(\Sigma;\mathbb{C})\to L^2(\Sigma;\mathbb{C})}\left(1+K_2\cdot c\cdot\frac{K_1}{c}\right),
\end{aligned}$$

and hence (3.24) holds.

Step 4 Finally, we show (3.20). Using again (3.25), the fact that $\mathcal{S}_z:H^{1/2}(\Sigma;\mathbb{C})\to H^{3/2}(\Sigma;\mathbb{C})$ is boundedly invertible, and the results from *Step 1* and *Step 3* we obtain

$$\begin{aligned}
&\|\widetilde{\mathcal{S}}_{z,c}^{-1}-\mathcal{S}_z^{-1}I_2\|_{H^{3/2}(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)}\\
&\quad\leq\|\widetilde{\mathcal{S}}_{z,c}^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)}\\
&\qquad\|\mathcal{S}_zI_2-\widetilde{\mathcal{S}}_{z,c}\|_{H^{1/2}(\Sigma;\mathbb{C}^2)\to H^{1/2}(\Sigma;\mathbb{C}^2)}\|\mathcal{S}_z^{-1}\|_{H^{3/2}(\Sigma;\mathbb{C})\to H^{1/2}(\Sigma;\mathbb{C})}\\
&\quad\leq\frac{K(z)}{c}.
\end{aligned}$$

This completes the proof of Proposition 3.4. □

Now we are ready to study the convergence of the inverse of $\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c$.

Proposition 3.5 *Let $z < 0$ and $c > \sqrt{|z|}$. Then, there exists a constant $K(z)$ such that for all c sufficiently large*

$$\left\| \left(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c\right)^{-1} - \begin{pmatrix} \mathcal{S}_z^{-1} I_2 & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4) \to H^{-1/2}(\Sigma;\mathbb{C}^4)} \leq \frac{K(z)}{\sqrt{c}}.$$

Moreover, $\left(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c\right)^{-1} : H^{1/2}(\Sigma;\mathbb{C}^4) \to H^{-1/2}(\Sigma;\mathbb{C}^4)$ is uniformly bounded in c.

Proof It follows from (3.18) that

$$\left(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c\right)^{-1} = F_1(c) F_2(c) F_3(c), \tag{3.26}$$

where

$$\begin{aligned}
F_1(c) &:= \begin{pmatrix} I_2 & 0 \\ -\frac{1}{\sqrt{c}} \left(a_- + \frac{z}{c} \mathcal{S}_{z+c^2/2}\right)^{-1} \mathcal{T}_{z+c^2/2} & I_2 \end{pmatrix}, \\
F_2(c) &:= \begin{pmatrix} \widetilde{\mathcal{S}}_{z,c}^{-1} & 0 \\ 0 & \left(a_- + \frac{z}{c} \mathcal{S}_{z+z^2/c^2}\right)^{-1} I_2 \end{pmatrix}, \\
F_3(c) &:= \begin{pmatrix} I_2 & -\frac{1}{\sqrt{c}} \mathcal{T}_{z+c^2/2} \left(a_- + \frac{z}{c} \mathcal{S}_{z+z^2/c^2}\right)^{-1} \\ 0 & I_2 \end{pmatrix}.
\end{aligned}$$

For $c > 0$ sufficiently large we use the uniform boundedness of $\mathcal{S}_{z+z^2/c^2}$ in $H^s(\Sigma;\mathbb{C})$, $s \in [-\frac{1}{2}, \frac{1}{2}]$, from Lemma 2.4 to estimate

$$\begin{aligned}
&\left\| \left(a_- + \frac{z}{c} \mathcal{S}_{z+z^2/c^2}\right)^{-1} - a_-^{-1} \right\|_{H^s(\Sigma;\mathbb{C}) \to H^s(\Sigma;\mathbb{C})} \\
&\quad = -\frac{1}{a_-} \left\| \sum_{n=1}^{\infty} \left(-\frac{z}{a_- c} \mathcal{S}_{z+z^2/c^2}\right)^n \right\|_{H^s(\Sigma;\mathbb{C}) \to H^s(\Sigma;\mathbb{C})} \\
&\quad \leq -\frac{1}{a_-} \left(\frac{\frac{z}{a_- c} \|\mathcal{S}_{z+z^2/c^2}\|_{H^s(\Sigma;\mathbb{C}) \to H^s(\Sigma;\mathbb{C})}}{1 - \frac{z}{a_- c} \|\mathcal{S}_{z+z^2/c^2}\|_{H^s(\Sigma;\mathbb{C}) \to H^s(\Sigma;\mathbb{C})}} \right) \\
&\quad \leq \frac{K_1}{c},
\end{aligned}$$

