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Journal of Differential Equations 462 (2026) 114131

Journal of Differential Equations

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On sesquilinear forms for lower semibounded (singular) Sturm–Liouville operators

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Received 8 September 2025; revised 6 January 2026; accepted 14 January 2026

Abstract

Any self-adjoint extension of a (singular) Sturm–Liouville operator bounded from below uniquely leads to an associated sesquilinear form. This form is characterized in terms of principal and nonprincipal solutions of the Sturm–Liouville operator by using generalized boundary values. We provide these forms in detail in all possible cases (explicitly, when both endpoints are limit circle, when one endpoint is limit circle, and when both endpoints are limit point).

MSC: primary 34B24, 34L40, 47A07; secondary 47E05

Keywords: Sturm–Liouville operator; Sesquilinear form

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https://doi.org/10.1016/j.jde.2026.114131

Contents

1. Introduction

The traditional three-coefficient Sturm–Liouville (generalized eigenvalue) problem on an arbitrary open interval $(a,b) \subseteq \mathbb{R}$ is of the form

$$-(p(x)f'(x))' + q(x)f(x) = zr(x)f(x) \text{ for a.e. } x \in (a,b),\ z \in \mathbb{C}, \tag{1.1}$$

where the coefficients p, q, r are real-valued (Lebesgue) a.e. on (a,b), $p, r > 0$ a.e. on (a,b), and $p^{-1}, q, r \in L^1_{loc}((a,b); dx)$. In addition, $z \in \mathbb{C}$ represents a (generally, complex-valued) spectral parameter, and f and pf' are assumed to be locally absolutely continuous on (a,b); see Section 2 for details. More precisely, the differential expression τ underlying (1.1),

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \text{ for a.e. } x \in (a,b), \tag{1.2}$$

naturally leads to a minimal closed symmetric operator T_{min} in the Hilbert space $L^2((a,b); r\,dx)$ (cf. (2.10) and (2.12)) and its deficiency indices are then given by $(0,0)$, $(1,1)$, or $(2,2)$. From the outset, the operator T_{min} is in general not lower semibounded. However, in this paper it will be assumed that equation (1.1) has solutions which are nonoscillatory at the endpoints a and b for some $z \in \mathbb{R}$ and in this case T_{min} turns out to be lower semibounded. As a consequence, all self-adjoint extensions of T_{min} in $L^2((a,b); r\,dx)$ are then lower semibounded, see Proposition 2.8. For example, in the special case of a one-dimensional Schrödinger operator where τ simplifies to $\tau = -(d^2/dx^2) + q(x)$ for a.e. $x \in (a,b)$, quantum mechanical considerations typically lead to the requirement of lower semibounded self-adjoint extensions of T_{min} and the characterization of the underlying quadratic forms (representing the sum of kinetic and potential energy) corresponding to them.

In this paper we consider the natural and nontrivial question of determining the form domains associated with general, that is, lower semibounded, self-adjoint, singular, three-coefficient Sturm–Liouville operators associated with $L^2((a,b); r\,dx)$-realizations of the differential expression τ in (1.2). The corresponding sesquilinear forms are then connected to integrals of the form

$$\int_a^b dx \left[p(x)\overline{f'(x)}g'(x) + q(x)\overline{f(x)}g(x)\right] \tag{1.3}$$

for “appropriate” elements $f, g \in L^2((a,b); r\,dx)$. However, if one of the functions f or g is not compactly supported in (a,b), there might well be a problem with the convergence of the integral in (1.3). This problem will be avoided when rewriting the integral by means of the nonoscillatory solutions of (1.1) mentioned above. These solutions will also be used to introduce generalized boundary values (see Proposition 2.10) that are associated to the particular self-adjoint extension of T_{min} under consideration. The main results in this paper are formulated in terms of proper interpretations of the integral (1.3) and, in particular, in terms of generalized boundary values, see Proposition 2.10.

The history of Sturm–Liouville problems, and, especially, the naturally associated spectral theory, is incredibly rich. Hence, we can only point to some of the classical contributions by Weyl [39–43], Titchmarsh [31–34], [35, Chs. I–VI], and Kodaira [19], [20], and, for more recent accounts, refer to the monographs [1, Sects. 127, 132], [3, Ch. 6], [4, Chs. 4, 6–8], [6, Ch. 9], [7, Sect. 13.6, 13.9, 13.10], [8, Ch. 2], [9, Sect. 3.10], [11, Chs. 4–10, 13], [13], [14, Parts II, III], [15, Ch. III], [21, Sect. 11.9], [22, Sect. 15-19], [24, Ch. 6], [27, Chs. 1–4, 6], [29, Ch. 15], [30, Ch. 9], [37, Sects. 3–7], [38, Ch. 13], [36, Sect. 8.4], [44, Ch. 5], and [45, Chs. 7–10].

The material in this paper is presented in a systematic and straightforward way. A brief review of Sturm–Liouville theory is given in Section 2. The description of all self-adjoint extensions by means of *generalized boundary values* can be found in Propositions 2.12, 2.13, and 2.14, depending on the endpoints being in the limit circle case or in the limit point case. Section 2 also briefly surveys the history of the notion of generalized boundary values (cf. Remark 2.15). In each of our principal Sections 3, 4, and 5, one can find a systematic description of the quadratic forms corresponding to the self-adjoint extensions in the various cases; see Theorems 3.8, 3.9, 4.5, and 5.4. The results are obtained via integration by parts of the expression $(f, T_{max}g)$ for $f, g \in \operatorname{dom}(T_{max})$, where T_{max} denotes the maximal operator associated to (1.2); see Lemma 3.4 and Lemma 4.4. This yields an alternative and very explicit formulation of the results in [3, Ch. 6] in terms of generalized boundary values. These results generalize those of [11, Sect. 4.5] in the special case where τ is regular at a and b. The presentation is for the most part self-contained. For completeness and convenience of the reader, we identify in Appendix A the boundary triplet and the boundary pair used in [3] to obtain the general formulation of the main results in Sections 3 and 4. In the appendix the emphasis is on the abstract analogue of Lemma 3.4 and Lemma 4.4. The abstract results also lead to a description of the Friedrichs extension in each of these sections by means of a boundary pair.

We conclude this introduction by briefly commenting on some of the notation employed in the bulk of this paper: The inner product in a separable (complex) Hilbert space $\mathcal{H}$ is denoted by $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ and is assumed to be linear with respect to the second argument. If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$, $\operatorname{ran}(T)$, and $\ker(T)$ denote the domain, range, and kernel (i.e., null space) of T, respectively. The analogous conventions are used for linear relations and sesquilinear forms (when applicable); in particular, the multi-valued part of a linear relation T is denoted by $\operatorname{mul}(T)$. Finally, $SL(2,\mathbb{R})$ denotes the set of all 2×2 matrices with real-valued entries and determinant one.

2. Sturm–Liouville operators, generalized boundary values, and self-adjoint realizations

The following hypothesis will be assumed throughout this paper.

Hypothesis 2.1. Let $-\infty \leqslant a < b \leqslant \infty$. Suppose that p, q, and r are Lebesgue measurable on (a,b) with $p^{-1}, q, r \in L^1_{loc}((a,b);dx)$ and real-valued a.e. on (a,b) with $r>0$ and $p>0$ a.e. on (a,b).

We recall the basic construction and properties of Sturm–Liouville differential expressions and their associated operators. For a full treatment with proofs of the assertions in this section, we refer to [11, Chapter 5].

Assuming Hypothesis 2.1, we introduce the set

$$\mathfrak{D}_\tau((a,b)) = \big\{g \in AC_{loc}((a,b)) \,\big|\, pg' \in AC_{loc}((a,b))\big\}. \tag{2.1}$$

The expression

$$f^{[1]} = pf', \quad f \in \mathfrak{D}_\tau((a,b)), \tag{2.2}$$

is called the *first quasi-derivative* of f. We note that $f \in \mathfrak{D}_\tau((a,b))$ implies $f^{[1]} \in AC_{loc}((a,b))$, so that $f^{[1]}$ is differentiable almost everywhere on (a,b). The differential expression τ is defined by

$$\tau f = \frac{1}{r}\big[-\big(f^{[1]}\big)' + qf\big] \in L^1_{loc}((a,b); r\,dx), \quad f \in \mathfrak{D}_\tau((a,b)). \tag{2.3}$$

For each $f, g \in \mathfrak{D}_\tau((a,b))$, the (modified) Wronskian of f and g is defined by

$$W(f,g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a,b). \tag{2.4}$$

Hence, $W(f,g)$ is locally absolutely continuous on (a,b) and its derivative is

$$W(f,g)'(x) = \big[g(x)(\tau f)(x) - f(x)(\tau g)(x)\big]r(x) \text{ for a.e. } x \in (a,b). \tag{2.5}$$

In particular, if $z \in \mathbb{C}$, then the Wronskian of two solutions $u_j(z,\cdot) \in \mathfrak{D}_\tau((a,b))$, $j \in \{1,2\}$, of $\tau u = zu$ on (a,b) is constant. Moreover, $W(u_1(z,\cdot), u_2(z,\cdot)) \neq 0$ if and only if $u_1(z,\cdot)$ and $u_2(z,\cdot)$ are linearly independent.

Definition 2.2. The differential expression τ is said to be *regular* on (a,b) if $-\infty < a < b < \infty$ (i.e., a and b are finite) and $p^{-1}, q, r \in L^1((a,b);dx)$; otherwise, τ is said to be *singular* on (a,b).

If τ is regular on (a,b), then for each $f \in \mathfrak{D}_\tau((a,b))$ the following limits exist and are finite:

$$\begin{aligned} f(a) &:= \lim_{x\downarrow a} f(x), \quad & f^{[1]}(a) &:= \lim_{x\downarrow a} f^{[1]}(x), \\ f(b) &:= \lim_{x\uparrow b} f(x), \quad & f^{[1]}(b) &:= \lim_{x\uparrow b} f^{[1]}(x). \end{aligned} \tag{2.6}$$

The differential expression τ gives rise to linear operators in the Hilbert space $L^2((a,b); r\,dx)$ equipped with the standard inner product

$$(f,g)_{L^2((a,b);r\,dx)} = \int_a^b r(x)\,dx\,\overline{f(x)}g(x), \quad f,g \in L^2((a,b); r\,dx). \tag{2.7}$$

The *maximal operator* associated to τ is denoted by T_{max} and is defined by

$$\begin{aligned} &T_{max} f = \tau f, \\ &f \in \operatorname{dom}(T_{max}) = \big\{g \in L^2((a,b); r\,dx) \,\big|\, g \in \mathfrak{D}_\tau((a,b)),\ \tau g \in L^2((a,b); r\,dx)\big\}. \end{aligned} \tag{2.8}$$

Furthermore, the Wronskian of any two functions $f, g \in \operatorname{dom}(T_{max})$ possesses finite boundary values at the endpoints of (a,b); that is, the following limits exist and are finite:

$$W(f,g)(a) := \lim_{x\downarrow a} W(f,g)(x), \quad W(f,g)(b) := \lim_{x\uparrow b} W(f,g)(x). \tag{2.9}$$

The *pre-minimal operator* associated to τ is denoted by $\dot{T}$ and is defined by

$$\begin{aligned} &\dot{T} f = \tau f, \\ &f \in \operatorname{dom}\big(\dot{T}\big) = \big\{g \in \operatorname{dom}(T_{max}) \,|\, g \text{ has compact support in } (a,b)\big\}. \end{aligned} \tag{2.10}$$

One can show that the operator $\dot{T}$ is densely defined and symmetric in the Hilbert space $L^2((a,b); r\,dx)$ and

$$\big(\dot{T}\big)^* = T_{max}. \tag{2.11}$$

The *minimal operator* associated to τ is denoted by T_{min} and is defined to be the closure of the pre-minimal operator:

$$T_{min} := \overline{\dot{T}}. \tag{2.12}$$

In addition, T_{min} and T_{max} are adjoint to one another:

$$T_{min}^* = T_{max} \quad \text{and} \quad T_{max}^* = T_{min}. \tag{2.13}$$

Definition 2.3. A measurable function $f : (a,b) \to \mathbb{C}$ is in $L^2((a,b); r\,dx)$ near a (resp., b) if $\chi_{(a,c)} f$ (resp., $\chi_{(c,b)} f$) belongs to $L^2((a,b); r\,dx)$ for some $c \in (a,b)$.

Theorem 2.4 *(Weyl's Alternative). Assume Hypothesis 2.1. Then the following alternative holds: Either*

(i) For every $z \in \mathbb{C}$, all solutions u of $\tau u = zu$ are in $L^2((a,b); r\,dx)$ near b (resp., near a),

or,

(ii) For every $z \in \mathbb{C}$, there exists at least one solution u of $\tau u = zu$ which is not in $L^2((a,b); r\,dx)$ near b (resp., near a). In this case, for each $z \in \mathbb{C}\backslash\mathbb{R}$, there exists precisely one solution ψ_b (resp., ψ_a) of $\tau u = zu$ (up to constant multiples) which lies in $L^2((a,b); r\,dx)$ near b (resp., near a).

Definition 2.5. Assume Hypothesis 2.1. In case (i) in Theorem 2.4, τ is said to be in the *limit circle case* at b (resp., a). In case (ii) in Theorem 2.4, τ is said to be in the *limit point case* at b (resp., a).

Remark 2.6. If τ is in the limit circle case at b (resp., a), then τ is frequently called *quasi-regular* at b (resp., a). If τ is in the limit circle case at both a and b, then τ is frequently also called *quasi-regular*. $\diamond$

We recall that T_{min} is *lower semibounded* or *bounded from below by* $\lambda_0 \in \mathbb{R}$, and one writes $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ (in this case, λ_0 is called a *lower bound* of T_{min}), if

$$(u, T_{min}u)_{L^2((a,b);r\,dx)} \geqslant \lambda_0 (u,u)_{L^2((a,b);r\,dx)}, \quad u \in \operatorname{dom}(T_{min}). \tag{2.14}$$

In particular, the *lower bound of T_{min}* is the largest of all the lower bounds λ_0 for which (2.14) holds.

The lower semiboundedness property of T_{min} (equivalently, $\dot{T}$) is connected to the existence of distinguished nonoscillatory solutions, the so-called *principal* and *nonprincipal* solutions, at the endpoints a and b (see Definition 2.9).

Definition 2.7. Assume Hypothesis 2.1 and fix $c \in (a,b)$ and $\lambda \in \mathbb{R}$. The differential expression $\tau - \lambda$ is called *nonoscillatory at a* (resp., b), if there exists a real-valued solution $u(\lambda, \cdot)$ of $\tau u = \lambda u$ that has finitely many zeros in (a,c) (resp., (c,b)). Otherwise, $\tau - \lambda$ is called *oscillatory at a* (resp., b). If $\tau - \lambda$ is nonoscillatory at a and b, one calls $\tau - \lambda$ *nonoscillatory on* (a,b). In addition, $\tau - \lambda$ is called *oscillatory on* (a,b) if it is oscillatory at least at one of the endpoints a or b.

Proposition 2.8. *Assume Hypothesis 2.1 and let $\lambda_0 \in \mathbb{R}$. Then the following items (i)– (iii) are equivalent:*

(i) T_{min} is bounded from below by λ_0; that is, $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$.

