The Krein-von Neumann Realization of Perturbed Laplacians on Bounded Lipschitz Domains

Jussi Behrndt, Fritz Gesztesy, Till Micheler and Marius Mitrea

Abstract. In this paper we study the self-adjoint Krein–von Neumann realization A_K of the perturbed Laplacian $-\Delta + V$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. We provide an explicit and self-contained description of the domain of A_K in terms of Dirichlet and Neumann boundary traces, and we establish a Weyl asymptotic formula for the eigenvalues of A_K.

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1. Introduction

The main objective of this note is to investigate the self-adjoint Krein–von Neumann realization associated to the differential expression $-\Delta + V$ in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$, $n > 1$, is assumed to be a bounded Lipschitz domain and V is a nonnegative bounded potential. In particular, we obtain an explicit description of the domain of A_K in terms of Dirichlet and Neumann boundary traces, and we prove the Weyl asymptotic formula

$$N(\lambda, A_K) \underset{\lambda \to \infty}{=} (2\pi)^{-n} v_n |\Omega| \, \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big). \qquad (1.1)$$

Here $N(\lambda, A_K)$ denotes the number of nonzero eigenvalues of A_K not exceeding λ, v_n is the volume of the unit ball in $\mathbb{R}^n$, and $|\Omega|$ is the (n-dimensional) Lebesgue measure of Ω.

Let us first recall the definition and some properties of the Krein–von Neumann extension in the abstract setting. Let S be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ and assume that S is strictly positive, that is, for some $c > 0$, $(Sf, f)_{\mathcal{H}} \geqslant c\|f\|_{\mathcal{H}}^2$ for all $f \in \mathrm{dom}(S)$. The Krein–von Neumann extension S_K of S is then given by

$$S_K f = S^* f, \quad f \in \mathrm{dom}(S_K) = \mathrm{dom}(S) \dot{+} \ker(S^*), \qquad (1.2)$$

see the original papers Krein [48] and von Neumann [62]. It follows that S_K is a nonnegative self-adjoint extension of S and that for all other nonnegative self-adjoint extensions S_Θ of S the operator inequality $S_K \leqslant S_\Theta$ holds in the sense of quadratic forms. As $\ker(S_K) = \ker(S^*)$, it is clear that 0 is an eigenvalue of S_K (except if S is self-adjoint, in which case $S_K = S^* = S$). Furthermore, if the self-adjoint Friedrichs extension S_F of S has purely discrete spectrum then the same is true for the spectrum of S_K with the possible exception of the eigenvalue 0, which may have infinite multiplicity. For further developments, extensive references, and a more detailed discussion of the properties of the Krein–von Neumann extension of a symmetric operator we refer the reader to [2, Sect. 109], [3], [4]–[6], [7, Chs. 9, 10], [8]–[13], [14], [15], [16], [17], [19], [27], [28], [29], [32, Sect. 15], [33, Sect. 3.3], [36], [38], [39, Sect. 13.2], [40], [41], [50], [55], [57], [58, Ch. 13], [59], [60], [61], [64], [65], [66], and the references cited therein.

In the concrete case considered in this paper, the symmetric operator S above is given by the minimal operator A_{min} associated to the differential expression $-\Delta + V$ in the Hilbert space $L^2(\Omega)$, that is,

$$A_{min} = -\Delta + V, \quad \mathrm{dom}(A_{min}) = \overset{\circ}{H}{}^2(\Omega), \qquad (1.3)$$

where $\overset{\circ}{H}{}^2(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^2(\Omega)$, and $0 \leqslant V \in L^\infty(\Omega)$. It can be shown that A_{min} is the closure of the symmetric operator $-\Delta + V$ defined on $C_0^\infty(\Omega)$. We point out that here Ω is a bounded Lipschitz domain and no further regularity assumptions on $\partial\Omega$ are imposed, thus it is remarkable that the functions in $\mathrm{dom}(A_{min})$ possess H^2-regularity. The adjoint A_{min}^* of A_{min} coincides with the maximal operator

$$A_{max} = -\Delta + V,$$
$$\mathrm{dom}(A_{max}) = \{f \in L^2(\Omega) \,\big|\, -\Delta f + Vf \in L^2(\Omega)\}, \qquad (1.4)$$

where Δf is understood in the sense of distributions. From (1.2) and (1.3) it is clear that the Krein–von Neumann extension A_K of A_{min} is then given by

$$A_K = -\Delta + V, \quad \mathrm{dom}(A_K) = \overset{\circ}{H}{}^2(\Omega) \dot{+} \ker(A_{max}). \qquad (1.5)$$

In the present situation A_{min} is a symmetric operator with infinite defect indices and therefore $\ker(A_{min}^*) = \ker(A_{max})$ is infinite-dimensional. In particular, 0 is an eigenvalue of A_K with infinite multiplicity, and hence belongs to the essential spectrum. It is also important to note that in general the functions in $\ker(A_K)$ do not possess any Sobolev regularity, that is, $\ker(A_K) \not\subset H^s(\Omega)$ for every $s > 0$. Moreover, since Ω is a bounded set, the Friedrichs extension of A_{min} (which coincides with the self-adjoint Dirichlet

operator associated to $-\Delta + V$) has compact resolvent and hence its spectrum is discrete. The abstract considerations above then yield that with the exception of the eigenvalue 0 the spectrum of A_K consists of a sequence of positive eigenvalues with finite multiplicity which tend to $+\infty$.