where $K_1 = K_1(z, s)$ is a constant; for the restriction onto $H^{3/2}(\Sigma;\mathbb{C})$ viewed as a mapping into $H^{-1/2}(\Sigma;\mathbb{C})$ this estimate yields

$$\left\| \left(a_- + \frac{z}{c} \mathcal{S}_{z+z^2/c^2}\right)^{-1} - a_-^{-1} \right\|_{H^{3/2}(\Sigma;\mathbb{C}) \to H^{-1/2}(\Sigma;\mathbb{C})} \leq \frac{K_1}{c}, \tag{3.27}$$

and also shows that $(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2})^{-1}$ is uniformly bounded in $H^s(\Sigma;\mathbb{C}^2)$, $s \in [-\frac{1}{2}, \frac{1}{2}]$, for $c > 0$ sufficiently large; cf. *Step 1* in the proof of Proposition 3.4. Together with Proposition 3.3 this implies with a constant $K_2 = K_2(z)$ that

$$\begin{aligned} \|F_1(c) - I_4\|_{H^{-1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} &\leq \frac{K_2}{\sqrt{c}}, \\ \|F_3(c) - I_4\|_{H^{1/2}(\Sigma;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} &\leq \frac{K_2}{\sqrt{c}}. \end{aligned} \tag{3.28}$$

In particular, $F_1(c)$ is uniformly bounded in $H^{-1/2}(\Sigma;\mathbb{C}^4)$ and $F_3(c)$ is uniformly bounded in $H^{1/2}(\Sigma;\mathbb{C}^4)$ in c, and the restrictions onto $H^{1/2}(\Sigma;\mathbb{C}^4)$ and $H^{3/2}(\Sigma;\mathbb{C}^4)$ satisfy the same bounds

$$\begin{aligned} \|F_1(c) - I_4\|_{H^{1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} &\leq \frac{K_2}{\sqrt{c}}, \\ \|F_3(c) - I_4\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} &\leq \frac{K_2}{\sqrt{c}}. \end{aligned} \tag{3.29}$$

Moreover, Proposition 3.4 and (3.27) imply

$$\left\| F_2(c) - \begin{pmatrix} \mathcal{S}_z^{-1} I_2 & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \leq \frac{K_3}{c}$$

with some constant $K_3 = K_3(z)$. Eventually, it follows from Proposition 3.4 and the uniform boundedness of $(a_- + \frac{z}{c}\mathcal{S}_{z+z^2/c^2})^{-1}$ as a mapping from $H^{1/2}(\Sigma;\mathbb{C})$ to $H^{-1/2}(\Sigma;\mathbb{C})$ that $F_2(c) : H^{1/2}(\Sigma;\mathbb{C}^4) \to H^{-1/2}(\Sigma;\mathbb{C}^4)$ is uniformly bounded. Combining this with (3.28) and (3.29) gives

$$\begin{aligned} &\left\| \left(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c\right)^{-1} - \begin{pmatrix} \mathcal{S}_z^{-1} & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \\ &\quad \leq \left\| F_1(c) F_2(c) (F_3(c) - I_4) \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \\ &\qquad + \left\| F_1(c) \left(F_2(c) - \begin{pmatrix} \mathcal{S}_z^{-1} I_2 & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right) \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \\ &\qquad + \left\| (F_1(c) - I_4) \begin{pmatrix} \mathcal{S}_z^{-1} I_2 & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \\ &\quad \leq \left\| F_1(c) F_2(c) \right\|_{H^{1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \left\| F_3(c) - I_4 \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} \\ &\qquad + \|F_1(c)\|_{H^{-1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \left\| F_2(c) - \begin{pmatrix} \mathcal{S}_z^{-1} I_2 & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \\ &\qquad + \|F_1(c) - I_4\|_{H^{1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)} \left\| \begin{pmatrix} \mathcal{S}_z^{-1} I_2 & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} \\ &\quad \leq \frac{K(z)}{\sqrt{c}}, \end{aligned}$$

which is exactly the claimed convergence result.

Finally, the claim about the uniform boundedness of the operator

$$(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c)^{-1} : H^{1/2}(\Sigma; \mathbb{C}^4) \to H^{-1/2}(\Sigma; \mathbb{C}^4)$$

follows from (3.26) and the above observations on the uniform boundedness of $F_1(c)$ in $H^{-1/2}(\Sigma; \mathbb{C}^4)$, $F_2(c)$ from $H^{1/2}(\Sigma; \mathbb{C}^4)$ to $H^{-1/2}(\Sigma; \mathbb{C}^4)$, and $F_3(c)$ in $H^{1/2}(\Sigma; \mathbb{C}^4)$. □

3.3 Nonrelativistic Limit of A_κ^Σ

With the preparations from the previous sections we are now ready to discuss the nonrelativistic limit of the Dirac operators A_κ^Σ.