(ii) For all $\lambda \leqslant \lambda_0$, $\tau - \lambda$ is nonoscillatory at a and b.

(iii) For all $\lambda \leqslant \lambda_0$, $\tau u = \lambda u$ has, for some $c_0, d_0 \in (a,b)$, real-valued nonvanishing solutions $u_a(\lambda, \cdot)$ and $\widehat{u}_a(\lambda, \cdot)$ in the interval $(a, c_0]$, and real-valued nonvanishing solutions $u_b(\lambda, \cdot)$ and $\widehat{u}_b(\lambda, \cdot)$ in the interval $[d_0, b)$, such that

$$W(u_a(\lambda, \cdot), \widehat{u}_a(\lambda, \cdot)) = 1, \quad u_a(\lambda, x) = o(\widehat{u}_a(\lambda, x)) \text{ as } x \downarrow a, \tag{2.15}$$

$$W(u_b(\lambda, \cdot), \widehat{u}_b(\lambda, \cdot)) = 1, \quad u_b(\lambda, x) = o(\widehat{u}_b(\lambda, x)) \text{ as } x \uparrow b, \tag{2.16}$$

and for all $c \in (a, c_0]$ and $d \in [d_0, b)$,

$$\int_a^c dx\, p(x)^{-1} u_a(\lambda, x)^{-2} = \int_d^b dx\, p(x)^{-1} u_b(\lambda, x)^{-2} = \infty, \tag{2.17}$$

$$\int_a^c dx\, p(x)^{-1} \widehat{u}_a(\lambda, x)^{-2} < \infty, \quad \int_d^b dx\, p(x)^{-1} \widehat{u}_b(\lambda, x)^{-2} < \infty. \tag{2.18}$$

In (2.15) and (2.16), we employ Landau's little-o notation; that is, $f(x) = o(g(x))$ as $x \downarrow a$ (resp., $x \uparrow b$) means that $f(x)/g(x) \to 0$ as $x \downarrow a$ (resp., $x \uparrow b$). For details on principal and nonprincipal solutions, we refer to [11, Sect. 8.2]. In particular, for a proof of Proposition 2.8, see [11, Theorem 8.3.6].

Definition 2.9. Assume Hypothesis 2.1, suppose that T_{min} is bounded from below by $\lambda_0 \in \mathbb{R}$ and let $\lambda \leqslant \lambda_0$. Then $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) in Proposition 2.8 (iii) is called a *principal* (or *minimal*) solution of $\tau u = \lambda u$ at a (resp., b). A real-valued solution $\widehat{u}_a(\lambda, \cdot)$ (resp., $\widehat{u}_b(\lambda, \cdot)$) of $\tau u = \lambda u$ linearly independent of $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) is called a *nonprincipal* solution of $\tau u = \lambda u$ at a (resp., b).

Following [10] and [11, Sect. 13.4], the next result introduces generalized boundary values at the endpoints a and b for functions belonging to $\operatorname{dom}(T_{max})$.

Proposition 2.10 *(Generalized boundary values). Assume Hypothesis 2.1 and let τ be in the limit circle case at a and b (i.e., τ is quasi-regular on (a,b)). In addition, assume that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$, and denote by $u_t(\lambda_0, \cdot)$ and $\widehat{u}_t(\lambda_0, \cdot)$ principal and nonprincipal solutions of $\tau u = \lambda_0 u$ on (a,b), respectively, at $t \in \{a,b\}$ that satisfy*

$$W(\widehat{u}_a(\lambda_0, \cdot), u_a(\lambda_0, \cdot)) = W(\widehat{u}_b(\lambda_0, \cdot), u_b(\lambda_0, \cdot)) = 1. \tag{2.19}$$

Introducing $v_j \in \operatorname{dom}(T_{max})$, $j = 1, 2$, via

$$v_1(x) = \begin{cases} \widehat{u}_a(\lambda_0, x), & \text{for } x \text{ near } a, \\ \widehat{u}_b(\lambda_0, x), & \text{for } x \text{ near } b, \end{cases} \quad v_2(x) = \begin{cases} u_a(\lambda_0, x), & \text{for } x \text{ near } a, \\ u_b(\lambda_0, x), & \text{for } x \text{ near } b, \end{cases} \tag{2.20}$$

then for each $g \in \operatorname{dom}(T_{max})$, the following limits exist and are finite:

$$\begin{aligned} \widetilde{g}(a) &:= -W(v_2, g)(a) = -W(u_a(\lambda_0, \cdot), g)(a) = \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)}, \\ \widetilde{g}(b) &:= -W(v_2, g)(b) = -W(u_b(\lambda_0, \cdot), g)(b) = \lim_{x \uparrow b} \frac{g(x)}{\widehat{u}_b(\lambda_0, x)}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \widetilde{g}'(a) &:= W(v_1, g)(a) = W(\widehat{u}_a(\lambda_0, \cdot), g)(a) = \lim_{x \downarrow a} \frac{g(x) - \widetilde{g}(a)\widehat{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)}, \\ \widetilde{g}'(b) &:= W(v_1, g)(b) = W(\widehat{u}_b(\lambda_0, \cdot), g)(b) = \lim_{x \uparrow b} \frac{g(x) - \widetilde{g}(b)\widehat{u}_b(\lambda_0, x)}{u_b(\lambda_0, x)}. \end{aligned} \tag{2.22}$$

Definition 2.11. The quantities $\widetilde{g}(c)$, $\widetilde{g}'(c)$, $c \in \{a, b\}$, defined by (2.21) and (2.22) are called the *generalized boundary values* of $g \in \operatorname{dom}(T_{max})$.

If τ is in the limit circle case at both endpoints of (a, b), then T_{min} has deficiency indices $(2, 2)$. In this case, the self-adjoint extensions of T_{min} are parametrized by boundary conditions at the endpoints of (a, b) according to the next proposition.

Proposition 2.12. *Assume Hypothesis 2.1 and let τ be in the limit circle case at a and b. In addition, assume that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$ and that $u_t(\lambda_0, \cdot)$ and $\widehat{u}_t(\lambda_0, \cdot)$ are principal and nonprincipal solutions of $\tau u = \lambda_0 u$ on (a, b), respectively, at $t \in \{a, b\}$ that satisfy (2.19). Then, given (2.21) and (2.22), the following items (i)–(v) hold:*

(i) *The minimal operator is characterized by*

$$\begin{aligned} &T_{min} f = \tau f, \\ &f \in \operatorname{dom}(T_{min}) = \big\{ g \in \operatorname{dom}(T_{max}) \,\big|\, \widetilde{g}(a) = \widetilde{g}'(a) = 0 = \widetilde{g}(b) = \widetilde{g}'(b) \big\}. \end{aligned} \tag{2.23}$$

(ii) *All self-adjoint extensions $T_{\alpha,\beta}$ of T_{min} with separated boundary conditions are of the form*

$$\begin{aligned} &T_{\alpha,\beta} f = \tau f, \quad \alpha, \beta \in [0, \pi), \\ &f \in \operatorname{dom}(T_{\alpha,\beta}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, \begin{array}{l} \sin(\alpha)\widetilde{g}'(a) + \cos(\alpha)\widetilde{g}(a) = 0 \\ \sin(\beta)\widetilde{g}'(b) + \cos(\beta)\widetilde{g}(b) = 0 \end{array} \right\}. \end{aligned} \tag{2.24}$$

(iii) *All self-adjoint extensions $T_{\varphi,R}$ of T_{min} with coupled boundary conditions are of the form*

$$\begin{aligned} &T_{\varphi,R} f = \tau f, \quad \varphi \in [0, \pi), \ R \in SL(2, \mathbb{R}), \\ &f \in \operatorname{dom}(T_{\varphi,R}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, \begin{pmatrix} \widetilde{g}(b) \\ \widetilde{g}'(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} \widetilde{g}(a) \\ \widetilde{g}'(a) \end{pmatrix} \right\}. \end{aligned} \tag{2.25}$$

(iv) *Every self-adjoint extension of T_{min} is either of type (ii) (i.e., with separated boundary conditions) or of type (iii) (i.e., with coupled boundary conditions).*

(v) *The operator $T_{\alpha=0,\beta=0}$ is the Friedrichs extension of T_{min}.*

In the case when exactly one endpoint is in the limit circle case, the deficiency indices of T_{min} are $(1, 1)$. The self-adjoint extensions of T_{min} are then characterized by a separated boundary condition at the limit circle endpoint. For simplicity of presentation, we assume in the following result that τ is in the limit circle case at a (the case when τ is in the limit circle case at b is entirely analogous).

Proposition 2.13. *Assume Hypothesis 2.1 and let τ be in the limit circle case at a and in the limit point case at b. In addition, assume that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$ and that $u_a(\lambda_0, \cdot)$ and $\widehat{u}_a(\lambda_0, \cdot)$ are principal and nonprincipal solutions of $\tau u = \lambda_0 u$ on (a, b), respectively, at a that satisfy (2.19). Introduce the corresponding generalized boundary values according to (2.21) and (2.22). Then the following statements (i)–(iii) hold:*

(i) The domain of T_{min} is characterized by

$$\operatorname{dom}(T_{min}) = \{g \in \operatorname{dom}(T_{max}) \,|\, \widetilde{g}(a) = \widetilde{g}'(a) = 0\}. \tag{2.26}$$

(ii) All self-adjoint extensions T_α of T_{min} are of the form

$$\begin{aligned} &T_\alpha f = \tau f, \quad \alpha \in [0, \pi), \\ &f \in \operatorname{dom}(T_\alpha) = \{g \in \operatorname{dom}(T_{max}) \,|\, \sin(\alpha)\widetilde{g}'(a) + \cos(\alpha)\widetilde{g}(a) = 0\}. \end{aligned} \tag{2.27}$$

(iii) The operator $T_{\alpha=0}$ is the Friedrichs extension of T_{min}.
Results analogous to (i)–(iii) hold if τ is in the limit point case at $x = a$ and in the limit circle case at $x = b$.

In the case when τ is in the limit point case at both a and b, the deficiency indices of T_{min} are $(0, 0)$. In this case, $T := T_{min} = T_{max}$ is self-adjoint.

Proposition 2.14. *Assume Hypothesis 2.1. If τ is in the limit point case at both a and b, then $T := T_{min} = T_{max}$ is self-adjoint.*

Remark 2.15. (i) The generalized boundary values associated with the Sturm–Liouville expression (2.3) as introduced in Proposition 2.10 by

$$\widetilde{g}(c) = \lim_{x \to c} \frac{g(x)}{\widehat{u}_c(\lambda_0, x)}, \tag{2.28}$$

$$\widetilde{g}'(c) = \lim_{x \to c} \frac{g(x) - \widetilde{g}(c)\widehat{u}_c(\lambda_0, x)}{u_c(\lambda_0, x)}, \tag{2.29}$$

especially, $\widetilde{g}(c)$ in (2.28), at an endpoint $c \in \{a, b\}$, have a longer history. They were originally introduced by Rellich [25] in connection with coefficients p, q, r that had a very particular behavior in a neighborhood of the endpoint c of the type

$$\begin{aligned} p(x) &= (x - c)^\sigma [p_0 + p_1(x - c) + p_2(x - c)^2 + \cdots], \\ q(x) &= (x - c)^{\sigma-2} [q_0 + q_1(x - c) + q_2(x - c)^2 + \cdots], \\ r(x) &= (x - c)^{\sigma-2} [r_0 + r_1(x - c) + r_2(x - c)^2 + \cdots], \end{aligned} \tag{2.30}$$

with $\sigma, p_0, p_1, \dots, q_0, q_1, \dots, r_0, r_1, \dots \in \mathbb{R}$, $p_0 \neq 0$, $r_k \neq 0$ for some $k \in \mathbb{N}_0$, $k_\ell = 0$ for $0 \leqslant \ell \leqslant k - 1$, etc. This was also recorded in [13, Ch. 15] and [15, Ch. III]. In 1951, Rellich [26] considerably generalized the hypotheses on p, q, r. The case of the Bessel equation was reconsidered in [12], and the case of Schrödinger operators on $(0, \infty)$ with potentials q satisfying

$$q(x) = (\gamma^2 - (1/4))x^{-2} + \eta x^{-1} + \omega x^{-a} + W(x) \text{ for a.e. } x > 0, \tag{2.31}$$

with $\gamma \geqslant 0$, $\eta, \omega \in \mathbb{R}$, $a \in (0, 2)$, and $W \in L^\infty((0, \infty); dx)$ real-valued a.e., was systematically treated in [5] and [18]. Under the general Hypothesis 2.1, the boundary value $\widetilde{g}(c)$ in (2.28) was studied in detail by Kalf [16, Remark 3] and subsequently by Rosenberger in [28, Theorem 3].

It was systematically employed by Niessen and Zettl [23]. In this context we also refer to [3, Propositions 6.11.1, 6.12.1], which discusses linearly independent boundary values in terms of boundary triplets and Wronskians $W(\widehat{u}_b(\lambda_0,\,\cdot),g)(c)$.

(ii) The difference quotient analogue of $\widetilde{g}'(c)$ in (2.29), on the other hand, apparently, was not considered in [3], [16], [23], and [28]. It is a new twist in [10] that offers an explicit description of boundary conditions for lower semibounded, self-adjoint, singular (quasi-regular) Sturm–Liouville operators.

(iii) We recall that for an element $g \in \operatorname{dom}(T_{max})$ the conditions $\widetilde{g}(a) = \widetilde{g}(b) = 0$ describe the Friedrichs extension in Proposition 2.12, and the condition $\widetilde{g}(a) = 0$ describes the Friedrichs extension in Proposition 2.13. It is worthwhile to observe that for $c \in \{a, b\}$ a condition of the form $\widetilde{g}(c) = 0$ is sometimes met in a different guise, such as

$$\lim_{x\to c} \frac{g(x)}{u_c(\lambda_0, x)} \text{ exists in } \mathbb{C}, \tag{2.32}$$

where $u_c(\lambda_0, \cdot)$ is a principal solution. For the special case of the Legendre operator see, for instance, [1, Sect. 132]. For the above and other alternative statements, see also [3, Corollary 6.11.9, Corollary 6.12.9] and [11, Sect. 13.4]. $\diamond$

3. Case one: two limit circle endpoints

In this section we investigate the situation when τ is in the limit circle case at both a and b. The main goal is to provide the sesquilinear forms corresponding to the lower semibounded self-adjoint extensions of T_{min} with separated and coupled boundary conditions from Proposition 2.12. The following hypothesis is assumed throughout this section.

Hypothesis 3.1. In addition to Hypothesis 2.1, assume that τ is in the limit circle case at a and b. Suppose that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$ and that $u_t(\lambda_0,\,\cdot)$ and $\widehat{u}_t(\lambda_0,\,\cdot)$ are principal and nonprincipal solutions of $\tau u = \lambda_0 u$ on (a, b), respectively, at $t \in \{a, b\}$ that satisfy (2.19).

