WEAK COUPLING AND SPECTRAL INSTABILITY FOR NEUMANN LAPLACIANS

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ABSTRACT. We prove an abstract criterion on spectral instability of nonnegative selfadjoint extensions of a symmetric operator and apply this to selfadjoint Neumann Laplacians on bounded Lipschitz domains, intervals, and graphs. Our results can be viewed as variants of the classical weak coupling phenomenon for Schrödinger operators in $L^2(\mathbb{R}^n)$ for $n = 1, 2$.

1. INTRODUCTION

We start by recalling the classical weak coupling phenomenon for Schrödinger operators, which goes back to Simon [41], [42]. For this purpose, let $-\Delta$ be the self-adjoint one-dimensional Laplacian in $L^2(\mathbb{R})$ defined on $H^2(\mathbb{R})$ and assume that the potential $V : \mathbb{R} \to \mathbb{R}$ satisfies $V \in L^1(\mathbb{R}; (1 + x^2)dx) \cap L^2(\mathbb{R}; dx)$, and V is not zero a.e. For $\alpha \in \mathbb{R}$ it follows that $-\Delta + \alpha V$ is self-adjoint in $L^2(\mathbb{R})$ and

$$\sigma_{ess}(-\Delta + \alpha V) = \sigma_{ess}(-\Delta) = \sigma(-\Delta) = [0, \infty). \tag{1.1}$$

It was shown in [41, Theorem 2.5] (see also [39, Theorem XIII.11]) that for any $\alpha < 0$ one has

$$\sigma_p(-\Delta + \alpha V) \cap (-\infty, 0) \neq \emptyset \quad \text{if and only if} \quad \int_{\mathbb{R}} V(x)\, dx \geq 0, \tag{1.2}$$

and hence, in particular, if $V \geq 0$, then $\sigma_p(-\Delta + \alpha V) \cap (-\infty, 0) \neq \emptyset$ for any $\alpha < 0$. The same result holds also for the self-adjoint Laplacian $-\Delta$ in $L^2(\mathbb{R}^2)$ under slightly different integrability conditions on the potential $V : \mathbb{R}^2 \to \mathbb{R}$, and it is also well known that the phenomenon of weakly coupled eigenvalues does not appear in dimensions $n \geq 3$. The works [41], [42] by Simon have inspired and influenced a lot of future research; they were followed by Klaus and Simon [28], [29], and Rauch [38]. A wealth of additional information can be found, for instance, in [6], [7], [12], [15], [21], [25]–[27], [33], [34], [36], [37], [39, Theorem XIII.11, p. 336–338]. For some other related more recent developments we refer the reader to [1], [8], [9], [10], [11], [13], [20], [30], [35], and the references cited therein.

The main objective of this note is to transfer these ideas from Schrödinger operators $-\Delta + \alpha V$ to an abstract setting that replaces the Laplacian by a nonnegative self-adjoint extension A of a densely defined closed nonnegative symmetric operator S in a Hilbert space $\mathfrak{H}$ and the potential by an appropriate nonnegative self-adjoint perturbation, also denoted by V, that is relatively form compact with respect to

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A. In our main abstract result Theorem 2.2 it is shown that under some additional mild assumptions A is spectrally unstable, that is, for any $\alpha < 0$ the perturbed self-adjoint operator $A + \alpha V$ has negative discrete eigenvalues. The proof of Theorem 2.2 is based on the Birman–Schwinger principle, see, for instance, [16, 31]. In fact, the essential assumptions to ensure the existence of weakly coupled negative eigenvalues of $A + \alpha V$, $\alpha < 0$, are $\ker(A) \neq \{0\}$ and $\ker(A) \not\subseteq \ker(V)$; roughly speaking the first assumption $\ker(A) \neq \{0\}$ ensures that the resolvent of A has a singularity at 0 and the second assumption $\ker(A) \not\subseteq \ker(V)$ is needed to preserve this singularity for the sandwiched resolvent $V^{1/2}(A - \mu I_{\mathfrak{H}})^{-1}V^{1/2}$ when $\mu < 0$ tends to 0. We note that for the special case where 0 is an isolated eigenvalue of finite multiplicity of A, our result would also follow from asymptotic perturbation theory.

Our general result applies directly to the Neumann Laplacian $-\Delta_N$ on a bounded interval (a, b) or on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, since in that case $\ker(-\Delta_N)$ is spanned by the constant function and if $V \geq 0$ is a multiplication operator which is relatively compact perturbation with respect to $-\Delta_N$, then certainly $\ker(-\Delta_N) \not\subseteq \ker(V)$ as otherwise $V = 0$ a.e. on (a, b) or Ω, respectively. As a consequence of our abstract result, Theorem 2.2, we conclude in Corollary 3.1 and Corollary 3.2 that for any $\alpha < 0$ and nonnegative function V, $V \neq 0$, such that $V \in L^p$ with $p \geq 2$ if $n = 1, 2$ and $p > 2n/3$ if $n \geq 3$, there exist weakly coupled negative bound states for the perturbed Neumann Laplacian $-\Delta_N + \alpha V$, that is,

$$\sigma(-\Delta_N + \alpha V) \cap (-\infty, 0) \neq \emptyset \quad \text{for any } \alpha < 0. \tag{1.3}$$

We note that weakly coupled bound states for Schrödinger operators in $\mathbb{R}^n$ exist only for $n = 1, 2$, whereas weakly coupled bound states for the perturbed Neumann Laplacian exist in *any* space dimension. We mention that our abstract result also applies to other self-adjoint nonnegative realizations A of the Laplacian on bounded domains with the property $\ker(A) \neq \{0\}$ (cf. Remark 3.3). The observations for the case of a bounded interval extend naturally to finite compact graphs, where the Neumann Laplacian corresponds to Kirchhoff or standard boundary conditions; see, Corollary 3.4. Furthermore, in Corollary 3.5 we consider a Sturm–Liouville operator with Neumann boundary conditions in $L^2((0, \infty))$ with 0 as embedded eigenvalue at the bottom of the essential spectrum.

