

# LOWER BOUNDS FOR SELF-ADJOINT STURM–LIOUVILLE OPERATORS

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ABSTRACT. In this note we provide estimates for the lower bound of the self-adjoint operator associated with the three-coefficient Sturm–Liouville differential expression

$$\frac{1}{r} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right)$$

in the weighted  $L^2$ -Hilbert space  $L^2(\mathbb{R}; r dx)$ .

## 1. INTRODUCTION

One-dimensional Schrödinger operators of the form

$$H = -\frac{d^2}{dx^2} + q \tag{1.1}$$

with real-valued potential  $q$  have been studied in the mathematical and physical literature intensively in the last century due to their particular importance in quantum mechanics. Typically one is interested in a suitable self-adjoint realization in  $L^2(\mathbb{R})$  and its spectral properties, among them estimates for lower bounds, numbers of negative eigenvalues, and Lieb–Thirring inequalities are particularly important, see, for instance, the recent survey [9].

The main objective of this note is to derive estimates on the lower bound of more general Sturm–Liouville operators of the type

$$T = \frac{1}{r} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right) \tag{1.2}$$

with real-valued coefficients under the standard assumptions  $r, 1/p, q \in L^1_{\text{loc}}(\mathbb{R})$  and  $r, p$  positive almost everywhere. We refer the reader to the textbooks [6], [11], [13], [15], [18], [21], [22], and [23] for an overview and detailed study of Sturm–Liouville (resp., Schrödinger) operators. The natural Hilbert space in this context is the weighted  $L^2$ -space  $L^2_r(\mathbb{R}) := L^2(\mathbb{R}; r dx)$  and under some mild additional assumptions on the coefficients one concludes that  $T$  is a semibounded self-adjoint operator in  $L^2_r(\mathbb{R})$ . As mentioned above, lower bounds for the spectrum of  $T$  are known for the special case  $r = p = 1$ , that is,  $T = H$ , and for completeness we provide a straightforward estimate as a warm up in Section 2.

In the general setting it seems that a systematic study is missing and it is the aim of this note to initiate and contribute to this circle of problems. It is clear that the coefficients  $r$  and  $p$  have an essential influence on the lower bound. If, for instance, the weight function  $r = r_0$  is constant and  $p = 1$  then formally  $T = (1/r_0)H$  and the lower bound  $\min \sigma(T)$  of  $T$  is simply given by  $(1/r_0) \min \sigma(H)$ . This already indicates that for a nonconstant weight function  $r$  the  $L^\infty$ -norm of  $1/r$  will appear in the lower bounds, and the situation becomes much more difficult if  $1/r \notin L^\infty(\mathbb{R})$ ,

in which case we require the existence of a function  $g$  that neutralizes the behaviour of the weight function  $r$  on subsets of  $\mathbb{R}$  where  $r$  is small. Furthermore, the norm of the coefficient  $p$  will enter in lower bound estimates and very roughly speaking  $1/p$  has to be considered in conjunction with the potential  $q$ . The methods and proofs in this paper are strongly inspired by [2], where bounds on nonreal eigenvalues of indefinite Sturm-Liouville operators are obtained.

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## 2. ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

As a warm up we discuss in this short section the special case  $p = r = 1$  and  $q \in L^s(\mathbb{R})$  real-valued a.e.,  $s \in [1, \infty]$ , and derive a lower bound for the self-adjoint Schrödinger operator  $H$  in (1.1) using the argument presented in [20, (3.5.30), p. 155–156].

We start by recalling that  $q \in L^s(\mathbb{R})$ ,  $s \in [1, \infty)$ , implies that  $q$  is relatively form compact with respect to the free Hamiltonian  $H_0$  in  $L^2(\mathbb{R})$ , where

$$H_0 f = -f'', \quad f \in D(H_0) = H^2(\mathbb{R}), \quad (2.1)$$

with  $H^\ell(\mathbb{R})$ ,  $\ell \in [0, \infty)$ , the standard scale of Sobolev spaces. This follows from the stronger statement that  $|q|^{1/2}(H_0 + I)^{-1/2}$  satisfies (see, e.g., [16, Theorem XI.20])

$$|q|^{1/2}(H_0 + I)^{-1/2} \in \mathcal{B}_{2s}(L^2(\mathbb{R})) \text{ if } q \in L^s(\mathbb{R}), \quad s \in [1, \infty), \quad (2.2)$$

where  $\mathcal{B}_t(\mathcal{H})$  represent the  $\ell^t(\mathbb{N})$ -based trace ideals of compact operators in the complex, separable Hilbert space  $\mathcal{H}$ . In particular,