Assuming Hypothesis 3.1, choose $a_0, b_0 \in (a, b)$ such that $a < a_0 < b_0 < b$ and

$$\begin{aligned} &u_a(\lambda_0, x) \neq 0, \quad \widehat{u}_a(\lambda_0, x) \neq 0, \quad x \in (a, a_0); \\ &u_b(\lambda_0, x) \neq 0, \quad \widehat{u}_b(\lambda_0, x) \neq 0, \quad x \in (b_0, b). \end{aligned} \tag{3.1}$$

Let $c \in (a, a_0)$ and $d \in (b_0, b)$ be fixed. Introducing the differential expressions $N_{\widehat{u}_a(\lambda_0,\,\cdot),c}$ and $N_{\widehat{u}_b(\lambda_0,\,\cdot),d}$ by

$$N_{\widehat{u}_a(\lambda_0,\,\cdot),c} f = p^{1/2} \widehat{u}_a(\lambda_0,\,\cdot) \left(\frac{f}{\widehat{u}_a(\lambda_0,\,\cdot)} \right)', \quad f \in AC_{loc}((a, c)); \tag{3.2}$$

$$N_{\widehat{u}_b(\lambda_0,\,\cdot),d}\, g = p^{1/2} \widehat{u}_b(\lambda_0,\,\cdot) \left(\frac{g}{\widehat{u}_b(\lambda_0,\,\cdot)} \right)', \quad g \in AC_{loc}((d, b)), \tag{3.3}$$

one defines the symmetric sesquilinear form $\mathfrak{Q}_{c,d}$ as follows, see, for instance, [3, Sect. 6.8], [11, Sect. 4.5],

$$\begin{aligned}\operatorname{dom}(\mathfrak{Q}_{c,d}) = \big\{h \in L^2((a,b); r\,dx) \,\big|\, h \in AC_{loc}((a,b)),\\ p^{-1/2}h^{[1]} \in L^2((c,d); dx),\ N_{\widehat{u}_a(\lambda_0,\cdot),c}h \in L^2((a,c); dx),\\ N_{\widehat{u}_b(\lambda_0,\cdot),d}h \in L^2((d,b); dx)\big\},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathfrak{Q}_{c,d}(f,g) &= \int_a^c dx\,\overline{(N_{\widehat{u}_a(\lambda_0,\cdot),c}f)(x)}(N_{\widehat{u}_a(\lambda_0,\cdot),c}g)(x)\\ &\quad + \int_d^b dx\,\overline{(N_{\widehat{u}_b(\lambda_0,\cdot),d}f)(x)}(N_{\widehat{u}_b(\lambda_0,\cdot),d}g)(x)\\ &\quad + \lambda_0\int_a^c r(x)\,dx\,\overline{f(x)}g(x) + \lambda_0\int_d^b r(x)\,dx\,\overline{f(x)}g(x)\\ &\quad + \int_c^d dx\,\Big[p(x)^{-1}\overline{f^{[1]}(x)}g^{[1]}(x) + q(x)\overline{f(x)}g(x)\Big]\\ &\quad + \frac{\widehat{u}_a^{[1]}(\lambda_0,c)}{\widehat{u}_a(\lambda_0,c)}\overline{f(c)}g(c) - \frac{\widehat{u}_b^{[1]}(\lambda_0,d)}{\widehat{u}_b(\lambda_0,d)}\overline{f(d)}g(d), \quad f,g \in \operatorname{dom}(\mathfrak{Q}_{c,d}).\end{aligned} \tag{3.5}$$

Several important properties of the sesquilinear form $\mathfrak{Q}_{c,d}$ are collected in the following result.

Proposition 3.2. *Assume Hypothesis 3.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a,a_0)$ and $d \in (b_0,b)$. Then the following statements (i)– (iv) hold:*

(i) The sesquilinear form $\mathfrak{Q}_{c,d}$ defined by (3.4) and (3.5) is densely defined, closed, and lower semibounded in $L^2((a,b); r\,dx)$.

(ii) $\operatorname{dom}(T_{max}) \subseteq \operatorname{dom}(\mathfrak{Q}_{c,d})$.

(iii) If $c' \in (a,a_0)$ and $d' \in (b_0,b)$, then $\mathfrak{Q}_{c,d} = \mathfrak{Q}_{c',d'}$. That is, the sesquilinear form defined by (3.4) and (3.5) is independent of the choices of $c \in (a,a_0)$ and $d \in (b_0,b)$.

(iv) If $g \in \operatorname{dom}(\mathfrak{Q}_{c,d})$, then the following limits exist:

$$\widetilde{g}(a) := \lim_{x\downarrow a}\frac{g(x)}{\widehat{u}_a(\lambda_0,x)}, \quad \widetilde{g}(b) := \lim_{x\uparrow b}\frac{g(x)}{\widehat{u}_b(\lambda_0,x)}. \tag{3.6}$$

In particular, the generalized boundary values $\widetilde{g}(a)$ and $\widetilde{g}(b)$ introduced in (2.21) for functions in $\operatorname{dom}(T_{max})$ extend to functions in $\operatorname{dom}(\mathfrak{Q}_{c,d})$.

Remark 3.3. The properties of $\mathfrak{Q}_{c,d}$ in Proposition 3.2 are discussed in detail in [3]; see [3, Theorem 6.10.9, Lemma 6.9.4, Corollary 6.11.2, Lemma 6.11.3]. $\diamond$

Lemma 3.4. *Assume Hypothesis 3.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a,a_0)$ and $d \in (b_0,b)$. If $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$, then*

$$(f, T_{max} g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}_{c,d}(f,g) + \overline{\widetilde{f}(a)}\widetilde{g}'(a) - \overline{\widetilde{f}(b)}\widetilde{g}'(b). \tag{3.7}$$

Proof. We recall Jacobi's factorization identity in the following form: If $g, h \in AC_{loc}((a,b))$ and $g^{[1]}, h^{[1]} \in AC_{loc}((a,b))$, then

$$-\big(g^{[1]}\big)' + \frac{\big(h^{[1]}\big)'}{h} g = -\frac{1}{h}\left[ph^2\left(\frac{g}{h}\right)'\right]' \text{ when } h \neq 0. \tag{3.8}$$

Since $\widehat{u}_t(\lambda_0, \cdot)$, $t \in \{a,b\}$, are solutions of $\tau u = \lambda_0 u$ on (a,b), one infers that:

$$q = \lambda_0 r + \frac{\big(\widehat{u}_a^{[1]}(\lambda_0, \cdot)\big)'}{\widehat{u}_a(\lambda_0, \cdot)} \text{ a.e. on } (a, a_0); \tag{3.9}$$

$$q = \lambda_0 r + \frac{\big(\widehat{u}_b^{[1]}(\lambda_0, \cdot)\big)'}{\widehat{u}_b(\lambda_0, \cdot)} \text{ a.e. on } (b_0, b). \tag{3.10}$$

To prove (3.7) one calculates for $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$ as follows:

$$\begin{aligned}
&(f, T_{max} g)_{L^2((a,b);r\,dx)} \\
&= \int_a^b dx\, \overline{f}\big[-\big(g^{[1]}\big)' + qg\big] \\
&= \lim_{a'\downarrow a} \int_{a'}^c dx\, \overline{f}\left[-\big(g^{[1]}\big)' + \lambda_0 r g + \frac{\big(\widehat{u}_a^{[1]}(\lambda_0, \cdot)\big)'}{\widehat{u}_a(\lambda_0, \cdot)} g\right] - \int_c^d dx\, \overline{f}\big(g^{[1]}\big)' \\
&\quad + \int_c^d dx\, q\overline{f}g + \lim_{b'\uparrow b} \int_d^{b'} dx\, \overline{f}\left[-\big(g^{[1]}\big)' + \lambda_0 r g + \frac{\big(\widehat{u}_b^{[1]}(\lambda_0, \cdot)\big)'}{\widehat{u}_b(\lambda_0, \cdot)} g\right] \\
&= \lim_{a'\downarrow a} \int_{a'}^c dx\, \overline{f}\left\{-\frac{1}{\widehat{u}_a(\lambda_0, \cdot)}\left[p\widehat{u}_a(\lambda_0, \cdot)^2\left(\frac{g}{\widehat{u}_a(\lambda_0, \cdot)}\right)'\right]'\right\} \\
&\quad + \lambda_0 \int_a^c r\,dx\, \overline{f}g - \overline{f}g^{[1]}\Big|_c^d + \int_c^d dx\,\big(p^{-1}\overline{f^{[1]}}g^{[1]} + q\overline{f}g\big) + \lambda_0 \int_d^b r\,dx\, \overline{f}g \\
&\quad + \lim_{b'\uparrow b} \int_d^{b'} dx\, \overline{f}\left\{-\frac{1}{\widehat{u}_b(\lambda_0, \cdot)}\left[p\widehat{u}_b(\lambda_0, \cdot)^2\left(\frac{g}{\widehat{u}_b(\lambda_0, \cdot)}\right)'\right]'\right\} \qquad (3.11) \\
&= \lim_{a'\downarrow a} \left\{-\frac{\overline{f}}{\widehat{u}_a(\lambda_0, \cdot)}\left[p\widehat{u}_a(\lambda_0, \cdot)^2\left(\frac{g}{\widehat{u}_a(\lambda_0, \cdot)}\right)'\right]\right|_{a'}^c
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_{a'}^{c} dx \left(\frac{\overline{f}}{\widehat{u}_a(\lambda_0,\cdot)}\right)' p\widehat{u}_a(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_a(\lambda_0,\cdot)}\right)' \Bigg\} \\
&+ \lambda_0 \int_a^c r\,dx\,\overline{f} g - \overline{f} g^{[1]}\big|_c^d + \int_c^d dx\,\big(p^{-1}\overline{f^{[1]}} g^{[1]} + q\overline{f} g\big) + \lambda_0 \int_d^b r\,dx\,\overline{f} g \\
&+ \lim_{b'\uparrow b} \Bigg\{ -\frac{\overline{f}}{\widehat{u}_b(\lambda_0,\cdot)} \Bigg[p\widehat{u}_b(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_b(\lambda_0,\cdot)}\right)' \Bigg]\Bigg|_d^{b'} \\
&\qquad + \int_d^{b'} dx \left(\frac{\overline{f}}{\widehat{u}_b(\lambda_0,\cdot)}\right)' p\widehat{u}_b(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_b(\lambda_0,\cdot)}\right)' \Bigg\}.
\end{aligned}
$$

The evaluation terms at c and d in the last equation in (3.11) are

$$
\begin{aligned}
&\Bigg\{ -\frac{\overline{f}}{\widehat{u}_a(\lambda_0,\cdot)} \Bigg[p\widehat{u}_a(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_a(\lambda_0,\cdot)}\right)' \Bigg] \Bigg\}(c) \\
&\quad - \Bigg\{ -\frac{\overline{f}}{\widehat{u}_b(\lambda_0,\cdot)} \Bigg[p\widehat{u}_b(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_b(\lambda_0,\cdot)}\right)' \Bigg] \Bigg\}(d) - \overline{f(d)} g^{[1]}(d) + \overline{f(c)} g^{[1]}(c) \\
&= -\frac{\overline{f(c)}}{\widehat{u}_a(\lambda_0,c)} \big\{ g^{[1]}(c)\widehat{u}_a(\lambda_0,c) - g(c)\widehat{u}_a^{[1]}(\lambda_0,c) \big\} - \overline{f(d)} g^{[1]}(d) + \overline{f(c)} g^{[1]}(c) \\
&\quad + \frac{\overline{f(d)}}{\widehat{u}_b(\lambda_0,d)} \big\{ g^{[1]}(d)\widehat{u}_b(\lambda_0,d) - g(d)\widehat{u}_b^{[1]}(\lambda_0,d) \big\} \\
&= \frac{\widehat{u}_a^{[1]}(\lambda_0,c)}{\widehat{u}_a(\lambda_0,c)} \overline{f(c)} g(c) - \frac{\widehat{u}_b^{[1]}(\lambda_0,d)}{\widehat{u}_b(\lambda_0,d)} \overline{f(d)} g(d).
\end{aligned}
\tag{3.12}
$$

Applying (2.22) and (3.6), one obtains for $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$,

$$
\begin{aligned}
&\lim_{a'\downarrow a} \Bigg[\frac{\overline{f}}{\widehat{u}_a(\lambda_0,\cdot)} p\widehat{u}_a(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_a(\lambda_0,\cdot)}\right)' \Bigg](a') \\
&\quad = \lim_{a'\downarrow a} \frac{\overline{f(a')}}{\widehat{u}_a(\lambda_0,a')} W(\widehat{u}_a(\lambda_0,\cdot), g)(a') = \overline{\widetilde{f}(a)} \widetilde{g}'(a),
\end{aligned}
\tag{3.13}
$$

and, similarly,

$$
\lim_{b'\uparrow b} \Bigg[-\frac{\overline{f}}{\widehat{u}_b(\lambda_0,\cdot)} p\widehat{u}_b(\lambda_0,\cdot)^2 \left(\frac{g}{\widehat{u}_b(\lambda_0,\cdot)}\right)' \Bigg](b') = -\overline{\widetilde{f}(b)} \widetilde{g}'(b). \tag{3.14}
$$

In light of (3.12), (3.13), and (3.14), (3.11) reduces to (3.7). □

Remark 3.5. The identity (3.7) may be found written in the language of boundary triplets in [3, Equation (6.11.5)]; see also Appendix A. $\diamond$

The following infinitesimal form boundedness result is a consequence of [3, Lemma 6.10.4].

Proposition 3.6. *Assume Hypothesis 3.1. For every* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that*

$$\left|\widetilde{f}(t)\right|^2 \leqslant \varepsilon \mathfrak{Q}_{c,d}(f, f) + C(\varepsilon)\|f\|^2_{L^2((a,b);r\,dx)}, \quad f \in \operatorname{dom}(\mathfrak{Q}_{c,d}),\ t \in \{a, b\}. \tag{3.15}$$

Remark 3.7. It is clear that the inequality in (3.15) remains valid with $\mathfrak{Q}_{c,d}(f, f)$ replaced by $|\mathfrak{Q}_{c,d}(f, f)|$, $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$. In particular, the sesquilinear forms

$$\mathfrak{q}_t(f, g) = \overline{\widetilde{f}(t)}\widetilde{g}(t), \quad f, g \in \operatorname{dom}(\mathfrak{q}_t) = \operatorname{dom}(\mathfrak{Q}_{c,d}),\ t \in \{a, b\}, \tag{3.16}$$

are infinitesimally bounded with respect to $\mathfrak{Q}_{c,d}$. $\diamond$

In the next theorem we provide the sesquilinear form corresponding to the self-adjoint extensions $T_{\alpha,\beta}$, $\alpha, \beta \in [0, \pi)$, of T_{min} with separated boundary conditions from Proposition 2.12 (ii).