The description of the domain of the Krein–von Neumann extension A_K in (1.5) is not satisfactory for applications involving boundary value problems. Instead, a more explicit description of $\mathrm{dom}(A_K)$ via boundary conditions seems to be natural and desirable. In the case of a bounded C^∞-smooth domain Ω, it is known that

$$\mathrm{dom}(A_K) = \big\{ f \in \mathrm{dom}(A_{max}) \,\big|\, \gamma_N f + M(0)\gamma_D f = 0 \big\} \qquad (1.6)$$

holds, where γ_D and γ_N denote the Dirichlet and Neumann trace operator, respectively, defined on the maximal domain $\mathrm{dom}(A_{max})$, and $M(0)$ is the Dirichlet-to-Neumann map or Weyl–Titchmarsh operator for $-\Delta + V$. The description (1.6) goes back to Višik [67] and Grubb [37], where certain classes of elliptic differential operators with smooth coefficients are discussed in great detail. Note that in contrast to the Dirichlet and Neumann boundary conditions the boundary condition in (1.6) is nonlocal, as it involves $M(0)$ which, when Ω is smooth, is a boundary pseudodifferential operator of order 1. It is essential for the boundary condition (1.6) that both trace operators γ_D and γ_N are defined on $\mathrm{dom}(A_{max})$. Even in the case of a smooth boundary $\partial\Omega$, the elements in $\mathrm{dom}(A_K)$, in general, do not possess any H^s-regularity for $s > 0$, and hence special attention has to be paid to the definition and the properties of the trace operators. In the smooth setting the classical analysis due to Lions and Magenes [49] ensures that $\gamma_D : \mathrm{dom}(A_{max}) \to H^{-1/2}(\partial\Omega)$ and $\gamma_N : \mathrm{dom}(A_{max}) \to H^{-3/2}(\partial\Omega)$ are well-defined continuous mappings when $\mathrm{dom}(A_{max})$ is equipped with the graph norm.

Let us now turn again to the present situation, where Ω is assumed to be a bounded Lipschitz domain. Our first main objective is to extend the description of $\mathrm{dom}(A_K)$ in (1.6) to the nonsmooth setting. The main difficulty here is to define appropriate trace operators on the domain of the maximal operator. We briefly sketch the strategy from [18], which is mainly based and inspired by abstract extension theory of symmetric operators. For this denote by A_D and A_N the self-adjoint realizations of $-\Delta + V$ corresponding to Dirichlet and Neumann boundary conditions, respectively. Recall that by [43] and [31] their domains $\mathrm{dom}(A_D)$ and $\mathrm{dom}(A_N)$ are both contained in $H^{3/2}(\Omega)$. Now consider the boundary spaces

$$\begin{aligned} \mathscr{G}_D(\partial\Omega) &:= \big\{ \gamma_D f \,\big|\, f \in \mathrm{dom}(A_N) \big\}, \\ \mathscr{G}_N(\partial\Omega) &:= \big\{ \gamma_N f \,\big|\, f \in \mathrm{dom}(A_D) \big\}, \end{aligned} \qquad (1.7)$$

equipped with suitable inner products induced by the Neumann-to-Dirichlet map and Dirichlet-to-Neumann map for $-\Delta + V - i$, see Section 3 for the details. It turns out that $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$ are both Hilbert spaces which are densely embedded in $L^2(\partial\Omega)$. It was shown in [18] that the Dirichlet trace operator γ_D and Neumann trace operator γ_N can be extended by continuity

to surjective mappings

$$\widetilde{\gamma}_D : \mathrm{dom}(A_{max}) \to \mathscr{G}_N(\partial\Omega)^* \quad \text{and} \quad \widetilde{\gamma}_N : \mathrm{dom}(A_{max}) \to \mathscr{G}_D(\partial\Omega)^*, \quad (1.8)$$

where $\mathscr{G}_D(\partial\Omega)^*$ and $\mathscr{G}_N(\partial\Omega)^*$ denote the adjoint (i.e., conjugate dual) spaces of $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$, respectively. Within the same process also the Dirichlet-to-Neumann map $M(0)$ of $-\Delta + V$ (originally defined as a mapping from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$) admits an extension to a mapping $\widetilde{M}(0)$ from $\mathscr{G}_N(\partial\Omega)^*$ to $\mathscr{G}_D(\partial\Omega)^*$. With the help of the trace maps $\widetilde{\gamma}_D$ and $\widetilde{\gamma}_N$, and the extended Dirichlet-to-Neumann operator $\widetilde{M}(0)$ we are then able to extend the description of the domain of the Krein–von Neumann extension for smooth domains in (1.6) to the case of Lipschitz domains. More precisely, we show in Theorem 3.3 that the Krein–von Neumann extension A_K of A_{min} is defined on

$$\mathrm{dom}(A_K) = \{f \in \mathrm{dom}(A_{max}) \mid \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0\}. \quad (1.9)$$

For an exhaustive treatment of boundary trace operators on bounded Lipschitz domains in $\mathbb{R}^n$ and applications to Schrödinger operators we refer to [17].

Our second main objective in this paper is to prove the Weyl asymptotic formula (1.1) for the nonzero eigenvalues of A_K. We mention that the study of the asymptotic behavior of the spectral distribution function of the Dirichlet Laplacian originates in the work by Weyl (cf. [68], [69], and the references in [70]), and that generalizations of the classical Weyl asymptotic formula were obtained in numerous papers - we refer the reader to [20], [21], [22], [23], [24], [25], [26], [56], [63], and the introduction in [16] for more details. There are relatively few papers available that treat the spectral asymptotics of the Krein Laplacian or the perturbed Krein Laplacian A_K. Essentially these considerations are inspired by Alonso and Simon who, at the end of their paper [3] posed the question if the asymptotics of the nonzero eigenvalues of the Krein Laplacian is given by Weyl's formula? In the case where Ω is bounded and C^∞-smooth, and $V \in C^\infty(\overline{\Omega})$, this has been shown to be the case three years later by Grubb [38], see also the more recent contributions [52], [53], and [40]. Following the ideas in [38] it was shown in [14] that for so-called quasi-convex domains (a nonsmooth subclass of bounded Lipschitz domains with the key feature that $\mathrm{dom}(A_D)$ and $\mathrm{dom}(A_N)$ are both contained in $H^2(\Omega)$) the Krein–von Neumann extension A_K is spectrally equivalent to the buckling of a clamped plate problem, which in turn can be reformulated with the help of the quadratic forms