$$|q|^{1/2}(H_0 + I)^{-1/2} \text{ is compact}, \quad (2.3)$$

and hence the form sum  $H$  of  $H_0$  and  $q$  is self-adjoint in  $L^2(\mathbb{R})$  and bounded from below. By a result of Hartman [12] and Rellich [17] (see also [11, Theorem 8.5.2]), the boundedness from below of the minimal operator associated with the differential expression  $-(d^2/dx^2) + q$  implies that the latter is in the limit point case at  $\pm\infty$  and hence the maximal operator associated with  $-(d^2/dx^2) + q$  is self-adjoint in  $L^2(\mathbb{R})$ , and thus necessarily coincides with  $H$ . It is clear that for  $s = \infty$  the same is true as  $q \in L^\infty(\mathbb{R})$  is a bounded perturbation of  $H_0$ . Consequently,  $H$  is given by

$$\begin{aligned} Hf &= -f'' + qf, \\ f \in D(H) &= \{g \in L^2(\mathbb{R}) \mid g, g' \in AC_{\text{loc}}(\mathbb{R}); (-g'' + qg) \in L^2(\mathbb{R})\}. \end{aligned}$$

Property (2.3) then implies

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0, \infty), \quad (2.4)$$

and hence it suffices to consider negative eigenvalues, which turn out to be simple as  $-(d^2/dx^2) + q$  is in the limit point case at  $\pm\infty$ . We consider an eigenvalue  $\lambda < 0$

of  $H$  and denote the corresponding eigenfunction by  $f_\lambda$ . From  $-f_\lambda'' + qf_\lambda = \lambda f_\lambda$  one concludes

$$f_\lambda = - \left( -\frac{d^2}{dx^2} - \lambda \right)^{-1} qf_\lambda$$

and using the corresponding Green's function we obtain

$$\|f_\lambda\|_2 = \frac{1}{2\sqrt{-\lambda}} \|e^{-\sqrt{-\lambda}|\cdot|} * qf_\lambda\|_2 \leq \frac{1}{2\sqrt{-\lambda}} \|e^{-\sqrt{-\lambda}|\cdot|}\|_t \|qf_\lambda\|_{t'}, \quad (2.5)$$

where Young's inequality,<sup>1</sup> with  $1/t + 1/t' = 1 + 1/2$  was applied in the last step. Hölder's inequality then yields  $\|qf_\lambda\|_{t'} \leq \|q\|_s \|f_\lambda\|_2$  for  $1/t' = 1/s + 1/2$  and hence,

$$\sqrt{-\lambda} \leq \frac{1}{2} \|e^{-\sqrt{-\lambda}|\cdot|}\|_t \|q\|_s = \frac{1}{2} \left( \frac{2}{t\sqrt{-\lambda}} \right)^{\frac{1}{t}} \|q\|_s \quad (2.6)$$

if  $t \in (1, \infty)$ , that is,  $s \in (1, \infty)$ . As  $1/t = 1 - 1/s$  it follows for  $s \in (1, \infty)$  that

$$(-\lambda)^{\frac{2s-1}{2s}} \leq 2^{-\frac{1}{s}} \left( \frac{s-1}{s} \right)^{\frac{s-1}{s}} \|q\|_s$$

and hence

$$\min \sigma(H) \geq -2^{-\frac{2}{2s-1}} \left( \frac{s-1}{s} \right)^{\frac{2(s-1)}{2s-1}} \|q\|_s^{\frac{2s}{2s-1}} \quad \text{for } s \in (1, \infty). \quad (2.7)$$

It is easy to see that the lower bound (2.7) also remains valid for  $s = 1$  (indeed, inequality (2.5) and the first inequality in (2.6) apply with  $s = 1$ ,  $t = \infty$ ,  $1/t' = 3/2$ ) in which case one obtains

$$\min \sigma(H) \geq -(1/4) \|q\|_1^2, \quad (2.8)$$

and obviously applies to  $s = \infty$ , implying

$$\min \sigma(H) \geq -\|q\|_\infty. \quad (2.9)$$

We mention that the bound (2.7) and (2.8) coincide with [5, Corollary 14.3.11 and Corollary 14.3.12] and that the above argument also leads to bounds for Schrödinger operators with complex potentials  $q \in L^s(\mathbb{R})$ ,  $s \in [1, \infty)$  (see, in particular, [1] for the case  $s = 1$ ). In the context of Schrödinger operators with complex-valued potentials we also refer, for instance, to [4], [7], [8], and [10].