Theorem 3.8. *Assume Hypothesis 3.1. Let* $a < a_0 < b_0 < b$ *with* a_0 *and* b_0 *chosen so that* (3.1) *holds and suppose* $c \in (a, a_0)$ *and* $d \in (b_0, b)$. *Then the following statements* (i)–(iv) *hold:*
(i) *If* $\alpha, \beta \in (0, \pi)$, *then the sesquilinear form* $\mathfrak{Q}^{\alpha,\beta}_{c,d}$ *defined by*

$$\begin{aligned} &\mathfrak{Q}^{\alpha,\beta}_{c,d}(f, g) = \mathfrak{Q}_{c,d}(f, g) + \cot(\beta)\overline{\widetilde{f}(b)}\widetilde{g}(b) - \cot(\alpha)\overline{\widetilde{f}(a)}\widetilde{g}(a), \\ &f, g \in \operatorname{dom}\left(\mathfrak{Q}^{\alpha,\beta}_{c,d}\right) = \operatorname{dom}(\mathfrak{Q}_{c,d}), \end{aligned} \tag{3.17}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_{\alpha,\beta}g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}^{\alpha,\beta}_{c,d}(f, g), \quad f \in \operatorname{dom}\left(\mathfrak{Q}^{\alpha,\beta}_{c,d}\right),\ g \in \operatorname{dom}(T_{\alpha,\beta}). \tag{3.18}$$

Hence, $\mathfrak{Q}^{\alpha,\beta}_{c,d}$ *is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to* $T_{\alpha,\beta}$, $\alpha, \beta \in (0, \pi)$, *by the First Representation Theorem (cf. [17, Theorem VI.2.1]).*
(ii) *If* $\alpha = 0$ *and* $\beta \in (0, \pi)$, *then the sesquilinear form defined by*

$$\begin{aligned} &\mathfrak{Q}^{0,\beta}_{c,d}(f, g) = \mathfrak{Q}_{c,d}(f, g) + \cot(\beta)\overline{\widetilde{f}(b)}\widetilde{g}(b), \\ &f, g \in \operatorname{dom}\left(\mathfrak{Q}^{0,\beta}_{c,d}\right) = \left\{h \in \operatorname{dom}(\mathfrak{Q}_{c,d}) \,\middle|\, \widetilde{h}(a) = 0\right\}, \end{aligned} \tag{3.19}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_{0,\beta}g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}^{0,\beta}_{c,d}(f, g), \quad f \in \operatorname{dom}\left(\mathfrak{Q}^{0,\beta}_{c,d}\right),\ g \in \operatorname{dom}(T_{0,\beta}). \tag{3.20}$$

Hence, $\mathfrak{Q}^{0,\beta}_{c,d}$ *is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to* $T_{0,\beta}$, $\beta \in (0, \pi)$, *by the First Representation Theorem.*

(iii) If $\alpha \in (0,\pi)$ and $\beta = 0$, then the sesquilinear form defined by

$$\begin{aligned} &\mathfrak{Q}_{c,d}^{\alpha,0}(f,g) = \mathfrak{Q}_{c,d}(f,g) - \cot(\alpha)\overline{\widetilde{f}(a)}\widetilde{g}(a), \\ &f,g \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\alpha,0}\big) = \big\{h \in \operatorname{dom}(\mathfrak{Q}_{c,d}) \,\big|\, \widetilde{h}(b) = 0\big\}, \end{aligned} \tag{3.21}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_{\alpha,0}g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}_{c,d}^{\alpha,0}(f,g), \quad f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\alpha,0}\big),\ g \in \operatorname{dom}(T_{\alpha,0}). \tag{3.22}$$

Hence, $\mathfrak{Q}_{c,d}^{\alpha,0}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to $T_{\alpha,0}$, $\alpha \in (0,\pi)$, by the First Representation Theorem.
(iv) If $\alpha = \beta = 0$, then the sesquilinear form defined by

$$\begin{aligned} &\mathfrak{Q}_{c,d}^{0,0}(f,g) = \mathfrak{Q}_{c,d}(f,g), \\ &f,g \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{0,0}\big) = \big\{h \in \operatorname{dom}(\mathfrak{Q}_{c,d}) \,\big|\, \widetilde{h}(a) = 0 = \widetilde{h}(b)\big\}, \end{aligned} \tag{3.23}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_{0,0}g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}_{c,d}^{0,0}(f,g), \quad f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{0,0}\big),\ g \in \operatorname{dom}(T_{0,0}). \tag{3.24}$$

Hence, $\mathfrak{Q}_{c,d}^{0,0}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to $T_{0,0}$ by the First Representation Theorem.

Proof. (i) It is clear by inspection that $\mathfrak{Q}_{c,d}^{\alpha,\beta}$ is symmetric. That $\mathfrak{Q}_{c,d}^{\alpha,\beta}$ is densely defined, closed, and lower semibounded follows from Remark 3.7 (specifically, the infinitesimal form boundedness of $\mathfrak{q}_t$, $t \in \{a,b\}$, with respect to $\mathfrak{Q}_{c,d}$). To establish (3.18), one applies Lemma 3.4 – specifically (3.7) – and the boundary conditions inherent in the definition of $\operatorname{dom}(T_{\alpha,\beta})$:

$$\begin{aligned} (f, T_{\alpha,\beta}g)_{L^2((a,b);r\,dx)} &= \mathfrak{Q}_{c,d}(f,g) + \overline{\widetilde{f}(a)}\widetilde{g}'(a) - \overline{\widetilde{f}(b)}\widetilde{g}'(b) \\ &= \mathfrak{Q}_{c,d}(f,g) - \cot(\alpha)\overline{\widetilde{f}(a)}\widetilde{g}(a) + \cot(\beta)\overline{\widetilde{f}(b)}\widetilde{g}(b) \\ &= \mathfrak{Q}_{c,d}^{\alpha,\beta}(f,g), \quad f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\alpha,\beta}\big),\ g \in \operatorname{dom}(T_{\alpha,\beta}). \end{aligned} \tag{3.25}$$

The proofs of (ii), (iii), and (iv) are all similar. We will provide a sketch of the proof of the claims in (ii) and omit the details for (iii) and (iv). To prove item (ii), one notes that the sesquilinear form $\mathfrak{Q}_{c,d}^{0,\beta}$ is densely defined since $\operatorname{dom}(T_{min}) \subseteq \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{0,\beta}\big)$ and T_{min} is densely defined. Moreover, $\mathfrak{Q}_{c,d}^{0,\beta}$ is lower semibounded since it is a restriction of $\mathfrak{Q}_{c,d}^{\pi/2,\beta}$, and the latter is lower semibounded by part (i). Let $\mathfrak{Q}'_{c,d}$ denote the restriction of $\mathfrak{Q}_{c,d}$ to $\operatorname{dom}\big(\mathfrak{Q}_{c,d}^{0,\beta}\big)$, where the latter domain is defined according to (3.19). Since $\mathfrak{Q}_{c,d}^{0,\beta}$ is an infinitesimally form bounded perturbation of $\mathfrak{Q}'_{c,d}$ by (3.15), to prove $\mathfrak{Q}_{c,d}^{0,\beta}$ is closed, it suffices to show that $\mathfrak{Q}'_{c,d}$ is closed. If $\{f_n\}_{n=1}^{\infty} \subset \operatorname{dom}(\mathfrak{Q}'_{c,d}) = \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{0,\beta}\big)$, $\|f_n - f\|_{L^2((a,b);r\,dx)} \to 0$ for some $f \in L^2((a,b);r\,dx)$, and $\mathfrak{Q}'_{c,d}(f_n - f_m, f_n - f_m) \to 0$, then the fact that $\mathfrak{Q}_{c,d}$ is closed (cf. Proposition 3.2) implies $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $\mathfrak{Q}_{c,d}(f_n - f, f_n - f) \to 0$. Using (3.15) one obtains

$$\begin{aligned} \big|\widetilde{f}(a)\big|^2 &= \big|\widetilde{f}_n(a) - \widetilde{f}(a)\big|^2 \\ &\leqslant \mathfrak{Q}_{c,d}(f_n - f, f_n - f) + C_0\|f_n - f\|^2_{L^2((a,b);r\,dx)}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.26}$$

for some scalar $C_0 \in (0,\infty)$ that does not depend on $n \in \mathbb{N}$. Taking $n \to \infty$ throughout (3.26), one obtains $\widetilde{f}(a) = 0$. Therefore, $f \in \operatorname{dom}\big(\mathfrak{Q}^{0,\beta}_{c,d}\big)$, and since $\mathfrak{Q}_{c,d}$ is an extension of $\mathfrak{Q}'_{c,d}$, $\mathfrak{Q}'_{c,d}(f_n - f, f_n - f) \to 0$. Hence, $\mathfrak{Q}'_{c,d}$ is closed, and it follows that $\mathfrak{Q}^{0,\beta}_{c,d}$ is closed and lower semibounded. That $\mathfrak{Q}^{0,\beta}_{c,d}$ is symmetric is clear by inspection. Finally, the verification of (3.20) is entirely analogous to that of (3.18) (invoking Lemma 3.4, etc.), so we omit the details. □

In the next theorem we provide the sesquilinear form corresponding to the self-adjoint extensions $T_{\varphi,R}$, $\varphi \in [0,\pi)$, $R \in SL(2,\mathbb{R})$, of T_{min} with coupled boundary conditions from Proposition 2.12 (iii).

Theorem 3.9. *Assume Hypothesis 3.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a, a_0)$ and $d \in (b_0, b)$. If $\varphi \in [0,\pi)$ and $R \in SL(2,\mathbb{R})$, then the following statements (i) and (ii) hold:*

(i) If $R_{1,2} \neq 0$, then the sesquilinear form $\mathfrak{Q}^{\varphi,R}_{c,d}$ defined by

$$\begin{aligned} \mathfrak{Q}^{\varphi,R}_{c,d}(f,g) = \mathfrak{Q}_{c,d}(f,g) - \frac{1}{R_{1,2}}\Big\{&R_{1,1}\overline{\widetilde{f}(a)}\widetilde{g}(a) - e^{-i\varphi}\overline{\widetilde{f}(a)}\widetilde{g}(b) \\ &- e^{i\varphi}\overline{\widetilde{f}(b)}\widetilde{g}(a) + R_{2,2}\overline{\widetilde{f}(b)}\widetilde{g}(b)\Big\}, \\ f, g \in \operatorname{dom}\big(\mathfrak{Q}^{\varphi,R}_{c,d}\big) = \operatorname{dom}(\mathfrak{Q}_{c,d}),& \end{aligned} \tag{3.27}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_{\varphi,R}\,g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}^{\varphi,R}_{c,d}(f,g), \quad f \in \operatorname{dom}\big(\mathfrak{Q}^{\varphi,R}_{c,d}\big),\ g \in \operatorname{dom}(T_{\varphi,R}). \tag{3.28}$$

Hence, $\mathfrak{Q}^{\varphi,R}_{c,d}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to $T_{\varphi,R}$ by the First Representation Theorem.

(ii) If $R_{1,2} = 0$, then the sesquilinear form $\mathfrak{Q}^{\varphi,R}_{c,d}$ defined by

$$\begin{aligned} &\mathfrak{Q}^{\varphi,R}_{c,d}(f,g) = \mathfrak{Q}_{c,d}(f,g) - R_{1,1}R_{2,1}\overline{\widetilde{f}(a)}\widetilde{g}(a), \\ &f, g \in \operatorname{dom}\big(\mathfrak{Q}^{\varphi,R}_{c,d}\big) = \big\{h \in \operatorname{dom}(\mathfrak{Q}_{c,d}) \,\big|\, \widetilde{h}(b) = e^{i\varphi}R_{1,1}\widetilde{h}(a)\big\}, \end{aligned} \tag{3.29}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_{\varphi,R}\,g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}^{\varphi,R}_{c,d}(f,g), \quad f \in \operatorname{dom}\big(\mathfrak{Q}^{\varphi,R}_{c,d}\big),\ g \in \operatorname{dom}(T_{\varphi,R}). \tag{3.30}$$

Hence, $\mathfrak{Q}_{c,d}^{\varphi,R}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to $T_{\varphi,R}$ by the First Representation Theorem.

Proof. The proof of item (i) begins by noting that $\mathfrak{Q}_{c,d}^{\varphi,R}$ is an infinitesimally form bounded perturbation of $\mathfrak{Q}_{c,d}$ by Proposition 3.6. Hence, $\mathfrak{Q}_{c,d}^{\varphi,R}$ is densely defined, closed, and lower semibounded by Proposition 3.2. To prove (3.28), let $f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\varphi,R}\big) = \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{\varphi,R})$. Using the boundary conditions for g given in (2.25), one obtains,

$$\begin{aligned} \widetilde{g}'(a) &= \frac{1}{R_{1,2}}\big[e^{-i\varphi}\widetilde{g}(b) - R_{1,1}\widetilde{g}(a)\big], \\ \widetilde{g}'(b) &= e^{i\varphi}\big[R_{2,1}\widetilde{g}(a) + R_{2,2}\widetilde{g}'(a)\big]. \end{aligned} \tag{3.31}$$

Therefore, using $\det_{\mathbb{C}^2}(R) = 1$ and (3.31), one computes,

$$\begin{aligned} &\overline{\widetilde{f}(a)}\widetilde{g}'(a) - \overline{\widetilde{f}(b)}\widetilde{g}'(b) \\ &= \frac{\overline{\widetilde{f}(a)}}{R_{1,2}}\big\{e^{-i\varphi}\widetilde{g}(b) - R_{1,1}\widetilde{g}(a)\big\} - e^{i\varphi}\overline{\widetilde{f}(b)}\big\{R_{2,1}\widetilde{g}(a) + R_{2,2}\widetilde{g}'(a)\big\} \\ &= \frac{\overline{\widetilde{f}(a)}}{R_{1,2}}\big\{e^{-i\varphi}\widetilde{g}(b) - R_{1,1}\widetilde{g}(a)\big\} \\ &\quad - e^{i\varphi}\overline{\widetilde{f}(b)}\bigg\{R_{2,1}\widetilde{g}(a) + \frac{R_{2,2}}{R_{1,2}}\big[e^{-i\varphi}\widetilde{g}(b) - R_{1,1}\widetilde{g}(a)\big]\bigg\} \\ &= -\frac{1}{R_{1,2}}\Big\{R_{1,1}\overline{\widetilde{f}(a)}\widetilde{g}(a) - e^{-i\varphi}\overline{\widetilde{f}(a)}\widetilde{g}(b) - e^{i\varphi}\overline{\widetilde{f}(b)}\widetilde{g}(a) + R_{2,2}\overline{\widetilde{f}(b)}\widetilde{g}(b)\Big\}, \end{aligned} \tag{3.32}$$

after taking a cancellation into account. The equality in (3.28) now follows from Lemma 3.4 and (3.32).