Proposition 3.6 *Let A_κ^Σ, $\kappa \in \mathbb{R}$, be as in (2.8) and $-\Delta_D := (-\Delta_D^{\Omega_+}) \oplus (-\Delta_D^{\Omega_-})$, where $-\Delta_D^{\Omega_\pm}$ is the Dirichlet Laplacian in $\Omega_\pm$ from (2.20). Let $z < 0$ and $c > \sqrt{|z|}$. Then, there exists a constant $K(z)$ such that for all c sufficiently large*

$$\left\| \left(A_\kappa^\Sigma - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta_D - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)} \leq \frac{K(z)}{\sqrt{c}}.$$

Proof Let $\mathcal{M}_c$ and ϑ_c be defined by (2.10). As $-c^2 < z < 0$, one has $z \in \rho(-\Delta_D)$ and $z + \frac{c^2}{2} \in (-\frac{c^2}{2}, \frac{c^2}{2}) \subset \rho(A_\kappa^\Sigma)$; cf Proposition 2.2. Furthermore, from Proposition 2.2 and Lemma 2.5 we obtain

$$\begin{aligned}
&\left(A_\kappa^\Sigma - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta_D - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad = \left(A_0 - \left(z + \frac{c^2}{2} \right) \right)^{-1} - \Phi_{z+c^2/2} \mathcal{M}_c \big(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c \big)^{-1} \mathcal{M}_c \Phi_{z+c^2/2}^* \\
&\qquad - \left((-\Delta - z)^{-1} - SL_z \mathcal{S}_z^{-1} SL_z^* \right) \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad = D_1(c) + D_2(c) + D_3(c) + D_4(c)
\end{aligned} \tag{3.30}$$

with

$$\begin{aligned}
D_1(c) &:= \left(A_0 - \left(z + \frac{c^2}{2} \right) \right)^{-1} - (-\Delta - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \\
D_2(c) &:= -\Phi_{z+c^2/2} \mathcal{M}_c \big(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c \big)^{-1} \left(\mathcal{M}_c \Phi_{z+c^2/2}^* - SL_z^* \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right), \\
D_3(c) &:= -\Phi_{z+c^2/2} \mathcal{M}_c \left(\big(\vartheta_c + \mathcal{M}_c \mathcal{C}_{z+c^2/2} \mathcal{M}_c \big)^{-1} - \begin{pmatrix} \mathcal{S}_z^{-1} & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} \right) SL_z^* \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \\
D_4(c) &:= -\left(\Phi_{z+c^2/2} \mathcal{M}_c - SL_z \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \mathcal{S}_z^{-1} & 0 \\ 0 & a_-^{-1} I_2 \end{pmatrix} SL_z^* \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

First, it follows from Proposition 3.1 that $\|D_1(c)\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)} \leq \frac{K_1}{c}$ for a constant $K_1 = K_1(z)$. To discuss $D_2(c)$ recall that $\Phi_{z+c^2/2}\mathcal{M}_c : H^{-1/2}(\Sigma;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)$ is uniformly bounded in c by Proposition 3.2 and $(\vartheta_c+\mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c)^{-1} : H^{1/2}(\Sigma;\mathbb{C}^4) \to H^{-1/2}(\Sigma;\mathbb{C}^4)$ is uniformly bounded in c by Proposition 3.5. Hence, we find with Proposition 3.2 that there exists a constant $K_2 = K_2(z)$ such that

$$\begin{aligned}
&\|D_2(c)\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)}\\
&\quad\leq \left\|\Phi_{z+c^2/2}\mathcal{M}_c\right\|_{H^{-1/2}(\Sigma;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)}\\
&\qquad\cdot \left\|\left(\vartheta_c+\mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c\right)^{-1}\right\|_{H^{1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)}\\
&\qquad\cdot \left\|\mathcal{M}_c\Phi^*_{z+c^2/2} - SL^*_z\begin{pmatrix} I_2 & 0\\ 0 & 0\end{pmatrix}\right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} \leq \frac{K_2}{\sqrt{c}}.
\end{aligned}$$