**Remark 2.1.** We mention that Lieb–Thirring inequalities (see, e.g., [9], [14] and the extensive literature cited therein) also lead to lower bounds of  $H$ . More specifically, for  $s = 3/2$  one can compare with the one-particle constant  $L_1^{(1)} = 4/[3^{3/2}\pi] = 0.24503$  in [9, Section 3.2]: The corresponding constant in (2.7) equals  $2^{-1}3^{-1/2} = 0.28867$ . Historically, we note that Barnes, Brascamp, and Lieb [3] derived a lower bound for the ground state energy of (multi-dimensional) Schrödinger operators already in 1976.

<sup>1</sup>Explicitly,  $\|f * g\|_\alpha \leq \|f\|_\beta \|g\|_\gamma$ ,  $1 \leq \alpha, \beta, \gamma \leq \infty$ ,  $1 + \alpha^{-1} = \beta^{-1} + \gamma^{-1}$ .

## 3. MAIN RESULTS

Now we consider the general Sturm-Liouville differential expression

$$\tau = \frac{1}{r} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right) \quad (3.1)$$

on  $\mathbb{R}$  with real-valued coefficients under the standard assumptions,

$$r, 1/p, q \in L^1_{\text{loc}}(\mathbb{R}), \quad (3.2)$$

and we assume from now on that Hypothesis 3.1 below is satisfied. In the following  $L^1_{\text{u}}(\mathbb{R})$  denotes the normed space of uniformly locally integrable functions, that is,

$$L^1_{\text{u}}(\mathbb{R}) = \{h \in L^1_{\text{loc}}(\mathbb{R}) : \|h\|_{\text{u}} < \infty\}, \quad \|h\|_{\text{u}} = \sup_{n \in \mathbb{Z}} \int_n^{n+1} |h(t)| dt.$$

**Hypothesis 3.1.** The real coefficients  $p$ ,  $q$  and  $r$  of  $\tau$  satisfy the following:

- (a)  $p(x) > 0$  for a. a.  $x \in \mathbb{R}$  and  $1/p \in L^\eta(\mathbb{R})$  for some  $\eta \in [1, \infty]$ ;
- (b)  $q \in L^1_{\text{u}}(\mathbb{R})$ ;
- (c)  $r(x) > 0$  for a. a.  $x \in \mathbb{R}$  and there exist  $a, b \in \mathbb{R}$  with  $a < b$  such that

$$\text{ess inf}_{t \in \mathbb{R} \setminus [a, b]} r(t) > 0. \quad (3.3)$$

It is known that the differential expression  $\tau$  is in the limit-point case at both singular endpoints  $\pm\infty$ , and the corresponding maximal operator

$$Tf = \tau f = \frac{1}{r} (-(pf)') + qf, \\ f \in D(T) = \{g \in L^2_r(\mathbb{R}) \mid g, pg' \in AC_{\text{loc}}(\mathbb{R}); \tau g \in L^2_r(\mathbb{R})\},$$

is self-adjoint in the weighted  $L^2$ -space  $L^2_r(\mathbb{R})$  and semibounded from below; cf. [2, Lemma A.2]. Our main goal is to derive estimates for the lower bound  $\min \sigma(T)$  of  $T$ . For a nonnegative function  $g \in L^\infty(\mathbb{R})$  the set

$$\Omega_g := \{x \in \mathbb{R} \mid r(x)g(x) < 1\}$$

and its Lebesgue measure  $\mu(\Omega_g)$  will appear in the lower bound estimates in our main results below. In the particular case  $1/r \in L^\infty(\mathbb{R})$  one can choose  $g = 1/r$  and hence  $\Omega_g$  becomes a Lebesgue null set, which leads to more explicit lower bound estimates. We also mention that for any  $\varepsilon > 0$  there exists a (constant) nonnegative function  $g_\varepsilon \in L^\infty(\mathbb{R})$  such that  $\mu(\Omega_{g_\varepsilon}) < \varepsilon$ ; this follows from

$$\lim_{n \rightarrow \infty} \mu(\{x \in \mathbb{R} \mid r(x) < 1/n\}) = \mu \left( \bigcap_{n=1}^{\infty} \{x \in \mathbb{R} \mid r(x) < 1/n\} \right) \\ = \mu(\{x \in \mathbb{R} \mid r(x) = 0\}) = 0.$$

The first main result requires only minimal assumptions on the potential in Hypothesis 3.1, that is,  $q \in L^1_{\text{u}}(\mathbb{R})$ , but we have to assume that  $1/p \in L^\infty(\mathbb{R})$ . We shall denote the negative part of the potential  $q$  by  $q_-$ .