To prove item (ii), one notes that $\operatorname{dom}(T_{min}) \subseteq \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\varphi,R}\big)$, so $\mathfrak{Q}_{c,d}^{\varphi,R}$ is densely defined since T_{min} is densely defined. Let $\mathfrak{Q}'_{c,d}$ denote the restriction of $\mathfrak{Q}_{c,d}$ to $\operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\varphi,R}\big)$, where the latter domain is defined according to (3.29). Since $\mathfrak{Q}_{c,d}^{\varphi,R}$ is an infinitesimally form bounded perturbation of $\mathfrak{Q}'_{c,d}$ by (3.15), to prove $\mathfrak{Q}_{c,d}^{\varphi,R}$ is closed, it suffices to show that $\mathfrak{Q}'_{c,d}$ is closed. If $\{f_n\}_{n=1}^{\infty} \subset \operatorname{dom}(\mathfrak{Q}'_{c,d}) = \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\varphi,R}\big)$, $\|f_n - f\|_{L^2((a,b);r\,dx)} \to 0$ for some $f \in L^2((a,b);r\,dx)$, and $\mathfrak{Q}'_{c,d}(f_n - f_m, f_n - f_m) \to 0$, then the fact that $\mathfrak{Q}_{c,d}$ is closed (cf. Proposition 3.2) implies $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $\mathfrak{Q}_{c,d}(f_n - f, f_n - f) \to 0$. Using (3.15) one obtains

$$\begin{aligned} \big|\widetilde{f}(b) - e^{i\varphi}R_{1,1}\widetilde{f}(a)\big|^2 &= \big|\big[\widetilde{f}_n(b) - \widetilde{f}(b)\big] - e^{i\varphi}R_{1,1}\big[\widetilde{f}_n(a) - \widetilde{f}(a)\big]\big|^2 \\ &\leqslant \mathfrak{Q}_{c,d}(f_n - f, f_n - f) + C_0\|f_n - f\|^2_{L^2((a,b);r\,dx)}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.33}$$

for some scalar $C_0 \in (0,\infty)$ that does not depend on $n \in \mathbb{N}$. Taking $n \to \infty$ throughout (3.33), one obtains $\widetilde{f}(b) = e^{i\varphi}R_{1,1}\widetilde{f}(a)$. Therefore, $f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\varphi,R}\big)$, and since $\mathfrak{Q}_{c,d}$ is an extension of $\mathfrak{Q}'_{c,d}$, $\mathfrak{Q}'_{c,d}(f_n - f, f_n - f) \to 0$. Hence, $\mathfrak{Q}'_{c,d}$ is closed, and it follows that $\mathfrak{Q}_{c,d}^{\varphi,R}$ is closed and lower semibounded.

To verify (3.30), let $f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\varphi,R}\big)$ and $g \in \operatorname{dom}(T_{\varphi,R})$. Using the relations

$$\widetilde{f}(b) = e^{i\varphi} R_{1,1}\widetilde{f}(a), \quad \widetilde{g}'(b) = e^{i\varphi}\big[R_{2,1}\widetilde{g}(a) + R_{2,2}\widetilde{g}'(a)\big], \tag{3.34}$$

and $1 = \det_{\mathbb{C}^2}(R) = R_{1,1}R_{2,2}$, one computes:

$$\begin{aligned}\overline{\widetilde{f}(a)}\widetilde{g}'(a) - \overline{\widetilde{f}(b)}\widetilde{g}'(b) &= \overline{\widetilde{f}(a)}\widetilde{g}'(a) - e^{-i\varphi}R_{1,1}\overline{\widetilde{f}(a)}e^{i\varphi}\big[R_{2,1}\widetilde{g}(a) + R_{2,2}\widetilde{g}'(a)\big] \\ &= \overline{\widetilde{f}(a)}\widetilde{g}'(a) - R_{1,1}R_{2,1}\overline{\widetilde{f}(a)}\widetilde{g}(a) - R_{1,1}R_{2,2}\overline{\widetilde{f}(a)}\widetilde{g}'(a) \\ &= -R_{1,1}R_{2,1}\overline{\widetilde{f}(a)}\widetilde{g}(a).\end{aligned} \tag{3.35}$$

The equality in (3.30) now follows from Lemma 3.4 and (3.35). □

Remark 3.10. (i) Since $\mathfrak{Q}_{c,d}$ is independent of the choices of $c \in (a, a_0)$ and $d \in (b_0, b)$ (cf. Proposition 3.2 (iii)), it follows that the sesquilinear forms $\mathfrak{Q}_{c,d}^{\alpha,\beta}$, $\alpha, \beta \in [0, \pi)$, and $\mathfrak{Q}_{c,d}^{\varphi,R}$, $\varphi \in [0, \pi)$, $R \in SL(2, \mathbb{R})$, are also independent of c and d.

(ii) It is clear that the sesquilinear forms for $T_{\alpha,\beta}$ and $T_{\varphi,R}$ in (3.17), (3.19), (3.21), (3.23), (3.27), and (3.29) depend on the choices of the principal and nonprincipal solutions $u_t(\lambda_0, \cdot)$ and $\widehat{u}_t(\lambda_0, \cdot)$, $t \in \{a, b\}$. However, this is to be expected, as the parametrizations of the self-adjoint extensions of T_{min} given in Proposition 2.12 also depend on the choices of the principal and nonprincipal solutions $u_t(\lambda_0, \cdot)$ and $\widehat{u}_t(\lambda_0, \cdot)$, $t \in \{a, b\}$. ◇

4. Case two: one limit circle endpoint

In this section we provide the sesquilinear forms corresponding to the lower semibounded self-adjoint realizations T_α from Proposition 2.13. We assume, in addition to Hypothesis 2.1, that the differential expression τ is in the limit circle case at exactly one endpoint of the interval (a, b) and that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$. For simplicity, we consider the case when τ is in the limit circle case at a and in the limit point case at b. The situation where τ is in the limit point case at a and in the limit circle case at b is entirely analogous. To be precise, we introduce the following hypothesis.

Hypothesis 4.1. In addition to Hypothesis 2.1, assume that τ is in the limit circle case at a and in the limit point case at b. Suppose that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$ and that $u_t(\lambda_0, \cdot)$ and $\widehat{u}_t(\lambda_0, \cdot)$ are principal and nonprincipal solutions of $\tau u = \lambda_0 u$ on (a, b), respectively, at $t \in \{a, b\}$ that satisfy (2.19).

Assuming Hypothesis 4.1, choose $a_0, b_0 \in (a, b)$ such that $a < a_0 < b_0 < b$ and (3.1) holds. Let $c \in (a, a_0)$ and $d \in (b_0, b)$ be fixed. Next, we formally replace the nonprincipal solution $\widehat{u}_b(\lambda_0, \cdot)$ in Section 3 with the principal solution $u_b(\lambda_0, \cdot)$. More precisely, introducing the differential expressions $N_{\widehat{u}_a(\lambda_0,\cdot),c}$ as in (3.2) and $N_{u_b(\lambda_0,\cdot),d}$ by

$$N_{u_b(\lambda_0,\cdot),d}\, g = p^{1/2} u_b(\lambda_0, \cdot)\left(\frac{g}{u_b(\lambda_0, \cdot)}\right)', \quad g \in AC_{loc}((d, b)), \tag{4.1}$$

one defines the symmetric sesquilinear form $\mathfrak{Q}_{c,d}$ as follows:

$$\begin{aligned}\operatorname{dom}(\mathfrak{Q}_{c,d}) = \big\{h \in L^2((a,b); r\,dx) \,\big|\, h \in AC_{loc}((a,b)),\\ p^{-1/2}h^{[1]} \in L^2((c,d); dx),\ N_{\widehat{u}_a(\lambda_0,\,\cdot\,),c}h \in L^2((a,c); dx),\\ N_{u_b(\lambda_0,\,\cdot\,),d}h \in L^2((d,b); dx)\big\},\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\mathfrak{Q}_{c,d}(f,g) = &\int_a^c dx\, \overline{(N_{\widehat{u}_a(\lambda_0,\,\cdot\,),c}f)(x)}(N_{\widehat{u}_a(\lambda_0,\,\cdot\,),c}g)(x)\\ &+ \int_d^b dx\, \overline{(N_{u_b(\lambda_0,\,\cdot\,),d}f)(x)}(N_{u_b(\lambda_0,\,\cdot\,),d}g)(x)\\ &+ \lambda_0 \int_a^c r(x)\,dx\, \overline{f(x)}g(x) + \lambda_0 \int_d^b r(x)\,dx\, \overline{f(x)}g(x)\\ &+ \int_c^d dx\, \Big[p(x)^{-1}\overline{f^{[1]}(x)}g^{[1]}(x) + q(x)\overline{f(x)}g(x)\Big]\\ &+ \frac{\widehat{u}_a^{[1]}(\lambda_0,c)}{\widehat{u}_a(\lambda_0,c)}\overline{f(c)}g(c) - \frac{u_b^{[1]}(\lambda_0,d)}{u_b(\lambda_0,d)}\overline{f(d)}g(d), \quad f,g \in \operatorname{dom}(\mathfrak{Q}_{c,d}).\end{aligned} \tag{4.3}$$

Several important properties of the sesquilinear form $\mathfrak{Q}_{c,d}$ are collected in the following result.

Proposition 4.2. *Assume Hypothesis 4.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a,a_0)$ and $d \in (b_0,b)$. Then the following statements (i)–(vi) hold:*
(i) *The sesquilinear form $\mathfrak{Q}_{c,d}$ defined by (4.2) and (4.3) is densely defined, closed, and lower semibounded in $L^2((a,b); r\,dx)$.*
(ii) $\operatorname{dom}(T_{max}) \subseteq \operatorname{dom}(\mathfrak{Q}_{c,d})$.
(iii) *If $c' \in (a,a_0)$ and $d' \in (b_0,b)$, then $\mathfrak{Q}_{c,d} = \mathfrak{Q}_{c',d'}$. That is, the sesquilinear form defined by (4.2) and (4.3) is independent of the choices of $c \in (a,a_0)$ and $d \in (b_0,b)$.*
(iv) *If $g \in \operatorname{dom}(\mathfrak{Q}_{c,d})$, then the following limit exists:*

$$\widetilde{g}(a) := \lim_{x\downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0,x)}. \tag{4.4}$$

In particular, the generalized boundary value $\widetilde{g}(a)$ introduced in (2.21) for functions in $\operatorname{dom}(T_{max})$ extends to functions in $\operatorname{dom}(\mathfrak{Q}_{c,d})$.
(v) *If $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$, then*

$$\lim_{b'\uparrow b} \frac{\overline{f(b')}}{u_b(\lambda_0,b')} W(u_b(\lambda_0,\,\cdot\,),g)(b') = 0. \tag{4.5}$$

(vi) *For every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\big|\widetilde{f}(a)\big|^2 \leqslant \varepsilon \mathfrak{Q}_{c,d}(f,f) + C(\varepsilon)\|f\|^2_{L^2((a,b);r\,dx)}, \quad f \in \operatorname{dom}(\mathfrak{Q}_{c,d}). \tag{4.6}$$

Remark 4.3. The properties of $\mathfrak{Q}_{c,d}$ summarized in Proposition 4.2 are discussed in detail in [3] (see [3, Lemma 6.9.4, Corollary 6.12.2, Lemma 6.12.3, Proof of Lemma 6.12.5]) and (4.6) is entirely analogous to Proposition 3.6. For the connection with [3, Sect. 6.12], see Appendix A. $\diamond$

Lemma 4.4. *Assume Hypothesis 4.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a, a_0)$ and $d \in (b_0, b)$. If $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$, then*

$$(f, T_{max}g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}_{c,d}(f,g) + \overline{\widetilde{f}(a)}\widetilde{g}'(a). \tag{4.7}$$

Proof. Repeating the calculations in (3.9)–(3.11) with $u_b(\lambda_0,\cdot)$ in place of $\widehat{u}_b(\lambda_0,\cdot)$, one obtains for $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$,

$$\begin{aligned}
&(f, T_{max}g)_{L^2((a,b);r\,dx)} \\
&= \lim_{a' \downarrow a} \Bigg\{ -\frac{\overline{f}}{\widehat{u}_a(\lambda_0,\cdot)}\Bigg[p\widehat{u}_a(\lambda_0,\cdot)^2\bigg(\frac{g}{\widehat{u}_a(\lambda_0,\cdot)}\bigg)'\Bigg]\Bigg|_{a'}^{c} \\
&\qquad + \int_{a'}^{c} dx \, \bigg(\frac{\overline{f}}{\widehat{u}_a(\lambda_0,\cdot)}\bigg)' p\widehat{u}_a(\lambda_0,\cdot)^2 \bigg(\frac{g}{\widehat{u}_a(\lambda_0,\cdot)}\bigg)'\Bigg\} \\
&\quad + \lambda_0 \int_a^c r\,dx\,\overline{f}g - \overline{f}g^{[1]}\big|_c^d + \int_c^d dx\,\big(p^{-1}\overline{f^{[1]}}g^{[1]} + q\overline{f}g\big) + \lambda_0\int_d^b r\,dx\,\overline{f}g \\
&\quad + \lim_{b'\uparrow b}\Bigg\{-\frac{\overline{f}}{u_b(\lambda_0,\cdot)}\Bigg[pu_b(\lambda_0,\cdot)^2\bigg(\frac{g}{u_b(\lambda_0,\cdot)}\bigg)'\Bigg]\Bigg|_d^{b'} \\
&\qquad + \int_d^{b'} dx\,\bigg(\frac{\overline{f}}{u_b(\lambda_0,\cdot)}\bigg)' pu_b(\lambda_0,\cdot)^2\bigg(\frac{g}{u_b(\lambda_0,\cdot)}\bigg)'\Bigg\}.
\end{aligned} \tag{4.8}$$

In analogy with (3.12), the evaluation terms at c and d in (4.8) are

$$\frac{\widehat{u}_a^{[1]}(\lambda_0,c)}{\widehat{u}_a(\lambda_0,c)}\overline{f(c)}g(c) - \frac{u_b^{[1]}(\lambda_0,d)}{u_b(\lambda_0,d)}\overline{f(d)}g(d). \tag{4.9}$$

Moreover, (3.13) remains valid. However, in lieu of (3.14), one now obtains, as a consequence of (4.5),

$$\begin{aligned}
&\lim_{b'\uparrow b}\Bigg[\frac{\overline{f}}{u_b(\lambda_0,\cdot)}pu_b(\lambda_0,\cdot)^2\bigg(\frac{g}{u_b(\lambda_0,\cdot)}\bigg)'\Bigg](b') \\
&= \lim_{b'\uparrow b}\frac{\overline{f(b')}}{u_b(\lambda_0,b')}W(u_b(\lambda_0,\cdot),g)(b') = 0.
\end{aligned} \tag{4.10}$$

Hence, (4.7) follows by combining (4.8), (4.9), and (4.10). □

In the next theorem we provide the sesquilinear form corresponding to the self-adjoint extensions T_α, $\alpha \in [0, \pi)$, of T_{min} with a separated boundary condition from Proposition 2.13 (ii).

Theorem 4.5. *Assume Hypothesis 4.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a, a_0)$ and $d \in (b_0, b)$. Then the following statements (i) and (ii) hold:*
(i) *If $\alpha \in (0, \pi)$, then the sesquilinear form $\mathfrak{Q}^{\alpha}_{c,d}$ defined by*

$$\begin{aligned} &\mathfrak{Q}^{\alpha}_{c,d}(f, g) = \mathfrak{Q}_{c,d}(f, g) - \cot(\alpha)\overline{\widetilde{f}(a)}\widetilde{g}(a), \\ &f, g \in \operatorname{dom}\left(\mathfrak{Q}^{\alpha}_{c,d}\right) = \operatorname{dom}(\mathfrak{Q}_{c,d}), \end{aligned} \tag{4.11}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_\alpha g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}^{\alpha}_{c,d}(f, g), \quad f \in \operatorname{dom}\left(\mathfrak{Q}^{\alpha}_{c,d}\right),\ g \in \operatorname{dom}(T_\alpha). \tag{4.12}$$

Hence, $\mathfrak{Q}^{\alpha}_{c,d}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to T_α by the First Representation Theorem.
(ii) *If $\alpha = 0$, then the sesquilinear form $\mathfrak{Q}^{0}_{c,d}$ defined by*

$$\mathfrak{Q}^{0}_{c,d}(f, g) = \mathfrak{Q}_{c,d}(f, g), \quad f, g \in \operatorname{dom}(\mathfrak{Q}^{0}_{c,d}) = \left\{h \in \operatorname{dom}(\mathfrak{Q}_{c,d}) \,\middle|\, \widetilde{h}(a) = 0\right\}, \tag{4.13}$$

is densely defined, closed, symmetric, and lower semibounded. In addition,

$$(f, T_0 g)_{L^2((a,b);r\,dx)} = \mathfrak{Q}^{0}_{c,d}(f, g), \quad f \in \operatorname{dom}\left(\mathfrak{Q}^{0}_{c,d}\right),\ g \in \operatorname{dom}(T_0). \tag{4.14}$$

Hence, $\mathfrak{Q}^{0}_{c,d}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to T_0 by the First Representation Theorem.