Next, as $SL^*_z : L^2(\mathbb{R}^3;\mathbb{C}) \to H^{3/2}(\Sigma;\mathbb{C})$ is bounded (see (2.17)), Proposition 3.5 implies that there exists a constant $K_3 = K_3(z)$ such that

$$\begin{aligned}
\|D_3(c)\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)} &\leq \left\|\Phi_{z+c^2/2}\mathcal{M}_c\right\|_{H^{-1/2}(\Sigma;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)}\\
&\quad\cdot\left\|\left(\vartheta_c+\mathcal{M}_c\mathcal{C}_{z+c^2/2}\mathcal{M}_c\right)^{-1} - \begin{pmatrix}\mathcal{S}_z^{-1} & 0\\ 0 & a_-^{-1}I_2\end{pmatrix}\right\|_{H^{3/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)}\\
&\quad\cdot\left\|SL^*_z\begin{pmatrix} I_2 & 0\\ 0 & 0\end{pmatrix}\right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^{3/2}(\Sigma;\mathbb{C}^4)} \leq \frac{K_3}{\sqrt{c}}.
\end{aligned}$$

In a similar way, as $\mathcal{S}_z^{-1} : H^{1/2}(\Sigma;\mathbb{C}) \to H^{-1/2}(\Sigma;\mathbb{C})$ is bounded (see (2.15)), we find with Proposition 3.2 that there exists a constant $K_4 = K_4(z)$ such that

$$\begin{aligned}
&\|D_4(c)\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)}\\
&\quad\leq \left\|\Phi_{z+c^2/2}\mathcal{M}_c - SL_z\begin{pmatrix} I_2 & 0\\ 0 & 0\end{pmatrix}\right\|_{H^{-1/2}(\Sigma;\mathbb{C}^4)\to L^2(\mathbb{R}^3;\mathbb{C}^4)}\\
&\qquad\cdot\left\|\begin{pmatrix}\mathcal{S}_z^{-1} & 0\\ 0 & a_-^{-1}I_2\end{pmatrix}\right\|_{H^{1/2}(\Sigma;\mathbb{C}^4)\to H^{-1/2}(\Sigma;\mathbb{C}^4)}\\
&\qquad\cdot\left\|SL^*_z\begin{pmatrix} I_2 & 0\\ 0 & 0\end{pmatrix}\right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} \leq \frac{K_4}{\sqrt{c}}.
\end{aligned}$$

Now the statement of the proposition follows by combining the above estimates for the operators $D_1(c)$, $D_2(c)$, $D_3(c)$, and $D_4(c)$ with (3.30). □

3.4 Proof of Theorem 1.2 and Corollary 1.3

In this section we complete the proof of our main result by combining Lemma 2.1 and Proposition 3.6. For this let $\kappa \in \mathbb{R}$, Ω, $\Omega_\pm$ be as in Hypothesis 1.1, and $\Sigma = \partial\Omega$. Let the Dirac operator A^Σ_κ be defined as in (2.8) and denote by $P_\Omega : L^2(\mathbb{R}^3;\mathbb{C}^4) \to L^2(\Omega;\mathbb{C}^4)$

the operator $P_\Omega f = f \restriction \Omega$; cf. Corollary 2.3. Then, the self-adjoint Dirac operator H_κ^Ω in $L^2(\Omega; \mathbb{C}^4)$ satisfies

$$H_\kappa^\Omega = P_\Omega A_\kappa^\Sigma P_\Omega^*$$

by Lemma 2.1 and since $\sigma(H_\kappa^\Omega) \subset (-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)$ by Corollary 2.3 it is clear that $z + c^2/2$ with $z \in \mathbb{C} \setminus [0, \infty)$ and $c > \sqrt{|z|}$ belongs to $\rho(H_\kappa^\Omega) \cap \rho(A_\kappa^\Sigma)$, so that

$$\left(H_\kappa^\Omega - \left(z + \frac{c^2}{2}\right)\right)^{-1} = P_\Omega \left(A_\kappa^\Sigma - \left(z + \frac{c^2}{2}\right)\right)^{-1} P_\Omega^*.$$

Therefore, if $z < 0$, then (1.3) follows from Proposition 3.6. In the general case $z \in \mathbb{C} \setminus [0, \infty)$ we obtain (1.3) by assuming that $c > 1$ and using the identity

$$\begin{aligned}
&\left(H_\kappa^\Omega - \left(z + \frac{c^2}{2}\right)\right)^{-1} - (-\Delta_D^\Omega - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \left(I_4 + (z+1)(-\Delta_D^\Omega - z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&\quad \cdot \left[\left(H_\kappa^\Omega - \left(-1 + \frac{c^2}{2}\right)\right)^{-1} - (-\Delta_D^\Omega + 1)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&\quad \cdot \left(I_4 + (z+1)\left(H_\kappa^\Omega - \left(z + \frac{c^2}{2}\right)\right)^{-1}\right).
\end{aligned}$$