**Theorem 3.2.** *In addition to Hypothesis 3.1, assume that  $1/p \in L^\infty(\mathbb{R})$ , let*

$$\alpha = 2\|q_-\|_{\text{u}} + 4\|1/p\|_{\infty}\|q_-\|_{\text{u}}^2 \quad \text{and} \quad \beta = (4\|1/p\|_{\infty}\alpha)^{1/2}, \quad (3.4)$$

*and choose a nonnegative function  $g \in L^\infty(\mathbb{R})$  such that  $\mu(\Omega_g)\beta < 1$ . Then*

$$\min \sigma(T) \geq \frac{-\alpha\|g\|_{\infty}}{1 - \mu(\Omega_g)\beta}$$

and in the special case  $1/r \in L^\infty(\mathbb{R})$  the choice  $g = 1/r$  implies  $\mu(\Omega_g) = 0$  and

$$\min \sigma(T) \geq -(2\|q_-\|_u + 4\|1/p\|_\infty \|q_-\|_u^2) \|1/r\|_\infty.$$

In the next result we consider the case  $q_- \in L^s(\mathbb{R})$ ,  $s \in [1, \infty]$ , and  $1/p \in L^\infty(\mathbb{R})$ .

**Theorem 3.3.** *In addition to Hypothesis 3.1, assume that  $1/p \in L^\infty(\mathbb{R})$  and  $q_- \in L^s(\mathbb{R})$  for some  $s \in [1, \infty]$ , let*

$$\alpha = \begin{cases} \|q_-\|_s \beta^{\frac{1}{s}} & \text{if } s \in [1, \infty), \\ \|q_-\|_\infty & \text{if } s = \infty, \end{cases} \quad \text{and} \quad \beta = \begin{cases} (4\|1/p\|_\infty \|q_-\|_s)^{\frac{s}{2s-1}} & \text{if } s \in [1, \infty), \\ (4\|1/p\|_\infty \|q_-\|_\infty)^{1/2} & \text{if } s = \infty, \end{cases} \quad (3.5)$$

and choose a nonnegative function  $g \in L^\infty(\mathbb{R})$  such that  $\mu(\Omega_g)\beta < 1$ . Then

$$\min \sigma(T) \geq \frac{-\alpha \|g\|_\infty}{1 - \mu(\Omega_g)\beta}$$

and in the special case  $1/r \in L^\infty(\mathbb{R})$  the choice  $g = 1/r$  implies  $\mu(\Omega_g) = 0$  and

$$\min \sigma(T) \geq -(4\|1/p\|_\infty)^{\frac{1}{2s-1}} \|q_-\|_s^{\frac{2s}{2s-1}} \|1/r\|_\infty \quad \text{if } s \in [1, \infty). \quad (3.6)$$

If  $s = \infty$  we have

$$\min \sigma(T) \geq -\|q_-\|_\infty \|1/r\|_\infty.$$

**Remark 3.4.** The bounds in Theorem 3.3 above are not optimal. In fact, in the special case  $p = r = 1$  and  $q_- \in L^s(\mathbb{R})$  for some  $s \in [1, \infty)$  the bound in (3.6) becomes

$$\min \sigma(T) \geq \begin{cases} -4\|q_-\|_1^2 & \text{if } s = 1, \\ -2^{\frac{2}{2s-1}} \|q_-\|_s^{\frac{2s}{2s-1}} & \text{if } s \in (1, \infty), \end{cases}$$

while (2.7) (or [5, Corollary 14.3.11 and Corollary 14.3.12]) show that

$$\min \sigma(T) \geq \begin{cases} -\|q\|_1^2/4 & \text{if } s = 1, \\ -2^{-\frac{2}{2s-1}} \left(\frac{s-1}{s}\right)^{\frac{2(s-1)}{2s-1}} \|q\|_s^{\frac{2s}{2s-1}} & \text{if } s \in (1, \infty). \end{cases} \quad (3.7)$$

◇

In the following theorem we deal with  $q_- \in L^s(\mathbb{R})$ ,  $s \in [1, \infty]$ , and  $1/p \in L^\eta(\mathbb{R})$ ,  $\eta \in [1, \infty)$ .