Proof. (i) It is clear by inspection that $\mathfrak{Q}^{\alpha}_{c,d}$ is symmetric, and $\operatorname{dom}(T_{min}) \subseteq \operatorname{dom}\left(\mathfrak{Q}^{\alpha}_{c,d}\right)$ shows that $\mathfrak{Q}^{\alpha}_{c,d}$ is densely defined in $L^2((a, b); r\,dx)$. That $\mathfrak{Q}^{\alpha}_{c,d}$ is closed and lower semibounded follows from the infinitesimal boundedness property summarized in (4.6). If $f \in \operatorname{dom}\left(\mathfrak{Q}^{\alpha}_{c,d}\right)$ and $g \in \operatorname{dom}(T_\alpha)$, then (4.7) and the boundary condition $\widetilde{g}'(a) = -\cot(\alpha)\widetilde{g}(a)$ yield:

$$\begin{aligned} (f, T_\alpha g)_{L^2((a,b);r\,dx)} &= (f, T_{max} g)_{L^2((a,b);r\,dx)} \\ &= \mathfrak{Q}_{c,d}(f, g) - \cot(\alpha)\overline{\widetilde{f}(a)}\widetilde{g}(a) = \mathfrak{Q}^{\alpha}_{c,d}(f, g). \end{aligned} \tag{4.15}$$

(ii) One notes that $\mathfrak{Q}^{0}_{c,d}$ is densely defined since $\operatorname{dom}(T_{min}) \subseteq \operatorname{dom}\left(\mathfrak{Q}^{0}_{c,d}\right)$, and $\mathfrak{Q}^{0}_{c,d}$ is lower semibounded since it is a restriction of $\mathfrak{Q}^{\pi/2}_{c,d}$, and the latter is lower semibounded by part (i). To prove that $\mathfrak{Q}^{0}_{c,d}$ is closed, let $\{f_n\}_{n=1}^{\infty} \subset \operatorname{dom}\left(\mathfrak{Q}^{0}_{c,d}\right)$ be a sequence such that $\|f_n - f\|_{L^2((a,b);r\,dx)} \to 0$ for some $f \in L^2((a, b); r\,dx)$ and $\mathfrak{Q}^{0}_{c,d}(f_n - f_m, f_n - f_m) \to 0$. Since $\mathfrak{Q}^{0}_{c,d}$ is a restriction of $\mathfrak{Q}^{\pi/2}_{c,d}$, and the latter is closed, it follows that $f \in \operatorname{dom}\left(\mathfrak{Q}^{\pi/2}_{c,d}\right)$ and

$\mathfrak{Q}_{c,d}^{\pi/2}(f_n - f, f_n - f) \to 0$. By (4.6), one obtains: For every $\varepsilon > 0$, there exists $\widehat{C}(\varepsilon) > 0$ such that

$$\big|\widetilde{g}(a)\big|^2 \leqslant \varepsilon \mathfrak{Q}_{c,d}^{\pi/2}(g,g) + \widehat{C}(\varepsilon)\|g\|^2_{L^2((a,b);r\,dx)}, \quad g \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^{\pi/2}\big). \tag{4.16}$$

In turn, (4.16) with $\varepsilon = 1$ yields:

$$\begin{aligned} \big|\widetilde{f}(a)\big|^2 &= \big|\widetilde{f_n}(a) - \widetilde{f}(a)\big|^2 \\ &\leqslant \mathfrak{Q}_{c,d}^{\pi/2}(f_n - f, f_n - f) + \widehat{C}(1)\|f_n - f\|^2_{L^2((a,b);r\,dx)}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.17}$$

Taking $n \to \infty$ throughout (4.17) yields $\widetilde{f}(a) = 0$, thereby implying $f \in \operatorname{dom}\big(\mathfrak{Q}_{c,d}^0\big)$. Using once more that $\mathfrak{Q}_{c,d}^0$ is a restriction of $\mathfrak{Q}_{c,d}^{\pi/2}$, it follows that $\mathfrak{Q}_{c,d}^0(f_n - f, f_n - f) \to 0$. Hence, one concludes that $\mathfrak{Q}_{c,d}^0$ is closed. Finally, (4.14) follows from (4.7) and the boundary condition $\widetilde{f}(a) = 0$. $\square$

5. Case three: two limit point endpoints

In this final section we provide the sesquilinear form corresponding to the unique, lower semibounded, self-adjoint realization from Proposition 2.14. We assume, in addition to Hypothesis 2.1, that τ is in the limit point case at both endpoints of the interval (a,b) and that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$. To be precise, we introduce the following hypothesis.

Hypothesis 5.1. In addition to Hypothesis 2.1, assume that τ is in the limit point case at both a and b. Suppose that $T_{min} \geqslant \lambda_0 I_{L^2((a,b);r\,dx)}$ for some $\lambda_0 \in \mathbb{R}$ and that $u_t(\lambda_0, \cdot)$ is a principal solution of $\tau u = \lambda_0 u$ on (a,b), respectively, at $t \in \{a,b\}$ that satisfies (2.19).

Under Hypothesis 5.1, the operator $T := T_{min} = T_{max}$ is self-adjoint (equivalently, $\dot{T}$ is essentially self-adjoint) by Proposition 2.14. In particular, T_{min} is self-adjoint and possesses no nontrivial self-adjoint extension.

Assuming Hypothesis 5.1, choose $a_0, b_0 \in (a,b)$ such that $a < a_0 < b_0 < b$ and (3.1) holds. Let $c \in (a, a_0)$ and $d \in (b_0, b)$ be fixed. Next, we formally replace the nonprincipal solutions $\widehat{u}_t(\lambda_0, \cdot)$, $t \in \{a,b\}$, in Section 3 with the principal solutions $u_t(\lambda_0, \cdot)$, $t \in \{a,b\}$. More precisely, introducing the differential expressions $N_{u_b(\lambda_0,\cdot),d}$ as in (4.1) and $N_{u_a(\lambda_0,\cdot),c}$ by

$$N_{u_a(\lambda_0,\cdot),c}\, g = p^{1/2} u_a(\lambda_0, \cdot)\left(\frac{g}{u_a(\lambda_0,\cdot)}\right)', \quad g \in AC_{loc}((a,c)), \tag{5.1}$$

one defines the symmetric sesquilinear form $\mathfrak{Q}_{c,d}$ as follows:

$$\begin{aligned} \operatorname{dom}(\mathfrak{Q}_{c,d}) = \big\{ h \in L^2((a,b);r\,dx) \,\big|\, & h \in AC_{loc}((a,b)), \\ & p^{-1/2} h^{[1]} \in L^2((c,d);dx),\ N_{u_a(\lambda_0,\cdot),c}\, h \in L^2((a,c);dx), \\ & N_{u_b(\lambda_0,\cdot),d}\, h \in L^2((d,b);dx) \big\}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\mathfrak{Q}_{c,d}(f,g) &= \int_a^c dx\, \overline{(N_{u_a(\lambda_0,\,\cdot\,),c} f)(x)} (N_{u_a(\lambda_0,\,\cdot\,),c}\, g)(x) \\ &\quad + \int_d^b dx\, \overline{(N_{u_b(\lambda_0,\,\cdot\,),d} f)(x)} (N_{u_b(\lambda_0,\,\cdot\,),d}\, g)(x) \\ &\quad + \lambda_0 \int_a^c r(x)\, dx\, \overline{f(x)} g(x) + \lambda_0 \int_d^b r(x)\, dx\, \overline{f(x)} g(x) \\ &\quad + \int_c^d dx \Big[p(x)^{-1} \overline{f^{[1]}(x)} g^{[1]}(x) + q(x) \overline{f(x)} g(x) \Big] \\ &\quad + \frac{u_a^{[1]}(\lambda_0,c)}{u_a(\lambda_0,c)} \overline{f(c)} g(c) - \frac{u_b^{[1]}(\lambda_0,d)}{u_b(\lambda_0,d)} \overline{f(d)} g(d), \quad f,g \in \operatorname{dom}(\mathfrak{Q}_{c,d}).\end{aligned} \tag{5.3}$$

Several important properties of the sesquilinear form $\mathfrak{Q}_{c,d}$ are collected in the following result.

Proposition 5.2. *Assume Hypothesis 5.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a, a_0)$ and $d \in (b_0, b)$. Then the following statements (i)–(iv) hold:*

(i) The sesquilinear form $\mathfrak{Q}_{c,d}$ defined by (5.2) and (5.3) is densely defined, closed, and lower semibounded in $L^2((a,b); r\,dx)$.

(ii) $\operatorname{dom}(T_{max}) \subseteq \operatorname{dom}(\mathfrak{Q}_{c,d})$.

(iii) If $c' \in (a, a_0)$ and $d' \in (b_0, b)$, then $\mathfrak{Q}_{c,d} = \mathfrak{Q}_{c',d'}$. That is, the sesquilinear form defined by (5.2) and (5.3) is independent of the choices of $c \in (a, a_0)$ and $d \in (b_0, b)$.

(iv) If $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T_{max})$, then

$$\lim_{a' \downarrow a} \frac{\overline{f(a')}}{u_a(\lambda_0,a')} W(u_a(\lambda_0,\,\cdot\,), g)(a') = \lim_{b' \uparrow b} \frac{\overline{f(b')}}{u_b(\lambda_0,b')} W(u_b(\lambda_0,\,\cdot\,), g)(b') = 0. \tag{5.4}$$

Remark 5.3. The proofs of items (i)–(iv) in Proposition 5.2 are entirely analogous to those of the corresponding facts in Proposition 4.2. $\diamond$

Theorem 5.4. *Assume Hypothesis 5.1. Let $a < a_0 < b_0 < b$ with a_0 and b_0 chosen so that (3.1) holds and suppose $c \in (a, a_0)$ and $d \in (b_0, b)$. If $T := T_{min} = T_{max}$, then*

$$(f, Tg)_{L^2((a,b);r\,dx)} = \mathfrak{Q}_{c,d}(f,g), \quad f \in \operatorname{dom}(\mathfrak{Q}_{c,d}),\ g \in \operatorname{dom}(T). \tag{5.5}$$

Hence, $\mathfrak{Q}_{c,d}$ is the unique densely defined, closed, symmetric, lower semibounded sesquilinear form associated to T by the First Representation Theorem.

Proof. Repeating the calculations in (4.11) with $u_a(\lambda_0,\,\cdot\,)$ in place of $\widehat{u}_a(\lambda_0,\,\cdot\,)$, one obtains for $f \in \operatorname{dom}(\mathfrak{Q}_{c,d})$ and $g \in \operatorname{dom}(T)$,

$$
\begin{aligned}
&(f, T_{max} g)_{L^2((a,b);r\,dx)} \\
&\quad = \lim_{a' \downarrow a} \bigg\{ - \frac{\overline{f}}{u_a(\lambda_0, \cdot)} \bigg[p u_a(\lambda_0, \cdot)^2 \bigg(\frac{g}{u_a(\lambda_0, \cdot)} \bigg)' \bigg] \bigg|_{a'}^{c} \\
&\qquad\qquad + \int_{a'}^{c} dx \, \bigg(\frac{\overline{f}}{u_a(\lambda_0, \cdot)} \bigg)' p u_a(\lambda_0, \cdot)^2 \bigg(\frac{g}{u_a(\lambda_0, \cdot)} \bigg)' \bigg\} \\
&\qquad + \lambda_0 \int_a^c r\,dx\, \overline{f} g - \overline{f} g^{[1]} \big|_c^d + \int_c^d dx\, \big(p^{-1} \overline{f^{[1]}} g^{[1]} + q \overline{f} g \big) + \lambda_0 \int_d^b r\,dx\, \overline{f} g \\
&\qquad + \lim_{b' \uparrow b} \bigg\{ - \frac{\overline{f}}{u_b(\lambda_0, \cdot)} \bigg[p u_b(\lambda_0, \cdot)^2 \bigg(\frac{g}{u_b(\lambda_0, \cdot)} \bigg)' \bigg] \bigg|_{d}^{b'} \\
&\qquad\qquad + \int_{d}^{b'} dx \, \bigg(\frac{\overline{f}}{u_b(\lambda_0, \cdot)} \bigg)' p u_b(\lambda_0, \cdot)^2 \bigg(\frac{g}{u_b(\lambda_0, \cdot)} \bigg)' \bigg\}.
\end{aligned} \tag{5.6}
$$

In analogy with (4.9), the evaluation terms at c and d in (5.6) are

$$
\frac{u_a^{[1]}(\lambda_0, c)}{u_a(\lambda_0, c)} \overline{f(c)} g(c) - \frac{u_b^{[1]}(\lambda_0, d)}{u_b(\lambda_0, d)} \overline{f(d)} g(d). \tag{5.7}
$$

Moreover, (4.10) remains valid. In addition, as a consequence of (5.4),

$$
\begin{aligned}
&\lim_{a' \downarrow a} \bigg[\frac{\overline{f}}{u_a(\lambda_0, \cdot)} p u_a(\lambda_0, \cdot)^2 \bigg(\frac{g}{u_a(\lambda_0, \cdot)} \bigg)' \bigg](a') \\
&\quad = \lim_{a' \downarrow a} \frac{\overline{f(a')}}{u_a(\lambda_0, a')} W(u_a(\lambda_0, \cdot), g)(a') = 0.
\end{aligned} \tag{5.8}
$$

Hence, (5.5) follows by combining (5.6), (5.7), and (5.8). □

Acknowledgments

The authors thank the referees for a very careful reading of the manuscript and various helpful suggestions to improve the text. J.B. is most grateful for a stimulating research stay at Baylor University, where some parts of this paper were written in May of 2025. F.G. and H.S. gratefully acknowledge kind invitations to the Institute of Applied Mathematics at the Graz University of Technology, Austria. This research was funded by the Austrian Science Fund (FWF) Grant-DOI: 10.55776/P33568.