$$\mathfrak{a}[f,g] := (A_{min}f, A_{min}g)_{L^2(\Omega)} \text{ and } \mathfrak{t}[f,g] := (f, A_{min}g)_{L^2(\Omega)}, \quad (1.10)$$

defined on $\mathrm{dom}(A_{min}) = \overset{\circ}{H}{}^2(\Omega)$. In the Hilbert space $(\overset{\circ}{H}{}^2(\Omega), \mathfrak{a}[\cdot, \cdot])$ the form $\mathfrak{t}$ can be expressed with the help of a nonnegative compact operator T, and it follows that

$$\lambda \in \sigma_p(A_K)\backslash\{0\} \text{ if and only if } \lambda^{-1} \in \sigma_p(T), \quad (1.11)$$

counting multiplicities. These considerations can be extended from quasi-convex domains to the more general setting of Lipschitz domains, see, for

instance, Section 4 and Lemma 4.2. Finally, the main ingredient in the proof of the Weyl asymptotic formula (1.1) for the Krein–von Neumann extension A_K of $-\Delta + V$ on a bounded Lipschitz domain Ω is then a more general Weyl type asymptotic formula due to Kozlov [46] (see also [45], [47]) which yields the asymptotics of the spectral distribution function of the compact operator T, and hence via (1.11) the asymptotics of the spectral distribution function of A_K. This reasoning in the proof of our second main result Theorem 4.1 is along the lines of [14, 15], where the special case of quasi-convex domains was treated. For perturbed Krein Laplacians this result completes work that started with Grubb more than 30 years ago and demonstrates that the question posed by Alonso and Simon in [3] regarding the validity of the Weyl asymptotic formula continues to have an affirmative answer for bounded Lipschitz domains – the natural end of the line in the development from smooth domains all the way to minimally smooth ones.

2. Schrödinger Operators on Bounded Lipschitz Domains

This section is devoted to a study of self-adjoint Schrödinger operators on a nonempty, bounded Lipschitz domain in $\mathbb{R}^n$ (automatically assumed to be open). We shall make the following general assumption.

Hypothesis 2.1. *Let $n \in \mathbb{N}\backslash\{1\}$, assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, and suppose that $0 \leqslant V \in L^\infty(\Omega)$.*

We consider operator realizations of the differential expression $-\Delta + V$ in the Hilbert space $L^2(\Omega)$. For this we define the *preminimal* realization A_p of $-\Delta + V$ by

$$A_p := -\Delta + V, \quad \mathrm{dom}(A_p) := C_0^\infty(\Omega). \tag{2.1}$$

It is clear that A_p is a densely defined, symmetric operator in $L^2(\Omega)$, and hence closable. The *minimal* realization A_{min} of $-\Delta + V$ is defined as the closure of A_p in $L^2(\Omega)$,

$$A_{min} := \overline{A_p}. \tag{2.2}$$

It follows that A_{min} is a densely defined, closed, symmetric operator in $L^2(\Omega)$. The *maximal* realization A_{max} of $-\Delta + V$ is given by

$$A_{max} := -\Delta+V, \quad \mathrm{dom}(A_{max}) := \big\{f \in L^2(\Omega)\,\big|-\Delta f+Vf \in L^2(\Omega)\big\}, \tag{2.3}$$

where the expression Δf, $f \in L^2(\Omega)$, is understood in the sense of distributions.

In the next lemma we collect some properties of the operators A_p, A_{min}, and A_{max}. The standard L^2-based Sobolev spaces of order $s \geqslant 0$ will be denoted by $H^s(\Omega)$; for the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ we write $\mathring{H}^s(\Omega)$.

Lemma 2.2. *Assume Hypothesis* 2.1 *and let A_p, A_{min}, and A_{max} be as introduced above. Then the following assertions hold:*

(i) A_{min} and A_{max} are adjoints of each other, that is,

$$A_{min}^* = A_p^* = A_{max} \ \ and \ \ A_{min} = \overline{A_p} = A_{max}^*. \tag{2.4}$$

(ii) A_{min} is defined on $\mathring{H}^2(\Omega)$, that is,

$$\operatorname{dom}(A_{min}) = \mathring{H}^2(\Omega), \tag{2.5}$$

and the graph norm of A_{min} and the H^2-norm are equivalent on the domain of A_{min}.

(iii) A_{min} is strictly positive, that is, for some $C > 0$ we have

$$(A_{min}f, f)_{L^2(\Omega)} \geqslant C\|f\|_{L^2(\Omega)}^2, \quad f \in \mathring{H}^2(\Omega). \tag{2.6}$$

(iv) A_{min} has infinite deficiency indices.

One recalls that the *Friedrichs extension* A_F of A_{min} is defined by

$$A_F := -\Delta + V, \quad \operatorname{dom}(A_F) := \{f \in \mathring{H}^1(\Omega) \,|\, \Delta f \in L^2(\Omega)\}. \tag{2.7}$$

It is well-known that A_F is a strictly positive self-adjoint operator in $L^2(\Omega)$ with compact resolvent (see, e.g. [30, Sect. VI.1]).