**Theorem 3.5.** *In addition to Hypothesis 3.1, assume that  $1/p \in L^\eta(\mathbb{R})$  for some  $\eta \in [1, \infty)$  and  $q_- \in L^s(\mathbb{R})$  for some  $s \in [1, \infty]$  such that  $\eta + s > 2$  if  $s \neq \infty$ . Let  $\alpha$  be as in (3.5) and*

$$\beta = \begin{cases} \left( \left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \|q_-\|_s \right)^{\frac{\eta s}{2\eta s - \eta - s}} & \text{if } s \in [1, \infty), \\ \left( \left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \|q_-\|_\infty \right)^{\frac{\eta}{2\eta-1}} & \text{if } s = \infty, \end{cases} \quad (3.8)$$

and choose a nonnegative function  $g \in L^\infty(\mathbb{R})$  such that  $\mu(\Omega_g)\beta < 1$ . Then

$$\min \sigma(T) \geq \frac{-\alpha \|g\|_\infty}{1 - \mu(\Omega_g)\beta}$$

and in the special case  $1/r \in L^\infty(\mathbb{R})$  the choice  $g = 1/r$  implies  $\mu(\Omega_g) = 0$  and

$$\min \sigma(T) \geq -\left( \left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \right)^{\frac{\eta}{2\eta s - \eta - s}} \|q_-\|_s^{\frac{2\eta s - s}{2\eta s - \eta - s}} \|1/r\|_\infty \quad \text{if } s \in [1, \infty).$$

If  $s = \infty$  we have

$$\min \sigma(T) \geq -\|q_-\|_\infty \|1/r\|_\infty.$$

The following proposition provides a simple condition for nonnegativity of  $T$ .

**Proposition 3.6.** *Suppose that Hypothesis 3.1 holds and assume that  $1/p, q_- \in L^1(\mathbb{R})$ . If  $\|1/p\|_1 \|q_-\|_1 < 1$  then  $\min \sigma(T) \geq 0$ .*

**Remark 3.7.** If the coefficients  $p$ ,  $q$  and  $r$  in Hypothesis 3.1 are restricted to the half line  $(0, \infty)$  and  $T_+$  denotes the self-adjoint realization of the (restricted) differential expression  $\tau$  in  $L^2((0, \infty))$  with Dirichlet boundary conditions at the regular endpoint 0, then the lower bound estimates above remain valid for  $T_+$ . In fact, the Dirichlet boundary condition at 0 ensures that the boundary term in the integration by parts formula for  $f \in D(T_+)$  vanishes and hence the proofs in the next section (see, e.g. (4.4)) extend directly to the half line case.

#### 4. PROOFS

In this section we prove our main results. It will always be assumed that the coefficients  $p, q, r$  satisfy Hypothesis 3.1. The first three items of the following useful statement can be found, for instance, in [2, Lemma A.2]. The last item follows from [2, Lemma A.1] and the first item.

**Lemma 4.1.** *Assume Hypothesis 3.1, then the following assertions hold for all  $f, g \in D(T)$ :*

- (i)  $f, \sqrt{p}f' \in L^2(\mathbb{R})$  and  $qf^2 \in L^1(\mathbb{R})$ ;
- (ii) there exists a sequence  $(x_n)_{n \in \mathbb{Z}}$  in  $\mathbb{R}$  satisfying  $\lim_{n \rightarrow \infty} x_n = \infty$  and  $\lim_{n \rightarrow -\infty} x_n = -\infty$  such that  $\lim_{|n| \rightarrow \infty} f(x_n) = 0$ ;
- (iii)  $\lim_{|x| \rightarrow \infty} (pf')(x)g(x) = 0$ .
- (iv)  $f \in L^\infty(\mathbb{R})$ .

For our estimates of the lower bound of  $T$  it is convenient to reduce the considerations to the set

$$D_-(T) := \{f \in D(T) \mid (Tf, f)_r \leq 0\}, \quad (4.1)$$

where  $(\cdot, \cdot)_r$  stands for the weighted inner product corresponding to  $L_r^2(\mathbb{R})$ . The potential  $q$  is decomposed in its positive part  $q_+$  and negative part  $q_-$ , i. e.

$$q = q_+ - q_-, \quad \text{where } q_+ := \frac{|q| + q}{2} \quad \text{and} \quad q_- := \frac{|q| - q}{2}. \quad (4.2)$$

**Lemma 4.2.** *Assuming Hypothesis 3.1, every function  $f \in D_-(T)$  satisfies*

$$\|\sqrt{p}f'\|_2^2 \leq \|q_- f^2\|_1 \quad \text{and} \quad \|qf^2\|_1 \leq 2\|q_- f^2\|_1. \quad (4.3)$$

Moreover, the inequality  $\|q_- f^2\|_1 \leq \|q_+ f^2\|_1$  implies  $\|\sqrt{p}f'\|_2 = 0$ .