This completes the proof of Theorem 1.2 and now we turn our attention to Corollary 1.3, which can be viewed as an immediate consequence of classical results on eigenvalues of Dirichlet Laplacians and convergence of spectra under operator norm convergence of resolvents. For the convenience of the reader and to keep the presentation self-contained we briefly provide the details of the arguments. In the present situation, it is convenient to apply [52, Satz 3.17 d)] or [31, Theorem 2.3.1] about the convergence of eigenvalues of nonnegative compact operators. More precisely, let B_c, $c \in (c_0, \infty]$ for a suitable $c_0 \in \mathbb{R}$, be a family of compact, self-adjoint, and nonnegative operators with eigenvalues $\mu_1(B_c) \geq \mu_2(B_c) \geq \cdots \geq 0$ taking multiplicities into account. If B_c converges to B_∞ in the operator norm, as $c \to \infty$, then by [52, Satz 3.17 d)] for all $j \in \mathbb{N}$ also $\mu_j(B_c)$ converges to $\mu_j(B_\infty)$, as $c \to \infty$. We apply this result to

$$B_c := f\left(\left(H_\kappa^\Omega - \left(-1 + \frac{c^2}{2}\right)\right)^{-1}\right) \quad \text{and} \quad B_\infty := f\left((-\Delta_D^\Omega + 1)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

where $f \in C(\mathbb{R})$ is a nonnegative function such that $f(x) = 0$ for $x \in (-\infty, 0] \cup [2, \infty)$, and $f(x) = x$ for $x \in (0, 1]$. It is clear that B_c and B_∞ are nonnegative bounded self-adjoint operators, and from the compactness of the resolvents of H_κ^Ω and $-\Delta_D^\Omega$, which holds as $\operatorname{dom} H_\kappa^\Omega$ and $\operatorname{dom}(-\Delta_D^\Omega)$ are compactly embedded in $L^2(\Omega; \mathbb{C}^4)$ and $L^2(\Omega; \mathbb{C})$, respectively, it follows that B_c and B_∞ are both compact.

In the following let $c > 1$. For the eigenvalues $\lambda_j^{\pm}(H_\kappa^\Omega)$ of H_κ^Ω ordered as in (1.2) we have

$$\left(\lambda_j^{-}(H_\kappa^\Omega) + 1 - \frac{c^2}{2}\right)^{-1} < 0 \quad \text{and} \quad \left(\lambda_j^{+}(H_\kappa^\Omega) + 1 - \frac{c^2}{2}\right)^{-1} \leq 1, \qquad j \in \mathbb{N};$$

cf. Corollary 2.3. From the choice of f it is clear that the positive eigenvalues of B_c are given by $\mu_j(B_c) = (\lambda_j^{+}(H_\kappa^\Omega) + 1 - \frac{c^2}{2})^{-1}$. Similarly, $\sigma(-\Delta_D^\Omega) \subset [0, \infty)$ and the fact that two copies of $(-\Delta_D^\Omega + 1)^{-1}$ appear in the definition of B_∞ leads to

$$\mu_{2j-1}(B_\infty) = \mu_{2j}(B_\infty) = (\lambda_j(-\Delta_D^\Omega) + 1)^{-1}$$

for $j \in \mathbb{N}$, where $0 < \lambda_1(-\Delta_D^\Omega) \leq \lambda_2(-\Delta_D^\Omega) \leq \ldots$ denote the discrete eigenvalues of $-\Delta_D^\Omega$ taking multiplicities into account. Now it follows from Theorem 1.2 and [49, Theorem VIII.20] that B_c converges to B_∞ in the operator norm, as $c \to \infty$. Using that all eigenvalues of H_κ^Ω have even multiplicity, see Corollary 2.3 (i), we conclude with [52, Satz 3.17 d)] from the above considerations that for any $j \in \mathbb{N}$

$$\mu_{2j-1}(B_c) = \mu_{2j}(B_c) = \left(\lambda_{2j-1}^{+}(H_\kappa^\Omega) - \frac{c^2}{2} + 1\right)^{-1} = \left(\lambda_{2j}^{+}(H_\kappa^\Omega) - \frac{c^2}{2} + 1\right)^{-1}$$

tends to

$$\mu_{2j-1}(B_\infty) = \mu_{2j}(B_\infty) = \big(\lambda_j(-\Delta_D^\Omega) + 1\big)^{-1}, \quad \text{as } c \to \infty,$$

which is equivalent to

$$\lambda_{2j-1}^{+}(H_\kappa^\Omega) - \frac{c^2}{2} = \lambda_{2j}^{+}(H_\kappa^\Omega) - \frac{c^2}{2} \to \lambda_j(-\Delta_D^\Omega), \quad \text{as } c \to \infty. \tag{3.31}$$

Eventually, to conclude Corollary 1.3 we note that (3.31) remains true if Ω is replaced by a ball B or the disjoint union of two balls $B_1 \cup B_2$. Thus, the claims follow immediately from the classical results for the Dirichlet Laplacian, which under the assumptions of Corollary 1.3 read as follows:

(i) Faber–Krahn inequality: $\lambda_1(-\Delta_D^B) \leq \lambda_1(-\Delta_D^\Omega)$ and equality holds if and only if Ω is a ball, see [29, 36] and also [31, Theorem 3.2.1 and Remark 3.2.2].