In this note we are particularly interested in the *Krein–von Neumann extension* A_K of A_{min}. According to (1.2), A_K is given by

$$A_K := -\Delta + V, \quad \operatorname{dom}(A_K) := \operatorname{dom}(A_{min}) \dotplus \ker(A_{max}). \tag{2.8}$$

In the following theorem we briefly collect some well-known properties of the Krein–von Neumann extension A_K in the present setting. For more details we refer the reader to the celebrated paper [48] by Krein and to [3], [4], [11], [14], [15], [16], [40], and [41] for further developments and references.

Theorem 2.3. *Assume Hypothesis 2.1 and let A_K be the Krein–von Neumann extension of A_{min}. Then the following assertions hold:*

(i) *A_K is a nonnegative self-adjoint operator in $L^2(\Omega)$ and $\sigma(A_K)$ consists of eigenvalues only. The eigenvalue 0 has infinite multiplicity,*

$$\dim(\ker(A_K)) = \infty,$$

and the restriction $A_K|_{(\ker(A_K))^\perp}$ is a strictly positive self-adjoint operator in the Hilbert space $(\ker(A_K))^\perp$ with compact resolvent.

(ii) *$\operatorname{dom}(A_K) \not\subset H^s(\Omega)$ for every $s > 0$.*

(iii) *A nonnegative self-adjoint operator B in $L^2(\Omega)$ is a self-adjoint extension of A_{min} if and only if for some (and, hence for all) $\mu < 0$,*

$$(A_F - \mu)^{-1} \leqslant (B - \mu)^{-1} \leqslant (A_K - \mu)^{-1}. \tag{2.9}$$

We also mention that the Friedrich extension A_F and the Krein–von Neumann extension A_K are relatively prime (or disjoint), that is,

$$\operatorname{dom}(A_F) \cap \operatorname{dom}(A_K) = \operatorname{dom}(A_{min}) = \mathring{H}^2(\Omega). \tag{2.10}$$

For later purposes we briefly recall some properties of the Dirichlet and Neumann trace operator and the corresponding self-adjoint Dirichlet and Neumann realizations of $-\Delta + V$ in $L^2(\Omega)$. We consider the space

$$H_\Delta^{3/2}(\Omega) := \{f \in H^{3/2}(\Omega) \,|\, \Delta f \in L^2(\Omega)\}, \tag{2.11}$$

equipped with the inner product

$$(f,g)_{H^{3/2}_\Delta(\Omega)} = (f,g)_{H^{3/2}(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)}, \quad f,g \in H^{3/2}_\Delta(\Omega). \quad (2.12)$$

One recalls that the Dirichlet and Neumann trace operators γ_D and γ_N defined by

$$\gamma_D f := f \restriction_{\partial\Omega} \text{ and } \gamma_N f := \mathfrak{n} \cdot \nabla f \restriction_{\partial\Omega}, \quad f \in C^\infty(\overline{\Omega}), \quad (2.13)$$

admit continuous extensions to operators

$$\gamma_D : H^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega) \text{ and } \gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega). \quad (2.14)$$

Here $H^1(\partial\Omega)$ denotes the usual L^2-based Sobolev space of order 1 on $\partial\Omega$; cf. [51, Chapter 3] and [54]. It is important to note that the extensions in (2.14) are both surjective, see [36, Lemma 3.1 and Lemma 3.2].

In the next theorem we collect some properties of the Dirichlet realization A_D and Neumann realization A_N of $-\Delta + V$ in $L^2(\Omega)$. We recall that the operators A_D and A_N are defined as the unique self-adjoint operators corresponding to the closed nonnegative forms

$$\begin{aligned}
\mathfrak{a}_D[f,g] &:= (\nabla f, \nabla g)_{(L^2(\Omega))^n} + (Vf,g)_{L^2(\Omega)}, \quad \mathrm{dom}(\mathfrak{a}_D) := \mathring{H}^1(\Omega), \\
\mathfrak{a}_N[f,g] &:= (\nabla f, \nabla g)_{(L^2(\Omega))^n} + (Vf,g)_{L^2(\Omega)}, \quad \mathrm{dom}(\mathfrak{a}_N) := H^1(\Omega).
\end{aligned} \quad (2.15)$$

In particular, one has $A_F = A_D$ by (2.7). In the next theorem we collect some well-known facts about the self-adjoint operators A_D and A_N. The $H^{3/2}$-regularity of the functions in their domains is remarkable, and a consequence of Ω being a bounded Lipschitz domain. We refer the reader to [35, Lemma 3.4 and Lemma 4.8] for more details, see also [42, 43] and [31].

Theorem 2.4. *Assume Hypothesis 2.1 and let A_D and A_N be the self-adjoint Dirichlet and Neumann realization of $-\Delta + V$ in $L^2(\Omega)$, respectively. Then the following assertions hold:*

(i) *The operator A_D coincides with the Friedrichs extension A_F and is given by*

$$A_D = -\Delta + V, \quad \mathrm{dom}(A_D) = \{f \in H^{3/2}_\Delta(\Omega) \,|\, \gamma_D f = 0\}. \quad (2.16)$$

The resolvent of A_D is compact, and the spectrum of A_D is purely discrete and contained in $(0,\infty)$.

(ii) *The operator A_N is given by*

$$A_N = -\Delta + V, \quad \mathrm{dom}(A_N) = \{f \in H^{3/2}_\Delta(\Omega) \,|\, \gamma_N f = 0\}. \quad (2.17)$$

The resolvent of A_N is compact, and the spectrum of A_N is purely discrete and contained in $[0,\infty)$.