*Proof.* For  $f \in D_-(T)$  integration by parts together with Lemma 4.1 (i) and (iii) yields

$$\begin{aligned} 0 \geq (Tf, f)_r &= \int_{\mathbb{R}} p(t)|f'(t)|^2 dt + \int_{\mathbb{R}} q(t)|f(t)|^2 dt \\ &= \|\sqrt{p}f'\|_2^2 + \|q_+ f^2\|_1 - \|q_- f^2\|_1. \end{aligned} \quad (4.4)$$

This implies  $\|\sqrt{p}f'\|_2^2 \leq \|q_- f^2\|_1$  and  $\|q_+ f^2\|_1 \leq \|q_- f^2\|_1$ . Therefore, with  $|q| = q_+ + q_-$  we have

$$\|qf^2\|_1 = \|q_+ f^2\|_1 + \|q_- f^2\|_1 \leq 2\|q_- f^2\|_1.$$

If  $\|q_- f^2\|_1 \leq \|q_+ f^2\|_1$  holds, then (4.4) implies  $\|\sqrt{p}f'\|_2 = 0$ .  $\square$

**Lemma 4.3.** *In addition to Hypothesis 3.1, assume that there are constants  $\alpha \geq 0$ ,  $\beta \geq 0$  and a nonnegative function  $g \in L^\infty(\mathbb{R})$  such that*

- (i)  $\|q_- f^2\|_1 \leq \alpha \|f\|_2^2$  and  $\|f\|_\infty^2 \leq \beta \|f\|_2^2$  for all  $f \in D_-(T)$ ;
- (ii)  $\mu(\Omega_g)\beta < 1$ .

Then one has

$$\min \sigma(T) \geq \frac{-\alpha \|g\|_\infty}{1 - \mu(\Omega_g)\beta}. \quad (4.5)$$

*Proof.* Let  $f \in D_-(T)$ . Then one has

$$\begin{aligned} \|g\|_\infty (f, f)_r &= \|g\|_\infty \int_{\mathbb{R}} |f(t)|^2 r(t) dt \geq \int_{\mathbb{R}} |f(t)|^2 r(t) g(t) dt \\ &\geq \int_{\mathbb{R} \setminus \Omega_g} |f(t)|^2 r(t) g(t) dt \geq \|f\|_2^2 - \int_{\Omega_g} |f(t)|^2 dt \\ &\geq \|f\|_2^2 - \mu(\Omega_g) \|f\|_\infty^2 \geq (1 - \mu(\Omega_g)\beta) \|f\|_2^2. \end{aligned} \quad (4.6)$$

Further, we have by (4.4)

$$(Tf, f)_r = \|\sqrt{p}f'\|_2^2 + \|q_+ f^2\|_1 - \|q_- f^2\|_1 \geq -\|q_- f^2\|_1 \geq -\alpha \|f\|_2^2.$$

This together with (4.6) yields

$$(Tf, f)_r \geq -\frac{\alpha \|g\|_\infty}{1 - \mu(\Omega_g)\beta} (f, f)_r. \quad (4.7)$$

Obviously, the inequality in (4.7) holds also for  $f \in D(T) \setminus D_-(T)$  and, thus, for all  $f \in D(T)$ . This implies (4.5)  $\square$

Next we recall estimates on the  $L^\infty$ -norm of functions in  $D(T)$  from [2].

**Lemma 4.4.** *Assume Hypothesis 3.1. Then the following assertions hold for all  $f \in D(T)$ .*

- (i) *If  $1/p \in L^\eta(\mathbb{R})$ , where  $\eta \in [1, \infty)$ , then*

$$\|f\|_\infty \leq \left( \frac{2\eta - 1}{\eta} \sqrt{\|1/p\|_\eta} \|\sqrt{p}f'\|_2 \right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{\eta-1}{2\eta-1}}. \quad (4.8)$$

- (ii) *If  $1/p \in L^\infty(\mathbb{R})$  then*

$$\|f\|_\infty \leq \left( 2\sqrt{\|1/p\|_\infty} \|\sqrt{p}f'\|_2 \|f\|_2 \right)^{1/2}. \quad (4.9)$$

Moreover, for every  $\varepsilon > 0$  and all  $n \in \mathbb{Z}$  one has

$$\sup_{t \in [n, n+1]} |f(t)|^2 \leq \varepsilon \|1/p\|_\infty \int_n^{n+1} p(t) |f'(t)|^2 dt + \left(1 + \frac{1}{\varepsilon}\right) \int_n^{n+1} |f(t)|^2 dt. \quad (4.10)$$