(ii) Hong–Krahn–Szegö inequality: $\lambda_2(-\Delta_D^{B_1 \cup B_2}) \leq \lambda_2(-\Delta_D^\Omega)$ and equality holds if and only if Ω is the union of two identical disjoint balls, see [33, 37] and also [31, Theorem 4.1.1 and Remark 4.1.2].

(iii) Payne–Pólya–Weinberger inequality: If Ω is connected, then

$$\frac{\lambda_1(-\Delta_D^B)}{\lambda_2(-\Delta_D^B)} \leq \frac{\lambda_1(-\Delta_D^\Omega)}{\lambda_2(-\Delta_D^\Omega)}$$

and equality holds if and only if Ω is a ball, see [8, 45].

Remark 3.7 Finally, let us remark that from Theorem 1.2 and Corollary 1.3 one gets information about the positive part of the spectrum of H_κ^Ω. Similar statements are also true for the negative part of the spectrum of H_κ^Ω. Indeed, consider the self-adjoint unitary matrix $U = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}$. Then, as $U\alpha_j + \alpha_j U = 0$, $j \in \{1, 2, 3\}$, and $U\beta + \beta U = 0$, it is not difficult to see that $H_\kappa^\Omega = -U H_{-\kappa}^\Omega U$, i.e. H_κ^Ω and $-H_{-\kappa}^\Omega$ are unitarily equivalent. Hence, it follows from Theorem 1.2 that for $z \in \mathbb{C} \setminus (-\infty, 0]$ also

$$
\begin{aligned}
&\left(H_\kappa^\Omega - \left(z - \frac{c^2}{2}\right)\right)^{-1} \\
&\quad = \left(-U H_{-\kappa}^\Omega U - \left(z - \frac{c^2}{2}\right)\right)^{-1} = -U\left(H_{-\kappa}^\Omega - \left(-z + \frac{c^2}{2}\right)\right)^{-1} U \\
&\quad \to -U(-\Delta_D^\Omega + z)^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} U = \left(-(-\Delta_D^\Omega) - z\right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}
\end{aligned}
$$

in the operator norm, as $c \to \infty$. This convergence is of interest by its own, but similarly as in the proof of Corollary 1.3 one can conclude spectral inequalities for the negative eigenvalues $\lambda_j^-(H_\kappa^\Omega)$ of H_κ^Ω; alternatively one can argue via the unitary equivalence $H_\kappa^\Omega = -U H_{-\kappa}^\Omega U$. More precisely, for a bounded C^2-domain $\Omega \subset \mathbb{R}^3$, a ball $B \subset \mathbb{R}^3$ with $|B| = |\Omega|$, and two identical and disjoint balls $B_1, B_2 \subset \mathbb{R}^3$ with $|B_1| + |B_2| = |\Omega|$ the following assertions follow for sufficiently large $c > 0$:

(i) $\lambda_j^-(H_\kappa^B) \geq \lambda_j^-(H_\kappa^\Omega)$ for $j \in \{1, 2\}$ and equality holds if and only if Ω is a ball.

(ii) $\lambda_j^-(H_\kappa^{B_1 \cup B_2}) \geq \lambda_j^-(H_\kappa^\Omega)$ for $j \in \{3, 4\}$ and equality holds if and only if Ω is the union of two identical disjoint balls.

(iii) If, in addition, Ω is connected, then

$$
\frac{\lambda_j^-(H_\kappa^B)}{\lambda_l^-(H_\kappa^B)} \leq \frac{\lambda_j^-(H_\kappa^\Omega)}{\lambda_l^-(H_\kappa^\Omega)}, \qquad j \in \{1, 2\},\ l \in \{3, 4\},
$$

and equality holds if and only if Ω is a ball.

Acknowledgements This research was funded in part by the Austrian Science Fund (FWF) 10.55776/P33568-N. For the purpose of open access, the author has applied a CC BY public copyright licence to any Author Accepted Manuscript version arising from this submission. This publication is based upon work from COST Action CA 18232 MAT-DYN-NET, supported by COST (European Cooperation in Science and Technology), https://www.cost.eu. The authors thank the referee for helpful comments and remarks that led to an improvement of the manuscript.