3. Boundary conditions for the Krein–von Neumann realization

Our goal in this section is to obtain an explicit description of the domain of the Krein–von Neumann extension A_K in terms of Dirichlet and Neumann boundary traces. For this we describe an extension procedure of the trace maps γ_D and γ_N in (2.14) onto $\mathrm{dom}(A_{max})$ from [18]. We recall that for $\varphi \in H^1(\partial\Omega)$ and $z \in \rho(A_D)$, the boundary value problem

$$-\Delta f + V f = z f, \quad \gamma_D f = \varphi, \tag{3.1}$$

admits a unique solution $f_z(\varphi) \in H^{3/2}_\Delta(\Omega)$. Making use of this fact and the trace operators (2.14) we define the Dirichlet-to-Neumann operator $M(z)$, $z \in \rho(A_D)$, as follows:

$$M(z) : L^2(\partial\Omega) \supset H^1(\partial\Omega) \to L^2(\partial\Omega), \quad \varphi \mapsto -\gamma_N f_z(\varphi), \tag{3.2}$$

where $f_z(\varphi) \in H^{3/2}_\Delta(\Omega)$ is the unique solution of (3.1). It can be shown that $M(z)$ is an unbounded operator in $L^2(\partial\Omega)$. Moreover, if $z \in \rho(A_D) \cap \rho(A_N)$ then $M(z)$ is invertible and the inverse $M(z)^{-1}$ is a bounded operator defined on $L^2(\partial\Omega)$. Considering $z = i$, we set

$$\Sigma := \mathrm{Im}\left(-M(i)^{-1}\right). \tag{3.3}$$

The imaginary part $\mathrm{Im}\, M(i)$ of $M(i)$ is a densely defined bounded operator in $L^2(\partial\Omega)$ and hence it admits a bounded closure

$$\Lambda := \overline{\mathrm{Im}(M(i))} \tag{3.4}$$

in $L^2(\partial\Omega)$. Both operators Σ and Λ are self-adjoint and invertible with unbounded inverses. Next we introduce the boundary spaces

$$\mathscr{G}_D(\partial\Omega) := \left\{\gamma_D f \,\middle|\, f \in \mathrm{dom}(A_N)\right\} \tag{3.5}$$

and

$$\mathscr{G}_N(\partial\Omega) := \left\{\gamma_N f \,\middle|\, f \in \mathrm{dom}(A_D)\right\}. \tag{3.6}$$

It turns out that

$$\mathscr{G}_D(\partial\Omega) = \mathrm{ran}\left(\Sigma^{1/2}\right) \quad \text{and} \quad \mathscr{G}_N(\partial\Omega) = \mathrm{ran}\left(\Lambda^{1/2}\right), \tag{3.7}$$

and hence the spaces $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$ can be equipped with the inner products

$$(\varphi, \psi)_{\mathscr{G}_D(\partial\Omega)} := \left(\Sigma^{-1/2}\varphi, \Sigma^{-1/2}\psi\right)_{L^2(\partial\Omega)}, \quad \varphi, \psi \in \mathscr{G}_D(\partial\Omega), \tag{3.8}$$

and

$$(\varphi, \psi)_{\mathscr{G}_N(\partial\Omega)} := \left(\Lambda^{-1/2}\varphi, \Lambda^{-1/2}\psi\right)_{L^2(\partial\Omega)}, \quad \varphi, \psi \in \mathscr{G}_N(\partial\Omega), \tag{3.9}$$

respectively. Then $\mathscr{G}_D(\partial\Omega)$ and $\mathscr{G}_N(\partial\Omega)$ both become Hilbert spaces which are dense in $L^2(\partial\Omega)$. The corresponding adjoint (i.e., conjugate dual) spaces will be denoted by $\mathscr{G}_D(\partial\Omega)^*$ and $\mathscr{G}_N(\partial\Omega)^*$, respectively. The following result can be found in [18, Section 4.1].

Theorem 3.1. *Assume Hypothesis* 2.1. *Then the Dirichlet trace operator γ_D and Neumann trace operator γ_N in* (2.14) *can be extended by continuity to surjective mappings*

$$\widetilde{\gamma}_D : \mathrm{dom}(A_{max}) \to \mathscr{G}_N(\partial\Omega)^* \ \text{ and } \ \widetilde{\gamma}_N : \mathrm{dom}(A_{max}) \to \mathscr{G}_D(\partial\Omega)^* \quad (3.10)$$

such that $\ker(\widetilde{\gamma}_D) = \ker(\gamma_D) = \mathrm{dom}(A_D)$ *and* $\ker(\widetilde{\gamma}_N) = \ker(\gamma_N) = \mathrm{dom}(A_N)$.

In a similar manner the boundary value problem (3.1) can be considered for all $\varphi \in \mathscr{G}_N(\partial\Omega)^*$ and the Dirichlet-to-Neumann operator $M(\cdot)$ in (3.2) can be extended. More precisely, the following statement holds.

Theorem 3.2. *Assume Hypothesis* 2.1 *and let $\widetilde{\gamma}_D$ and $\widetilde{\gamma}_N$ be the extended Dirichlet and Neumann trace operator from Theorem* 3.1. *Then the following are true:*

(i) *For $\varphi \in \mathscr{G}_N(\partial\Omega)^*$ and $z \in \rho(A_D)$ the boundary value problem*

$$-\Delta f + V f = z f, \quad \widetilde{\gamma}_D f = \varphi, \qquad (3.11)$$

admits a unique solution $f_z(\varphi) \in \mathrm{dom}(A_{max})$.

(ii) *For $z \in \rho(A_D)$ the Dirichlet-to-Neumann operator $M(z)$ in* (3.2) *admits a continuous extension*

$$\widetilde{M}(z) : \mathscr{G}_N(\partial\Omega)^* \to \mathscr{G}_D(\partial\Omega)^*, \quad \varphi \mapsto -\widetilde{\gamma}_N f_z(\varphi), \qquad (3.12)$$

where $f_z(\varphi) \in \mathrm{dom}(A_{max})$ is the unique solution of (3.11).