*Proof.* The estimates (4.8) and (4.9) are proved in [2, Lemma 4.1]. For the convenience of the reader we verify the estimate (4.10), which is a variant of [19, Lemma 9.32]. Let  $\varepsilon > 0$  and  $n \in \mathbb{Z}$ . Then for  $f \in D(T)$  and  $x, y \in [n, n+1]$

$$|f(x)|^2 = |f(y)|^2 + 2 \operatorname{Re} \int_y^x f'(t) \overline{f(t)} dt.$$

By the mean value theorem we can choose  $y$  such that  $|f(y)|^2 = \int_n^{n+1} |f(t)|^2 dt$ . Thus, by the Cauchy–Schwarz inequality and  $2\alpha\beta \leq \alpha^2 + \beta^2$  for  $\alpha, \beta \in \mathbb{R}$  we obtain

$$\begin{aligned} |f(x)|^2 &\leq \int_n^{n+1} |f(t)|^2 dt \\ &\quad + 2 \left( \frac{1}{\varepsilon} \int_n^{n+1} |f(t)|^2 dt \right)^{1/2} \cdot \left( \|1/p\|_\infty \varepsilon \int_n^{n+1} p(t) |f'(t)|^2 dt \right)^{1/2} \\ &\leq \varepsilon \|1/p\|_\infty \int_n^{n+1} p(t) |f'(t)|^2 dt + \left(1 + \frac{1}{\varepsilon}\right) \int_n^{n+1} |f(t)|^2 dt, \end{aligned}$$

which leads to (4.10).  $\square$

For the proofs of Theorems 3.2 – 3.5 it is no restriction to consider  $f \in D_-(T) \setminus \{0\}$  and to assume that  $q_-$  is positive on a set of positive Lebesgue measure.

*Proof of Theorem 3.2.* Let  $1/p \in L^\infty(\mathbb{R})$  and consider  $\alpha, \beta$  as in (3.4). Choose  $\varepsilon = (2\|q_- \|_{\mathfrak{u}} \|1/p\|_\infty)^{-1} > 0$ . The estimate in (4.10) of Lemma 4.4 yields

$$\begin{aligned} \|q_- f^2\|_1 &= \int_{\mathbb{R}} q_-(t) |f(t)|^2 dt \\ &\leq \|q_- \|_{\mathfrak{u}} \sum_{n \in \mathbb{Z}} \sup_{t \in [n, n+1]} |f(t)|^2 \\ &\leq \|q_- \|_{\mathfrak{u}} \left( \varepsilon \|1/p\|_\infty \|\sqrt{p} f'\|_2^2 + \left(1 + \frac{1}{\varepsilon}\right) \|f\|_2^2 \right) \quad (4.11) \\ &= \frac{1}{2} \|\sqrt{p} f'\|_2^2 + (\|q_- \|_{\mathfrak{u}} + 2\|1/p\|_\infty \|q_- \|_{\mathfrak{u}}^2) \|f\|_2^2 \\ &= \frac{1}{2} \|\sqrt{p} f'\|_2^2 + \frac{\alpha}{2} \|f\|_2^2. \end{aligned}$$

Together with Lemma 4.2 we obtain

$$\|\sqrt{p} f'\|_2^2 = 2\|\sqrt{p} f'\|_2^2 - \|\sqrt{p} f'\|_2^2 \leq 2\|q_- f^2\|_1 - \|\sqrt{p} f'\|_2^2 \leq \alpha \|f\|_2^2.$$

With (4.9) in Lemma 4.4 and (4.11) we see

$$\|f\|_\infty^2 \leq 2\sqrt{\|1/p\|_\infty \alpha} \|f\|_2^2 = \beta \|f\|_2^2 \quad \text{and} \quad \|q_- f^2\|_1 \leq \alpha \|f\|_2^2$$

and hence Lemma 4.3 leads to the statements in Theorem 3.2.  $\square$

*Proof of Theorem 3.5.* Suppose that  $1/p \in L^\eta(\mathbb{R})$  and  $q_- \in L^s(\mathbb{R})$ , where  $\eta, s \in [1, \infty)$  with  $\eta + s > 2$ . Since  $\eta + s > 2$  we obtain

$$2\eta s - \eta - s = \eta(s-1) + s(\eta-1) \geq s-1 + \eta-1 > 0.$$

Let  $\alpha$  and  $\beta$  as in (3.5) and (3.8), respectively. From Hölder's inequality we obtain

$$\begin{aligned} \|q_- f^2\|_1 &\leq \|f\|_\infty^{\frac{2}{s}} \int_{\mathbb{R}} |q_-(t)| |f(t)|^{\frac{2(s-1)}{s}} dt \\ &\leq \|f\|_\infty^{\frac{2}{s}} \left( \int_{\mathbb{R}} |q_-(t)|^s dt \right)^{\frac{1}{s}} \left( \int_{\mathbb{R}} |f(t)|^2 dt \right)^{\frac{s-1}{s}} \\ &= \|q_-\|_s \|f\|_\infty^{\frac{2}{s}} \|f\|_2^{\frac{2(s-1)}{s}}. \end{aligned} \quad (4.12)$$