Funding Open access funding provided by Graz University of Technology.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

References

1. Alpay, D., Behrndt, J.: Generalized Q-functions and Dirichlet-to-Neumann maps for elliptic differential operators. J. Funct. Anal. **257**, 1666–1694 (2009)
2. Antunes, P., Benguria, R., Lotoreichik, V., Ourmières-Bonafos, T.: A variational formulation for Dirac operators in bounded domains. Applications to spectral geometric inequalities. Commun. Math. Phys. **386**(2), 781–818 (2021)
3. Arrizabalaga, N., Le Treust, L., Raymond, N.: On the MIT bag model in the non-relativistic limit. Commun. Math. Phys. **354**(2), 641–669 (2017)
4. Arrizabalaga, N., Le Treust, L., Raymond, N.: Extension operator for the MIT bag model. Ann. Fac. Sci. Toulouse Math. **29**(1), 135–147 (2020)
5. Arrizabalaga, N., Le Treust, L., Mas, A., Raymond, N.: The MIT bag model as an infinite mass limit. J. Éc. Polytech. Math. **6**, 329–365 (2019)
6. Arrizabalaga, N., Mas, A., Vega, L.: Shell interactions for Dirac operators: on the point spectrum and the confinement. SIAM J. Math. Anal. **47**(2), 1044–1069 (2015)
7. Arrizabalaga, N., Mas, A., Sanz-Perela, T., Vega, L.: Eigenvalue curves for generalized MIT bag models. Commum. Math. Phys. **397**(1), 337–392 (2023)
8. Ashbaugh, M.S., Benguria, R.D.: A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions. Ann. Math. **135**, 601–628 (1992)
9. Behrndt, J.: On compressed resolvents of Schrödinger operators with complex potentials. Complex Anal. Oper. Theory **15**, 12 (9 pages) (2021)
10. Behrndt, J., Exner, P., Holzmann, M., Lotoreichik, V.: On the spectral properties of Dirac operators with electrostatic δ-shell interactions. J. Math. Pures Appl. **111**, 47–78 (2018)
11. Behrndt, J., Exner, P., Holzmann, M., Lotoreichik, V.: On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar δ-shell interactions. Quantum Stud. **6**, 295–314 (2019)
12. Behrndt, J., Exner, P., Holzmann, M., Lotoreichik, V.: The Landau Hamiltonian with δ-potentials supported on curves. Rev. Math. Phys. **32**, 2050010 (51 pages) (2020)
13. Behrndt, J., Exner, P., Holzmann, M., Tušek, M.: On two-dimensional Dirac operators with δ-shell interactions supported on unbounded curves with straight ends. To appear in Singularities, Asymptotics, and Limiting Models, Springer INdAM Series (2024)
14. Behrndt, J., Hassi, S., de Snoo, H.: Boundary Value Problems, Weyl Functions, and Differential Operators, Volume 108 of Monographs in Mathematics. Birkhäuser/Springer, Cham (2020)
15. Behrndt, J., Holzmann, M.: On Dirac operators with electrostatic δ-shell interactions of critical strength. J. Spectr. Theory **10**, 147–184 (2020)
16. Behrndt, J., Holzmann, M., Mas, A.: Self-adjoint Dirac operators on domains in $\mathbb{R}^3$. Ann. Henri Poincaré **21**, 2681–2735 (2020)
17. Behrndt, J., Holzmann, M., Stelzer, C., Stenzel, G.: Boundary triples and Weyl functions for Dirac operators with singular interactions. Rev. Math. Phys. **36**(2), 2350036 (65 pages) (2024)
18. Behrndt, J., Holzmann, M., Stenzel, G.: Schrödinger operators with oblique transmission conditions in $\mathbb{R}^2$. Commun. Math. Phys. **401**, 3149–3167 (2023)
19. Benguria, R.D., Fournais, S., Stockmeyer, E., Van Den Bosch, H.: Self-adjointness of two-dimensional Dirac operators on domains. Ann. Henri Poincaré **18**(4), 1371–1383 (2017)
20. Benguria, R.D., Fournais, S., Stockmeyer, E., Van Den Bosch, H.: Spectral gaps of Dirac operators describing graphene quantum dots. Math. Phys. Anal. Geom. **20**(2), 11 (12 pages) (2017)