Now we are able to state our main result in this section: A description of the domain of the Krein–von Neumann extension A_K in terms of Dirichlet and Neumann boundary traces. The extended Dirichlet-to-Neumann map at $z = 0$ will enter as a regularization parameter in the boundary condition. For C^∞-smooth domains this result goes back to Grubb [37], where a certain class of elliptic differential operators with smooth coefficients is discussed systematically. For the special case of a so-called quasi-convex domains Theorem 3.3 reduces to [15, Theorem 5.5] and [36, Theorem 13.1]. In an abstract setting the Krein–von Neumann extension appears in a similar form in [18, Example 3.9].

Theorem 3.3. *Assume Hypothesis* 2.1 *and let $\widetilde{\gamma}_D$, $\widetilde{\gamma}_N$ and $\widetilde{M}(0)$ be as in Theorem* 3.1 *and Theorem* 3.2. *Then the Krein–von Neumann extension A_K of A_{min} is given by*

$$A_K = -\Delta + V,$$
$$\mathrm{dom}(A_K) = \big\{ f \in \mathrm{dom}(A_{max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0 \big\}. \qquad (3.13)$$

Proof. We recall that the Krein–von Neumann extension A_K of A_{min} is defined on

$$\mathrm{dom}(A_K) = \mathrm{dom}(A_{min}) \,\dot{+}\, \ker(A_{max}). \qquad (3.14)$$

Thus, from Lemma 2.2 (ii) one concludes

$$\mathrm{dom}(A_K) = \mathring{H}^2(\Omega) \,\dot{+}\, \ker(A_{max,\Omega}). \qquad (3.15)$$

Next, we show the inclusion

$$\operatorname{dom}(A_K) \subseteq \big\{ f \in \operatorname{dom}(A_{max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0 \big\}. \tag{3.16}$$

Fix $f \in \operatorname{dom}(A_K)$ and decompose f in the form $f = f_{min} + f_0$, where $f_{min} \in \mathring{H}^2(\Omega)$ and $f_0 \in \ker(A_{max})$ (cf. (3.15)). Thus,

$$\gamma_D f_{min} = \widetilde{\gamma}_D f_{min} = 0 \quad \text{and} \quad \gamma_N f_{min} = \widetilde{\gamma}_N f_{min} = 0, \tag{3.17}$$

and hence it follows from Theorem 3.2 (ii) that

$$\widetilde{M}(0)\widetilde{\gamma}_D f = \widetilde{M}(0)\widetilde{\gamma}_D(f_{min} + f_0) = \widetilde{M}(0)\widetilde{\gamma}_D f_0 = -\widetilde{\gamma}_N f_0 = -\widetilde{\gamma}_N f. \tag{3.18}$$

Thus, $\widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0$ and the inclusion (3.16) holds.

Next we verify the opposite inclusion

$$\operatorname{dom}(A_K) \supseteq \big\{ f \in \operatorname{dom}(A_{max}) \,\big|\, \widetilde{\gamma}_N f + \widetilde{M}(0)\widetilde{\gamma}_D f = 0 \big\}. \tag{3.19}$$

We use the direct sum decomposition

$$\operatorname{dom}(A_{max}) = \operatorname{dom}(A_D) \dotplus \ker(A_{max}), \tag{3.20}$$

which is not difficult to check. Assuming that $f \in \operatorname{dom}(A_{max,\Omega})$ satisfies the boundary condition

$$\widetilde{M}(0)\widetilde{\gamma}_D f + \widetilde{\gamma}_N f = 0, \tag{3.21}$$

according to the decomposition (3.20) we write f in the form $f = f_D + f_0$, where $f_D \in \operatorname{dom}(A_D)$ and $f_0 \in \ker(A_{max})$. Thus, $\gamma_D f_D = \widetilde{\gamma}_D f_D = 0$ by Theorem 3.1 and with the help of Theorem 3.2 (ii) one obtains

$$\widetilde{M}(0)\widetilde{\gamma}_D f = \widetilde{M}(0)\widetilde{\gamma}_D(f_D + f_0) = \widetilde{M}(0)\widetilde{\gamma}_D f_0 = -\widetilde{\gamma}_N f_0. \tag{3.22}$$

Taking into account the boundary condition (3.21) one concludes

$$-\widetilde{\gamma}_N f = \widetilde{M}(0)\widetilde{\gamma}_D f = -\widetilde{\gamma}_N f_0, \tag{3.23}$$

and hence

$$0 = \widetilde{\gamma}_N(f - f_0) = \widetilde{\gamma}_N f_D. \tag{3.24}$$

Together with Theorem 3.1 this implies $f_D \in \ker(\widetilde{\gamma}_N) = \ker(\gamma_N) = \operatorname{dom}(A_N)$. Thus, one arrives at

$$f_D \in \operatorname{dom}(A_D) \cap \operatorname{dom}(A_N) = \operatorname{dom}(A_{min}) = \mathring{H}^2(\Omega), \tag{3.25}$$

where (2.10) and $A_D = A_F$ was used (cf. Theorem 2.4 (i)). Summing up, one has

$$f = f_D + f_0 \in \mathring{H}^2(\Omega) \dotplus \ker(A_{max}) = \operatorname{dom}(A_K), \tag{3.26}$$

which shows (3.19) and completes the proof of Theorem 3.3. $\qquad \square$

4. Spectral asymptotics of the Krein–von Neumann extension

As the main result in this section we derive the following Weyl-type spectral asymptotics for the Krein–von Neumann extension A_K of A_{min}.