Thus, together with Lemma 4.4 (i) and Lemma 4.2 we obtain

$$\begin{aligned} \|f\|_\infty^2 &= \left( \frac{\|f\|_\infty^{\frac{2(2\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}} \right)^{\frac{\eta s}{2\eta s - \eta - s}} \leq \left( \frac{\left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \|\sqrt{p}f'\|_2^2 \|f\|_2^{\frac{2(\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}} \right)^{\frac{\eta s}{2\eta s - \eta - s}} \\ &\leq \left( \left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \|q_-\|_s \right)^{\frac{\eta s}{2\eta s - \eta - s}} \|f\|_2^2 = \beta \|f\|_2^2. \end{aligned}$$

The estimate from (4.12) yields

$$\|q_- f^2\|_1 \leq \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha \|f\|_2^2.$$

Now consider the case  $q_- \in L^\infty(\mathbb{R})$ . Choose  $\alpha$  and  $\beta$  as in (3.5) and (3.8), respectively. Observe that

$$\|q_- f^2\|_1 \leq \|q_-\|_\infty \|f\|_2^2 = \alpha \|f\|_2^2. \quad (4.13)$$

Lemma 4.4 (i) in combination with Lemma 4.2 and (4.13) leads to

$$\begin{aligned} \|f\|_\infty^2 &\leq \left( \left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \|\sqrt{p}f'\|_2^2 \right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{2(\eta-1)}{2\eta-1}} \\ &\leq \left( \left( \frac{2\eta-1}{\eta} \right)^2 \|1/p\|_\eta \|q_-\|_\infty \right)^{\frac{\eta}{2\eta-1}} \|f\|_2^2 = \beta \|f\|_2^2. \end{aligned}$$

Now Theorem 3.5 follows from Lemma 4.3.  $\square$

*Proof of Theorem 3.3.* Consider first the case  $1/p \in L^\infty(\mathbb{R})$  and  $q_- \in L^s(\mathbb{R})$ , where  $s \in [1, \infty)$ . Let  $\alpha$  and  $\beta$  be as in (3.5). Again Hölder's inequality yields (4.12). Lemma 4.4 (ii), (4.12) and Lemma 4.2 imply

$$\begin{aligned} \|f\|_\infty^2 &= \left( \frac{\|f\|_\infty^4}{\|f\|_\infty^{\frac{2}{s}}} \right)^{\frac{s}{2s-1}} \leq \left( \frac{4 \|1/p\|_\infty \|\sqrt{p}f'\|_2^2 \|f\|_2^2}{\|f\|_\infty^{\frac{2}{s}}} \right)^{\frac{s}{2s-1}} \\ &\leq (4 \|1/p\|_\infty \|q_-\|_s)^{\frac{s}{2s-1}} \|f\|_2^2 = \beta \|f\|_2^2. \end{aligned}$$

By applying this to the estimate in (4.12) we arrive at

$$\|q_- f^2\|_1 \leq \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha \|f\|_2^2,$$

and again the statements in Theorem 3.3 follow from Lemma 4.3.

The assertion for  $1/p, q_- \in L^\infty(\mathbb{R})$  follows in a similar way. Consider  $\alpha, \beta$  in (3.5). As before (4.13) holds. Lemma 4.4 (ii) in combination with Lemma 4.2 and (4.13) implies

$$\|f\|_\infty^2 \leq 2\sqrt{\|1/p\|_\infty} \|\sqrt{p}f'\|_2 \|f\|_2 \leq 2\sqrt{\|1/p\|_\infty \|q_-\|_\infty} \|f\|_2^2 = \beta \|f\|_2^2. \quad \square$$

*Proof of Proposition 3.6.* Let  $f \in D_-(T)$ . In the case  $1/p, q_- \in L^1(\mathbb{R})$  Lemma 4.2 and Lemma 4.4 (i) yield

$$\|f\|_\infty^2 \leq \|1/p\|_1 \|\sqrt{p}f'\|_2^2 \leq \|1/p\|_1 \|q_- f^2\|_1 \leq \|1/p\|_1 \|q_-\|_1 \|f\|_\infty^2.$$

If  $\|1/p\|_1 \|q_-\|_1 < 1$ , then  $\|f\|_\infty = 0$  and hence  $D_-(T) = \{0\}$ . This implies  $(Tf, f)_r \geq 0$  for all  $f \in D(T)$  and hence  $\min \sigma(T) \geq 0$ .  $\square$

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