Appendix A. Approach via boundary triplets and boundary pairs

In this appendix we briefly provide the background of the results in Section 3 and Section 4 of this paper in terms of the boundary triplets and boundary pairs following the extensive treatment in [3, Chs. 2, 5, 6]. By means of boundary pairs one can systematically treat the semibounded forms that are associated with the lower semibounded self-adjoint extensions of lower semibounded symmetric operators. In this paper inner products and sesquilinear forms are linear in the second entry and anti-linear in the first entry; in the references to [3] one should be aware of the present convention. Thus, when a sesquilinear form $\mathfrak{t}$ in a Hilbert space $\mathfrak{H}$ is densely defined, closed, and lower semibounded, then there exists a unique self-adjoint operator H in $\mathfrak{H}$, such that

$$\mathfrak{t}[f,g]=(f,Hg)_{\mathfrak{H}}, \quad f\in \operatorname{dom}(\mathfrak{t}),\ g\in\operatorname{dom}(H)\subseteq\operatorname{dom}(\mathfrak{t}),$$

by the First Representation Theorem. The notation $\mathfrak{t}=\mathfrak{t}_H$ is used to indicate the connection with the self-adjoint operator H.

Boundary Triplets. Let S be a closed, densely defined, symmetric operator in a Hilbert space $\mathfrak{H}$ and assume that the defect numbers of S are equal to (n,n), $n\in\mathbb{N}$. A triplet $\{\mathbb{C}^n,\Gamma_0,\Gamma_1\}$ is called a *boundary triplet* for S^* if the linear mappings $\Gamma_0,\Gamma_1:\operatorname{dom}(S^*)\to\mathbb{C}^n$ satisfy the abstract Green identity,

$$(f,S^*g)_{\mathfrak{H}}-(S^*f,g)_{\mathfrak{H}}=(\Gamma_0 f,\Gamma_1 g)_{\mathbb{C}^n}-(\Gamma_1 f,\Gamma_0 g)_{\mathbb{C}^n}, \quad f,g\in\operatorname{dom}(S^*),$$

and $(\Gamma_0,\Gamma_1)^\top:\operatorname{dom}(S^*)\to\mathbb{C}^{2n}$ is onto, see [3, Definition 2.1.1]. If $\{\mathbb{C}^n,\Gamma_0,\Gamma_1\}$ is a boundary triplet for S^*, then one has

$$\operatorname{dom}(S)=\big\{g\in\operatorname{dom}(S^*)\,\big|\,\Gamma_0 g=\Gamma_1 g=0\big\},$$

and the mapping $(\Gamma_0,\Gamma_1)^\top:\operatorname{dom}(S^*)\to\mathbb{C}^{2n}$ is continuous if $\operatorname{dom}(S^*)$ is equipped with the graph norm. The self-adjoint extensions A_Θ of S are parametrized over the self-adjoint relations (multi-valued operators) Θ in $\mathbb{C}^n$ via

$$A_\Theta g=S^*g, \quad g\in\operatorname{dom}(A_\Theta)=\big\{h\in\operatorname{dom}(S^*)\,\big|\,\{\Gamma_0 h,\Gamma_1 h\}\in\Theta\big\}, \tag{A.1}$$

see [3, Theorem 2.1.3]. For more details on (self-adjoint) linear relations, their adjoints, and further operations and notions we refer to [3, Chapter 1]; the special case of self-adjoint relations in finite dimensional spaces will be discussed below. We also note that if Θ is a self-adjoint relation in $\mathbb{C}^n$, then $\operatorname{dom}(\Theta)=(\operatorname{mul}(\Theta))^\perp$ and one has the decomposition

$$\mathbb{C}^n=\operatorname{dom}(\Theta)\oplus\operatorname{mul}(\Theta).$$

In this context we recall that the multi-valued part $\operatorname{mul}(\Theta)$ is given by $\{h\in\mathbb{C}^n\,|\,\{0,h\}\in\Theta\}$. Let P be the orthogonal projection onto $\operatorname{dom}(\Theta)$ and define the orthogonal operator part $\Theta_{\rm op}=P\Theta$. Then there is the componentwise orthogonal decomposition

$$\Theta=\Theta_{\rm op}\,\widehat{\oplus}\,(\{0\}\times\operatorname{mul}(\Theta)), \tag{A.2}$$

where $\Theta_{\rm op}$ is a self-adjoint operator in $\operatorname{dom}(\Theta)$ and the second summand in the right-hand side is a purely multi-valued self-adjoint relation in $\operatorname{mul}(\Theta)$.

Boundary Pairs. Assume in addition that the closed densely defined symmetric operator S with defect numbers (n,n) is lower semibounded. In this case all self-adjoint extensions of S are lower semibounded. Recall that the form $\mathfrak{s}[f,g]=(f,Sg)$, $f,g\in\operatorname{dom}(S)$, is closable and that the Friedrichs extension S_F of S is the unique self-adjoint operator that is associated with the closure $\bar{\mathfrak{s}}\,(=\mathfrak{t}_{S_F})$ via the First Representation Theorem. Moreover, let S_1 be a self-adjoint extension of S which satisfies

$$\operatorname{dom}(S^*)\subseteq\operatorname{dom}(\mathfrak{t}_{S_1}), \tag{A.3}$$

where $\mathfrak{t}_{S_1}$ is the closed semibounded form associated with S_1 via the First Representation Theorem. The condition (A.3) is equivalent to

$$\operatorname{dom}(\mathfrak{t}_{S_1})=\ker(S^*-cI_{\mathfrak{H}})\dot{+}\operatorname{dom}(\mathfrak{t}_{S_F}),\quad \text{a direct sum}, \tag{A.4}$$

where c is below the lower bound of S_1, and due to finite defect, (A.3) is also equivalent to the simple condition

$$\operatorname{dom}(S)=\operatorname{dom}(S_F)\cap\operatorname{dom}(S_1),$$

see [3, Theorem 5.3.8]. The next lemma involves the notion of a boundary pair for S with finite defect numbers; see [3, Lemma 5.6.5] for the general case.

Lemma A.1. *Let $\{\mathbb{C}^n,\Gamma_0,\Gamma_1\}$ be an arbitrary boundary triplet for S^* and let S_1 be a self-adjoint extension of S which satisfies (A.3). Let $\Lambda:\operatorname{dom}(\mathfrak{t}_{S_1})\to\mathbb{C}^n$ be a linear mapping which is bounded when $\operatorname{dom}(\mathfrak{t}_{S_1})$ is provided with the inner product associated with $\mathfrak{t}_{S_1}-c$, where c is below the lower bound of S_1. If Λ extends Γ_0, then the self-adjoint extension S_0, $\operatorname{dom}(S_0)=\ker(\Gamma_0)$, coincides with the Friedrichs extension S_F and the following equalities hold:*

$$\ker(\Lambda)=\operatorname{dom}(\mathfrak{t}_{S_F})\ \textit{and}\ \operatorname{ran}(\Lambda)=\mathbb{C}^n.$$

Proof. Since Λ is an extension of Γ_0, one concludes that $\operatorname{ran}(\Lambda)=\operatorname{ran}(\Gamma_0)=\mathbb{C}^n$ and also $\operatorname{dom}(S_0)=\ker(\Gamma_0)\subseteq\ker(\Lambda)$. In particular, $\operatorname{dom}(S)\subseteq\ker(\Lambda)$ and hence by continuity of Λ and the definition of the Friedrichs extension S_F one concludes that $\operatorname{dom}(\mathfrak{t}_{S_F})\subseteq\ker(\Lambda)$. On the other hand, since the sum in (A.4) is direct and $\dim(\ker(S^*-cI_{\mathfrak{H}}))=n<\infty$ it follows that $\ker(\Lambda)=\operatorname{dom}(\mathfrak{t}_{S_F})$ and that Λ maps $\ker(S^*-cI_{\mathfrak{H}})$ bijectively onto $\mathbb{C}^n$. Combining this with the stated inclusion $\operatorname{dom}(S_0)\subseteq\ker(\Lambda)$ gives the inclusion $\operatorname{dom}(S_0)\subseteq\operatorname{dom}(\mathfrak{t}_{S_F})$. This implies that $S_0=S_F$ by [3, Theorem 5.3.3]. □

The pair $\{\mathbb{C}^n,\Lambda\}$, where $\Lambda:\operatorname{dom}(\mathfrak{t}_{S_1})\to\mathbb{C}^n$ is bounded in the form topology on $\mathfrak{t}_{S_1}$ is called a *boundary pair* for S if $\ker(\Lambda)=\operatorname{dom}(\mathfrak{t}_{S_F})$, see [3, Definition 5.6.1]. If, in addition, one has $\operatorname{dom}(S_1)=\ker(\Gamma_1)$, then the boundary triplet $\{\mathbb{C}^n,\Gamma_0,\Gamma_1\}$ and the boundary pair $\{\mathbb{C}^n,\Lambda\}$ are *compatible*, see [3, Definition 5.6.4], and the identity

$$(f,S^*g)_{\mathfrak{H}}=\mathfrak{t}_{S_1}[f,g]+(\Lambda f,\Gamma_1 g)_{\mathbb{C}^n},\quad f\in\operatorname{dom}(\mathfrak{t}_{S_1}),\ g\in\operatorname{dom}(S^*),$$

holds, see [3, Corollary 5.6.7]. Hence, Lemma A.1 offers general sufficient conditions needed to construct a compatible boundary pair $\{\mathbb{C}^n,\Lambda\}$ for S corresponding to S_1. Boundary pairs offer a general tool to describe forms generated by semibounded self-adjoint extensions of lower semibounded symmetric operators via boundary conditions.

Now let $\{\mathbb{C}^n, \Lambda\}$ be a compatible boundary pair corresponding to S_1. Then the closed semibounded form $\mathfrak{t}_\Theta$ associated with the self-adjoint extension A_Θ in (A.1) can be expressed in terms of the form $\mathfrak{t}_{S_1}$ and the boundary pair $\{\mathbb{C}^n, \Lambda\}$ as follows

$$\begin{aligned}&\mathfrak{t}_\Theta[f,g] = \mathfrak{t}_{S_1}[f,g] + (\Lambda f, \Theta_{\text{op}}\Lambda g)_{\mathbb{C}^n},\\ &f,g \in \text{dom}(\mathfrak{t}_\Theta) = \big\{h \in \text{dom}(\mathfrak{t}_{S_1}) \,\big|\, \Lambda h \in \text{dom}(\Theta_{\text{op}})\big\},\end{aligned} \tag{A.5}$$

see [3, Corollary 5.6.14]. Hence, if the self-adjoint relation Θ is the graph of a matrix, then (A.5) reads

$$\mathfrak{t}_\Theta[f,g] = \mathfrak{t}_{S_1}[f,g] + (\Lambda f, \Theta\Lambda g)_{\mathbb{C}^n}, \quad f,g \in \text{dom}(\mathfrak{t}_\Theta) = \text{dom}(\mathfrak{t}_{S_1}). \tag{A.6}$$

Moreover, if $\text{mul}(\Theta) = \mathbb{C}^n$, then

$$\mathfrak{t}_\Theta \subseteq \mathfrak{t}_{S_1}, \quad \text{dom}(\mathfrak{t}_\Theta) = \big\{h \in \text{dom}(\mathfrak{t}_{S_1}) \,\big|\, \Lambda h = 0\big\},$$

which corresponds to the Friedrichs extension. In particular, for the case $n = 1$ it is clear that $\Theta \in \mathbb{R} \cup \{\infty\}$. One notes that for $\Theta \in \mathbb{R}$ the decomposition reads

$$\mathfrak{t}_\Theta[f,g] = \mathfrak{t}_{S_1}[f,g] + (\Lambda f, \Theta\Lambda g)_{\mathbb{C}}, \quad f,g \in \text{dom}(\mathfrak{t}_\Theta) = \text{dom}(\mathfrak{t}_{S_1}),$$

while for $\Theta = \infty$ one has

$$\mathfrak{t}_\Theta \subseteq \mathfrak{t}_{S_1}, \quad \text{dom}(\mathfrak{t}_\Theta) = \big\{h \in \text{dom}(\mathfrak{t}_{S_1}) \,\big|\, \Lambda h = 0\big\}.$$

Self-adjoint Linear Relations in $\mathbb{C}^n$. The structure of the self-adjoint extensions in (A.1) is clarified next. It follows from [3, Theorem 1.10.5, Corollary 1.10.8, Proposition 1.10.3] that any self-adjoint relation Θ in $\mathbb{C}^n$ can be expressed as

$$\Theta = \big\{\{\mathbf{u},\mathbf{v}\} \in \mathbb{C}^n \times \mathbb{C}^n \,\big|\, \mathcal{B}\mathbf{u} = \mathcal{A}\mathbf{v}\big\}, \tag{A.7}$$

where the $n \times n$ matrices $\mathcal{A}$ and $\mathcal{B}$ satisfy

$$\mathcal{A}\mathcal{B}^* = \mathcal{B}\mathcal{A}^*, \quad \text{rank}(\mathcal{B}\ \ \mathcal{A}) = n, \tag{A.8}$$

and $(\mathcal{B}\ \ \mathcal{A})$ stands for the $n \times 2n$ matrix of the columns of $\mathcal{B}$ and $\mathcal{A}$. The multi-valued part of Θ is given by

$$\text{mul}(\Theta) = \big\{\mathbf{v} \in \mathbb{C}^n \,\big|\, \mathcal{A}\mathbf{v} = \mathbf{0}\big\} = \ker(\mathcal{A}),$$

so that it follows from (A.2) and (A.7) that

$$\mathcal{B}\mathbf{u} = \mathcal{A}\Theta_{\text{op}}\mathbf{u}, \quad \mathbf{u} \in \text{dom}(\Theta) = (\text{mul}(\Theta))^\perp = (\ker(\mathcal{A}))^\perp = \text{ran}(\mathcal{A}^*). \tag{A.9}$$

Therefore, $\Theta_{\rm op}$ can be expressed as

$$\Theta_{\rm op} = \mathcal{A}^{[-1]}\mathcal{B} \restriction \operatorname{ran}(\mathcal{A}^*), \tag{A.10}$$

where $\mathcal{A}^{[-1]}$ stands for the Moore–Penrose inverse of $\mathcal{A}$. Hence, if $\ker(\mathcal{A}) = \{0\}$, then one has $\operatorname{dom}(\Theta) = \mathbb{C}^n$ and $\Theta = \mathcal{A}^{-1}\mathcal{B}$ is an $n \times n$ self-adjoint matrix. Moreover, if $\ker(\mathcal{A}) = \mathbb{C}^n$, then $\operatorname{dom}(\Theta) = \{0\}$ and Θ is a purely multi-valued self-adjoint relation in $\mathbb{C}^n$ given by $\Theta = \{0\} \times \mathbb{C}^n$.

In the case $n = 2$ and $\dim(\ker(\mathcal{A})) = 1$ the selfadjoint operator $\Theta_{\rm op}$, acting in the invariant one-dimensional subspace $\operatorname{dom}(\Theta)$, is multiplication by the unique real number c_Θ given by

$$\mathcal{B}\mathbf{u} = c_\Theta \mathcal{A}\mathbf{u}, \quad \mathbf{u} \in \operatorname{dom}(\Theta) = (\ker(\mathcal{A}))^\perp, \quad \mathbf{u} \neq 0. \tag{A.11}$$

In the case $n = 1$ the self-adjoint relation Θ can be expressed as

$$\Theta = \big\{\{\mathbf{u}, \mathbf{v}\} \in \mathbb{C} \times \mathbb{C} \,\big|\, \cos(\gamma)\mathbf{u} + \sin(\gamma)\mathbf{v} = \mathbf{0}\big\}, \tag{A.12}$$

with $\gamma \in [0, \pi)$. If $\gamma = 0$, then one has $\operatorname{mul}(\Theta) = \mathbb{C}$ and $\Theta = \{0\} \times \mathbb{C}$, whereas if $\gamma \neq 0$, then one has $\operatorname{mul}(\Theta) = \{0\}$ and $\Theta = \Theta_{\rm op}$ is multiplication by $-\cot(\gamma)$.