Theorem 4.1. *Assume Hypothesis 2.1. Let $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ be the strictly positive eigenvalues of the Krein–von Neumann extension A_K enumerated in nondecreasing order counting multiplicity, and let*

$$N(\lambda, A_K) := \#\{j \in \mathbb{N} : 0 < \lambda_j \leqslant \lambda\}, \quad \lambda > 0, \tag{4.1}$$

be the eigenvalue distribution function for A_K. Then the following Weyl asymptotic formula holds,

$$N(\lambda, A_K) \underset{\lambda\to\infty}{=} \frac{v_n \, |\Omega|}{(2\pi)^n} \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big), \tag{4.2}$$

where $v_n = \pi^{n/2}/\Gamma((n/2)+1)$ denotes the (Euclidean) volume of the unit ball in $\mathbb{R}^n$ (with $\Gamma(\cdot)$ the classical Gamma function [1, Sect. 6.1]) and $|\Omega|$ represents the (n-dimensional) Lebesgue measure of Ω.

The proof of Theorem 4.1 follows along the lines of [14, 15], where the case of quasi-convex domains was investigated. The main ingredients are a general Weyl type asymptotic formula due to Kozlov [46] (see also [45], [47] for related results) and the connection between the eigenvalues of the so-called buckling operator and the positive eigenvalues of the Krein–von Neumann extension A_K (cf. [15], [16]). We first consider the quadratic forms $\mathfrak{a}$ and $\mathfrak{t}$ on $\text{dom}(A_{min}) = \mathring{H}^2(\Omega)$ defined by

$$\mathfrak{a}[f,g] := \big(A_{min}f, A_{min}g\big)_{L^2(\Omega)}, \quad f,g \in \text{dom}(\mathfrak{a}) := \mathring{H}^2(\Omega), \tag{4.3}$$

$$\mathfrak{t}[f,g] := \big(f, A_{min}g\big)_{L^2(\Omega)}, \quad f,g \in \text{dom}(\mathfrak{t}) := \mathring{H}^2(\Omega). \tag{4.4}$$

Since the graph norm of A_{min} and the H^2-norm are equivalent on $\text{dom}\, A_{min} = \mathring{H}^2(\Omega)$ by Lemma 2.2 (ii), it follows that $\mathcal{W} := (\text{dom}(\mathfrak{a}); (\cdot,\cdot)_{\mathcal{W}})$, where the inner product is defined by

$$(f,g)_{\mathcal{W}} := \mathfrak{a}[f,g] = \big(A_{min}f, A_{min}g\big)_{L^2(\Omega)}, \quad f,g \in \text{dom}(\mathfrak{a}), \tag{4.5}$$

is a Hilbert space. One observes that the embedding $\iota : \mathcal{W} \to L^2(\Omega)$ is compact; this is a consequence of Ω being bounded. Next, we consider for fixed $g \in \mathcal{W}$ the functional

$$\mathcal{W} \ni f \mapsto \mathfrak{t}[\iota f, \iota g], \tag{4.6}$$

which is continuous on the Hilbert space $\mathcal{W}$ and hence can be represented with the help of a bounded operator T in $\mathcal{W}$ in the form

$$(f, Tg)_{\mathcal{W}} = \mathfrak{t}[\iota f, \iota g], \quad f,g \in \mathcal{W}. \tag{4.7}$$

The nonnegativity of the form $\mathfrak{t}$ implies that T is a self-adjoint and nonnegative operator in $\mathcal{W}$. Furthermore, one obtains for $f,g \in \mathcal{W}$ from (4.4) that

$$(f, Tg)_{\mathcal{W}} = \mathfrak{t}[\iota f, \iota g] = \big(\iota f, A_{min}\iota g\big)_{L^2(\Omega)} = \big(f, \iota^* A_{min}\iota g\big)_{\mathcal{W}}, \tag{4.8}$$

and hence,

$$T = \iota^* A_{min} \iota. \tag{4.9}$$

In particular, since $A_{min}\iota : \mathcal{W} \to L^2(\Omega)$ is defined on the whole space $\mathcal{W}$ and is closed as an operator from $\mathcal{W}$ to $L^2(\Omega)$, it follows that $A_{min}\iota$ is bounded and hence the compactness of $\iota^* : L^2(\Omega) \to \mathcal{W}$ implies that $T = \iota^* A_{min}\iota$ is a compact operator in the Hilbert space $\mathcal{W}$.

The next useful lemma shows that the eigenvalues of T are precisely the reciprocals of the nonzero eigenvalues of A_K. Lemma 4.2 is inspired by the connection of the Krein–von Neumann extension to the buckling of a clamped plate problem (cf. [15, Theorem 6.2] and [14, 16, 38]).

Lemma 4.2. *Assume Hypothesis 2.1 and let T be the nonnegative compact operator in $\mathcal{W}$ defined by (4.7). Then*

$$\lambda \in \sigma_p(A_K)\backslash\{0\} \ \text{if and only if} \ \lambda^{-1} \in \sigma_p(T), \tag{4.10}$$

counting multiplicities.