21. Benhellal, B.: Spectral analysis of Dirac operators with delta interactions supported on the boundaries of rough domains. J. Math. Phys. **63**(1), 011507 (34 pages) (2022)
22. Bogolioubov, P.N.: Sur un modèle à quarks quasi-indépendants. Ann. Inst. Henri Poincaré. Sect. A **8**(2), 163–189 (1968)
23. Briet, P., Krejčiřík, D.: Spectral optimization of Dirac rectangles. J. Math. Phys. **63**(1), 013502 (11 pages) (2022)
24. Budyika, V., Malamud, M., Posilicano, A.: Nonrelativistic limit for $2p \times 2p$-Dirac operators with point interactions on a discrete set. Russ. J. Math. Phys. **24**(4), 426–435 (2017)
25. Carlone, R., Malamud, M., Posilicano, A.: On the spectral theory of Gesztesy-Šeba realizations of 1-D Dirac operators with point interactions on a discrete set. J. Differ. Equ. **254**(9), 3835–3902 (2013)
26. Chodos, A., Jaffe, R.L., Johnson, K., Thorn, C.B., Weisskopf, V.F.: New extended model of hadrons. Phys. Rev. D **9**(12), 3471–3495 (1974)
27. Cuenin, J.-C.: Estimates on complex eigenvalues for Dirac operators on the half-line. Integr. Equ. Oper. Theory **79**(3), 377–388 (2014)
28. DeGrand, T., Jaffe, R.L., Johnson, K., Kiskis, J.: Masses and other parameters of the light hadrons. Phys. Rev. D **12**(7), 2060–2076 (1975)
29. Faber, G.: Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. Sitzungsber. Bayer. Akad. Wiss. München Math.-Phys. Kl. 169–172 (1923)
30. Gesztesy, F., Šeba, P.: New analytically solvable models of relativistic point interactions. Lett. Math. Phys. **13**(4), 345–358 (1987)
31. Henrot, A.: Extremum Problems for Eigenvalues of Elliptic Operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2006)
32. Heriban, L., Tušek, M.: Non-self-adjoint relativistic point interaction in one dimension. J. Math. Anal. Appl. **516**(2), 126536 (28 pages) (2022)
33. Hong, I.: On an inequality concerning the eigenvalue problem of membrane. Kodai Math. Sem. Rep. **6**, 113–114 (1954)
34. Johnson, K.: The MIT bag model. Acta Physica Pol. B **6**, 865–892 (1975)
35. Kato, T.: Perturbation Theory for Linear Operators. Classics in Mathematics. Springer-Verlag, Berlin (1995). (**Reprint of the 1980 edition**)
36. Krahn, E.: Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. Math. Ann. **94**, 97–100 (1925)
37. Krahn, E.: Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen. Acta Univ. Dorpat. A **9**, 1–44 (1926)
38. Levitin, M., Mangoubi, D., Polterovich, I.: Topics in Spectral Geometry. Graduate Studies in Mathematics, vol. 237. American Mathematical Society, Providence (2023)
39. Lotoreichik, V., Ourmières-Bonafos, T.: A sharp upper bound on the spectral gap for graphene quantum dots. Math. Phys. Anal. Geom. **22**, 13 (30 pages) (2019)
40. Lotoreichik, V., Ourmières-Bonafos, T.: Spectral asymptotics of the Dirac operator in a thin shell. arXiv:2307.09033
41. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
42. Moroianu, A., Ourmières-Bonafos, T., Pankrashkin, K.: Dirac operators on hypersurfaces as large mass limits. Commun. Math. Phys. **374**(3), 1963–2013 (2020)
43. Nédélec, J.C.: Acoustic and Electromagnetic Equations. Integral Representations for Harmonic Problems. Springer-Verlag, New York (2001)
44. Ourmières-Bonafos, T., Vega, L.: A strategy for self-adjointness of Dirac operators: application to the MIT bag model and δ-shell interactions. Publ. Mat. **62**, 397–437 (2018)
45. Pólya, G.: On the characteristic frequencies of a symmetric membrane. Math. Z. **63**, 331–337 (1955)
46. Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematics Studies, vol 27. Princeton University Press, Princeton (1951)
47. Rabinovich, V.S.: Boundary problems for three-dimensional Dirac operators and generalized MIT bag models for unbounded domains. Russ. J. Math. Phys. **27**(4), 500–516 (2020)
48. Rabinovich, V.S.: Boundary value problems for 3D-Dirac operators and MIT bag model. Springer Proc. Math. Stat. **357**, 479–495 (2021)
49. Reed, M., Simon, B.: Methods of Modern Mathematical Physics I. Functional Analysis. Academic Press, Cambridge (1972)

50. Thaller, B.: The Dirac Equation. Texts and Monographs in Physics. Springer-Verlag, Berlin (1992)
51. Vu, T.: Spectral inequality for Dirac right triangles. J. Math. Phys. **64**(4), 041502 (18 pages) (2023)
52. Weidmann, J.: Lineare Operatoren in Hilberträumen. Teil I. Grundlagen. Mathematische Leitfäden. B. G. Teubner, Stuttgart (2000)