Summarizing, for a pair of $n \times n$ matrices $\mathcal{A}$ and $\mathcal{B}$ satisfying (A.8) and Θ given by (A.7), the self-adjoint extension A_Θ of S in (A.1) is given by

$$A_\Theta g = S^* g, \quad g \in \operatorname{dom}(A_\Theta) = \big\{h \in \operatorname{dom}(S^*) \,\big|\, \mathcal{B}\Gamma_0 h = \mathcal{A}\Gamma_1 h\big\}. \tag{A.13}$$

In this case the formula (A.5) can be written as

$$\begin{aligned} &\mathfrak{t}_\Theta[f, g] = \mathfrak{t}_{S_1}[f, g] + \big(\Lambda f, \mathcal{A}^{[-1]}\mathcal{B}\Lambda g\big)_{\mathbb{C}^n}, \\ &f, g \in \operatorname{dom}(\mathfrak{t}_\Theta) = \big\{h \in \operatorname{dom}(\mathfrak{t}_{S_1}) \,\big|\, \Lambda h \in (\ker(\mathcal{A}))^\perp\big\}. \end{aligned} \tag{A.14}$$

The expression (A.14) can be further simplified in the situations described in (A.11) and (A.12).

Two Limit Circle Endpoints. Return to the situation of Proposition 2.12. Then choose the boundary triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, defined on $\operatorname{dom}(T_{max})$, by

$$\Gamma_0 g = \begin{pmatrix} \widetilde{g}(a) \\ \widetilde{g}(b) \end{pmatrix}, \quad \Gamma_1 g = \begin{pmatrix} \widetilde{g}'(a) \\ -\widetilde{g}'(b) \end{pmatrix}, \quad g \in \operatorname{dom}(T_{max}). \tag{A.15}$$

Furthermore, introduce the form $\mathfrak{t}[f, g] = \mathfrak{Q}_{c,d}(f, g)$, $f, g \in \operatorname{dom}(\mathfrak{t}) = \operatorname{dom}(\mathfrak{Q}_{c,d})$, as in (3.4) and (3.5); cf. [3, Equation (6.11.2)]. Then it is easy to see that

$$(f, T_{min} g)_{L^2((a,b); r\,dx)} = \mathfrak{t}[f, g], \quad f \in \operatorname{dom}(\mathfrak{t}), \; g \in \operatorname{dom}(T_{min}) \subseteq \operatorname{dom}(\mathfrak{t}), \tag{A.16}$$

see [3, Corollary 6.11.2], and

$$\operatorname{dom}(T_{max}) \subseteq \operatorname{dom}(\mathfrak{t}), \tag{A.17}$$

see [3, Lemma 6.11.3]. Define $\Lambda : \operatorname{dom}(\mathfrak{t}) \to \mathbb{C}^2$ by

$$\Lambda g = \begin{pmatrix} \widetilde{g}(a) \\ \widetilde{g}(b) \end{pmatrix}, \quad g \in \operatorname{dom}(\mathfrak{t}). \tag{A.18}$$

For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|\Lambda g\|^2_{\mathbb{C}^2} \leqslant \varepsilon \mathfrak{t}[g,g] + C_\varepsilon \|g\|^2_{L^2((a,b);r\,dx)}, \quad g \in \operatorname{dom}(\mathfrak{t}), \tag{A.19}$$

see [3, Lemma 6.11.4]. It now follows from (A.16)–(A.19) that $\{\mathbb{C}^2, \Lambda\}$ is a boundary pair which is compatible with the boundary triplet in (A.15), see [3, Lemma 6.11.5]. Thus we can apply (A.5), see [3, Theorem 6.11.6]. Note that $\mathfrak{Q}_{c,d} = \mathfrak{t}_{S_1}$, where $\operatorname{dom}(S_1) = \ker(\Gamma_1)$; cf. (A.15). The self-adjoint extensions of T_{min} are parametrized via (A.13), given (A.7) and (A.8). As before, our treatment will distinguish between separated and coupled boundary conditions.

First, consider the case of separated boundary conditions in Theorem 3.8, where $\mathcal{A}$ and $\mathcal{B}$ are 2×2 matrices of the form

$$\mathcal{A} = \begin{pmatrix} -\sin(\alpha) & 0 \\ 0 & \sin(\beta) \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \cos(\alpha) & 0 \\ 0 & \cos(\beta) \end{pmatrix}. \tag{A.20}$$

Note that (A.8) is satisfied. There are three subcases to be discussed. First, consider the case $\alpha \neq 0$ and $\beta \neq 0$. Then $\mathcal{A}$ is invertible and it follows from (A.20) that Θ is given by

$$\Theta = \mathcal{A}^{-1}\mathcal{B} = \begin{pmatrix} -\cot(\alpha) & 0 \\ 0 & \cot(\beta) \end{pmatrix}. \tag{A.21}$$

Substitution of (A.18) and (A.21) into (A.14) leads to (3.17) in Theorem 3.8. The second case is that either $\alpha = 0$ or $\beta = 0$ (without equality simultaneously). Assume $\alpha = 0$. Then $\ker(\mathcal{A})$ is one-dimensional and, in fact, it follows from $\operatorname{mul}(\Theta) = \ker(\mathcal{A})$ and $\operatorname{dom}(\Theta) = (\operatorname{mul}(\Theta))^\perp$ that

$$\operatorname{mul}(\Theta) = \operatorname{lin.span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \operatorname{dom}(\Theta) = \operatorname{lin.span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Therefore, one sees from

$$\mathcal{B}\mathbf{u} = \begin{pmatrix} 0 \\ \cos(\beta) \end{pmatrix}, \quad \mathcal{A}\mathbf{u} = \begin{pmatrix} 0 \\ \sin(\beta) \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

together with (A.11), that $c_\Theta = \cot(\beta)$. Hence the operator Θ_{op} acting in $\operatorname{dom}(\Theta) = \operatorname{lin.span}(\mathbf{u})$ is given by

$$\Theta_{\mathrm{op}} = \cot(\beta), \quad \operatorname{dom}(\mathfrak{t}_\Theta) = \big\{h \in \operatorname{dom}(\mathfrak{t}) \,\big|\, \widetilde{h}(a) = 0\big\}.$$

This together with (A.14) and (A.11) leads to (3.19). Likewise, when $\beta = 0$, then $c_\Theta = -\cot(\alpha)$ and hence

$$\Theta_{\mathrm{op}} = -\cot(\alpha), \quad \operatorname{dom}(\mathfrak{t}_\Theta) = \big\{h \in \operatorname{dom}(\mathfrak{t}) \,\big|\, \widetilde{h}(b) = 0\big\},$$

and this leads to (3.21). The third case concerns $\alpha = \beta = 0$. Then $\operatorname{mul}(\Theta) = \ker(\mathcal{A}) = \mathbb{C}^2$ and $\operatorname{dom}(\Theta) = \{0\}$. Thus Θ_{op} is trivial and

$$\mathfrak{t}_\Theta \subseteq \mathfrak{t}, \quad \operatorname{dom}(\mathfrak{t}_\Theta) = \big\{ h \in \operatorname{dom}(\mathfrak{t}) \,\big|\, \widetilde{h}(a) = 0 = \widetilde{h}(b) \big\},$$

see (3.23), which corresponds to the Friedrichs extension. This treats all cases of Theorem 3.8.

Secondly, consider the case of coupled boundary conditions in Theorem 3.9, where $\mathcal{A}$ and $\mathcal{B}$ are 2×2 matrices of the form

$$\mathcal{A} = -\begin{pmatrix} e^{i\varphi} R_{1,2} & 0 \\ e^{i\varphi} R_{2,2} & 1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} e^{i\varphi} R_{1,1} & -1 \\ e^{i\varphi} R_{2,1} & 0 \end{pmatrix}, \tag{A.22}$$

and hence (A.8) is satisfied. There are two subcases to be discussed.

The first subcase is when $R_{1,2} \neq 0$. Then $\mathcal{A}$ is invertible and it follows from (A.22) and $\det_{\mathbb{C}^2}(R) = 1$, that Θ is given by

$$\Theta = \mathcal{A}^{-1}\mathcal{B} = -\frac{1}{R_{1,2}} \begin{pmatrix} R_{1,1} & -e^{-i\varphi} \\ -e^{i\varphi} & R_{2,2} \end{pmatrix}. \tag{A.23}$$

It follows from (A.5) and the expression in (A.23) that

$$(\Lambda f, \Theta \Lambda g)_{\mathbb{C}^2} = -\frac{1}{R_{1,2}} \begin{pmatrix} \widetilde{f}(a) \\ \widetilde{f}(b) \end{pmatrix}^* \begin{pmatrix} R_{1,1} & -e^{-i\varphi} \\ -e^{i\varphi} & R_{2,2} \end{pmatrix} \begin{pmatrix} \widetilde{g}(a) \\ \widetilde{g}(b) \end{pmatrix}, \quad f, g \in \operatorname{dom}(\mathfrak{t}).$$

Together with (A.14) this implies Theorem 3.9 (i).

The second subcase occurs when $R_{1,2} = 0$, which implies that $1 = \det_{\mathbb{C}^2}(R) = R_{1,1}R_{2,2}$. Then one has that $\ker(\mathcal{A})$ is one-dimensional and, in fact, it follows from $\operatorname{mul}(\Theta) = \ker(\mathcal{A})$ and $\operatorname{dom}(\Theta) = (\operatorname{mul}(\Theta))^\perp$ that

$$\operatorname{mul}(\Theta) = \operatorname{lin.span}\left(\begin{pmatrix} 1 \\ -e^{i\varphi} R_{2,2} \end{pmatrix} \right), \quad \operatorname{dom}(\Theta) = \operatorname{lin.span}\left(\begin{pmatrix} e^{-i\varphi} R_{2,2} \\ 1 \end{pmatrix} \right).$$

Therefore, one sees from

$$\mathcal{B}\mathbf{u} = \begin{pmatrix} R_{1,1}R_{2,2} - 1 \\ R_{2,1}R_{2,2} \end{pmatrix}, \quad \mathcal{A}\mathbf{u} = \begin{pmatrix} 0 \\ -R_{2,2}^2 - 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} e^{-i\varphi} R_{2,2} \\ 1 \end{pmatrix},$$

together with (A.11), that

$$c_\Theta = -\frac{R_{2,1}}{R_{1,1} + R_{2,2}}. \tag{A.24}$$

Thus by (A.5) and the expression in (A.24) it is clear that

$$\begin{aligned} &(\Lambda f, \Theta_{\rm op} \Lambda g)_{\mathbb{C}^2} = -R_{1,1}R_{2,1}\overline{\widetilde{f}(a)}\widetilde{g}(a), \\ &f, g \in \operatorname{dom}(\mathfrak{t}_\Theta) = \big\{ h \in \operatorname{dom}(\mathfrak{t}) \,\big|\, \widetilde{h}(b) = e^{i\varphi} R_{1,1}\widetilde{h}(a) \big\}. \end{aligned}$$

Together with (A.14) this implies Theorem 3.9 (ii).

One Limit Circle Endpoint. Return to the situation of Proposition 2.13. Choose the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, defined on $\operatorname{dom}(T_{max})$, by

$$\Gamma_0 g = \widetilde{g}(a), \quad \Gamma_1 g = \widetilde{g}'(a), \quad g \in \operatorname{dom}(T_{max}). \tag{A.25}$$

Furthermore, introduce the form $\mathfrak{t}[f, g] = \mathfrak{Q}_{c,d}(f, g)$, $f, g \in \operatorname{dom}(\mathfrak{t}) = \operatorname{dom}(\mathfrak{Q}_{c,d})$, as in (4.2) and (4.3); see [3, Equation (6.12.2)]. Then it is easy to see that

$$(f, T_{min} g)_{\mathfrak{H}} = \mathfrak{t}[f, g], \quad f \in \operatorname{dom}(\mathfrak{t}),\ g \in \operatorname{dom}(T_{min}) \subseteq \operatorname{dom}(\mathfrak{t}), \tag{A.26}$$

see [3, Corollary 6.12.2], and

$$\operatorname{dom}(T_{max}) \subseteq \operatorname{dom}(\mathfrak{t}), \tag{A.27}$$

see [3, Lemma 6.12.3]. Define $\Lambda : \operatorname{dom}(\mathfrak{t}) \to \mathbb{C}$ by

$$\Lambda g = \widetilde{g}(a), \quad g \in \operatorname{dom}(\mathfrak{t}). \tag{A.28}$$

For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|\Lambda g\|_{\mathbb{C}}^2 \leqslant \varepsilon \mathfrak{t}[g, g] + C_\varepsilon \|g\|^2_{L^2((a,b); r\,dx)}, \quad g \in \operatorname{dom}(\mathfrak{t}), \tag{A.29}$$

see [3, Lemma 6.12.4]. It now follows from (A.26)–(A.29) that $\{\mathbb{C}, \Lambda\}$ is a boundary pair which is compatible with the boundary triplet in (A.25), see [3, Lemma 6.12.5]. Thus we can apply (A.5), see [3, Theorem 6.12.6]. Note that $\mathfrak{Q}_{c,d} = \mathfrak{t}_{S_1}$, where $\operatorname{dom}(S_1) = \ker(\Gamma_1)$; see (A.25).

The self-adjoint extensions of T_{min} are now parametrized via (A.12)

$$\cos(\alpha)\Gamma_0 g + \sin(\alpha)\Gamma_1 g = 0, \quad g \in \operatorname{dom}(T_{max}),$$

over $\alpha \in [0, \pi)$, and denoted by T_α, see Proposition 2.13. Therefore, one has for $\alpha \in (0, \pi)$, that

$$\mathfrak{t}_\alpha[f, g] = \mathfrak{t}[f, g] - \cot(\alpha)(\Lambda f, \Lambda g)_{\mathbb{C}}, \quad f, g \in \operatorname{dom}(\mathfrak{t}_\alpha) = \operatorname{dom}(\mathfrak{t}).$$

Moreover, if $\alpha = 0$, then

$$\mathfrak{t}_\alpha \subseteq \mathfrak{t}, \quad \operatorname{dom}(\mathfrak{t}_\alpha) = \big\{h \in \operatorname{dom}(\mathfrak{t}) \,\big|\, \widetilde{h}(a) = 0\big\},$$

which corresponds to the Friedrichs extension. This implies Theorem 4.5, cf. [3, Theorem 6.12.6].

For a succinct treatment of boundary triplets and Weyl–Titchmarsh functions tailored towards ordinary differential operators (a.k.a., “boundary triplets in a nutshell”), see also [11, App. D.7]. Likewise, a treatment of boundary pairs, going back to [2], can be found in [3, Ch. 5].

Data availability

No data was used for the research described in the article.

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