Proof. Assume first that $\lambda \neq 0$ is an eigenvalue of A_K and let g be a corresponding eigenfunction. We decompose g in the form

$$g = g_{min} + g_0, \quad g_{min} \in \text{dom}(A_{min}), \ g_0 \in \ker(A_{max}) \tag{4.11}$$

(cf. (2.8)), where $g_{min} \neq 0$ as $\lambda \neq 0$. Then one concludes

$$A_{min}g_{min} = A_K(g_{min} + g_0) = A_K g, \tag{4.12}$$

and hence,

$$A_{min}g_{min} - \lambda g_{min} = A_K g - \lambda g_{min} = \lambda g - \lambda g_{min} = \lambda g_0 \in \ker(A_{max}), \tag{4.13}$$

so that

$$A_{max}A_{min}g_{min} = \lambda A_{max}g_{min} = \lambda A_{min}g_{min}. \tag{4.14}$$

This yields

$$\begin{aligned}
(f, \lambda^{-1}g_{min})_{\mathcal{W}} &= \mathfrak{a}[f, \lambda^{-1}g_{min}] \\
&= \left(A_{min}f, \lambda^{-1}A_{min}g_{min}\right)_{L^2(\Omega)} \\
&= \left(f, \lambda^{-1}A_{max}A_{min}g_{min}\right)_{L^2(\Omega)} \\
&= (f, A_{min}g_{min})_{L^2(\Omega)} \\
&= \mathfrak{t}[f, g_{min}] \\
&= (f, Tg_{min})_{\mathcal{W}}, \quad f \in \mathcal{W},
\end{aligned} \tag{4.15}$$

where, for simplicity, we have identified elements in $\mathcal{W}$ with those in $\text{dom}(\mathfrak{a})$, and hence omitted the embedding map ι. From (4.15) we then conclude

$$Tg_{min} = \frac{1}{\lambda}g_{min}, \tag{4.16}$$

which shows that $\lambda^{-1} \in \sigma_p(T)$.

Conversely, assume that $h \in \mathcal{W}\backslash\{0\}$ and $\lambda \neq 0$ are such that

$$Th = \frac{1}{\lambda}h \tag{4.17}$$

holds. Then it follows for $f \in \mathrm{dom}(\mathfrak{a})$ from (4.5) and (4.7) that

$$\mathfrak{a}[f, h] = \mathfrak{a}[f, \lambda T h] = (f, \lambda T h)_{\mathcal{W}} = \mathfrak{t}[f, \lambda h] = \big(f, \lambda A_{min} h\big)_{L^2(\Omega)}. \qquad (4.18)$$

As a consequence of the first representation theorem for quadratic forms [44, Theorem VI.2.1 (iii), Example VI.2.13], one concludes that $A_{max} A_{min}$ is the representing operator for $\mathfrak{a}$, and therefore,

$$h \in \mathrm{dom}(A_{max} A_{min}) \ \text{ and } \ A_{max} A_{min} h = \lambda A_{min} h. \qquad (4.19)$$

In particular, $h \in \mathrm{dom}(A_{min})$ and

$$\begin{aligned} A_{max}(A_{min} - \lambda)h &= A_{max} A_{min} h - \lambda A_{max} h \\ &= A_{max} A_{min} h - \lambda A_{min} h \\ &= 0. \end{aligned} \qquad (4.20)$$

Let us define

$$g := \frac{1}{\lambda} A_{min} h = h + \frac{1}{\lambda} \big(A_{min} - \lambda\big)h. \qquad (4.21)$$

As $h \in \mathrm{dom}(A_{min})$ and $(A_{min} - \lambda)h \in \ker(A_{max})$ by (4.20), we conclude from (2.8) that $g \in \mathrm{dom}\, A_K$. Moreover, $g \neq 0$ since A_{min} is positive. Furthermore,

$$A_K g = A_{max} g = \frac{1}{\lambda} A_{max} A_{min} h = A_{min} h = \lambda g, \qquad (4.22)$$

shows that $\lambda \in \sigma_p(A_K)$. $\qquad \square$

Proof of Theorem 4.1. Let T be the nonnegative compact operator in $\mathcal{W}$ defined by (4.7). We order the eigenvalues of T in the form

$$0 \leqslant \cdots \leqslant \mu_{j+1}(T) \leqslant \mu_j(T) \leqslant \cdots \leqslant \mu_1(T), \qquad (4.23)$$

listed according to their multiplicity, and set

$$\mathcal{N}(\lambda, T) := \# \big\{ j \in \mathbb{N} : \mu_j(T) \geqslant \lambda^{-1} \big\}, \quad \lambda > 0. \qquad (4.24)$$

It follows from Lemma 4.2 that

$$\mathcal{N}(\lambda, T) = N(\lambda, A_K), \quad \lambda > 0, \qquad (4.25)$$

and hence [46] yields the asymptotic formula,

$$N(\lambda, A_K) = \mathcal{N}(\lambda, T) \underset{\lambda \to \infty}{=} \omega \, \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big), \qquad (4.26)$$

with

$$\begin{aligned} \omega &:= \frac{1}{n(2\pi)^n} \int_\Omega \left(\int_{S^{n-1}} \left[\frac{\sum_{j=1}^n \xi_j^2}{\sum_{j,k=1}^n \xi_j^2 \xi_k^2} \right]^{\frac{n}{2}} d\omega_{n-1}(\xi) \right) d^n x \\ &= \frac{v_n \, |\Omega|}{(2\pi)^n}. \end{aligned} \qquad (4.27)$$

$\qquad \square$

For bounds on $N(\cdot, A_K)$ in the case of $\Omega \subset \mathbb{R}^n$ open and of finite (n-dimensional) Lebesgue measure, we refer to [34].

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Jussi Behrndt
Institut für Numerische Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
e-mail: `behrndt@tugraz.at`
URL: `http://www.math.tugraz.at/ behrndt/`

Fritz Gesztesy
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
e-mail: `gesztesyf@missouri.edu`
URL: `http://www.math.missouri.edu/personnel/faculty/gesztesyf.html`

Till Micheler
Department of Mathematics, Technische Universität Berlin, 10623 Berlin, Germany
e-mail: `micheler@math.tu-berlin.de`

Marius Mitrea
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
e-mail: `mitream@missouri.edu`
URL: `http://www.math.missouri.edu/personnel/faculty/mitream.html`