Finally, a few remarks about the notation employed: Given a separable complex Hilbert space $\mathfrak{H}$, $(\,\cdot\,, \cdot\,)_{\mathfrak{H}}$ denotes the scalar product in $\mathfrak{H}$ (linear in the second factor), $\|\cdot\|_{\mathfrak{H}}$ the norm in $\mathfrak{H}$, and $I_{\mathfrak{H}}$ represents the identity operator on $\mathfrak{H}$. The domain and range of a linear operator T in $\mathfrak{H}$ are abbreviated by $\operatorname{dom}(T)$ and $\operatorname{ran}(T)$. The kernel (null space) of T is denoted by $\ker(T)$. The spectrum, point spectrum (i.e., the set of eigenvalues), essential spectrum, and resolvent set of a self-adjoint operator in $\mathfrak{H}$ will be abbreviated by $\sigma(\,\cdot\,)$, $\sigma_p(\,\cdot\,)$, $\sigma_{ess}(\,\cdot\,)$, and $\rho(\,\cdot\,)$, respectively. The space of compact linear operators in $\mathfrak{H}$ is denoted by $\mathcal{B}_\infty(\mathfrak{H})$. For $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, we will abbreviate $L^2(\Omega; d^n x)$ for simplicity by $L^2(\Omega)$, and $I_{L^2(\Omega)}$ for convenience by I.

2. Spectral instability of nonnegative self-adjoint extensions

Throughout this section suppose that S is a densely defined closed symmetric operator in a Hilbert space $\mathfrak{H}$ and assume that S is semibounded from below with the lower bound $\kappa \geq 0$, that is,

$$(Sf, f)_{\mathfrak{H}} \geq \kappa (f, f)_{\mathfrak{H}}, \quad f \in \operatorname{dom}(S). \tag{2.1}$$

Hypothesis 2.1. Let A be a nonnegative self-adjoint extension of S in $\mathfrak{H}$ such that $\ker A \neq \{0\}$ and let $V \geq 0$ be a self-adjoint operator in $\mathfrak{H}$ which is relatively compact with respect to A, that is,

$$\operatorname{dom}(A) \subseteq \operatorname{dom}(V) \quad \text{and} \quad V(A + I_{\mathfrak{H}})^{-1} \in \mathcal{B}_\infty(\mathfrak{H}). \tag{2.2}$$

One notes that the lower bound of A is $\kappa \geq 0$ and recalls that the Friedrichs extension A_F of S has the same lower bound as S. In the case of differential operators (see the next section) the reader may think of A in Hypothesis 2.1 as the self-adjoint Laplacian with Neumann boundary conditions. Another typical example for a self-adjoint extension of S satisfying Hypothesis 2.1 is the Krein–von Neumann extension A_K, the smallest nonnegative extension of S, which in the case $\kappa > 0$ is defined by

$$A_K = S^* \restriction \operatorname{dom}(A_K), \quad \operatorname{dom}(A_K) = \operatorname{dom}(S) \dotplus \ker(S^*) \tag{2.3}$$

(see, e.g., [3, Sect. 5.4], [14] and the references cited therein). We also note that the self-adjoint extension theory point of view is not strictly necessary for the following arguments and Theorem 2.2 below, however we find it useful to compare A in Hypothesis 2.1 with the extremal nonnegative self-adjoint extensions A_F and A_K. We will return to this topic elsewhere.

Our goal is to show that the lower bound 0 for A in Hypothesis 2.1 is not stable under arbitrary small negative perturbations αV. The relative compactness assumption in Hypothesis 2.1 ensures that the operators $A + \alpha V$, $\alpha \in \mathbb{R}$, are self-adjoint and that

$$\sigma_{ess}(A + \alpha V) = \sigma_{ess}(A), \tag{2.4}$$

see, for instance, [39, Theorem XIII.14 and Corollary 2]. Furthermore,

- (i) If $\alpha \geq 0$, then $A + \alpha V \geq 0$ and, in particular, $\sigma(A + \alpha V) \cap (-\infty, 0) = \emptyset$.
- (ii) If $\alpha < 0$, then $\sigma(A + \alpha V) \cap (-\infty, 0)$ is either empty or consists of discrete eigenvalues.

From Hypothesis 2.1 one obtains $V(A - zI_{\mathfrak{H}})^{-1} \in \mathcal{B}_\infty(\mathfrak{H})$, $z \in \rho(A)$, by using the resolvent identity. We also note that

$$V^{1/2}(A + I_{\mathfrak{H}})^{-1/2} \in \mathcal{B}_\infty(\mathfrak{H}) \tag{2.5}$$

by [17, Theorem 3.5 (i)]. Then one has $V^{1/2}(A + I_{\mathfrak{H}})^{-1} \in \mathcal{B}_\infty(\mathfrak{H})$,

$$V^{1/2}(A - zI_{\mathfrak{H}})^{-1} \in \mathcal{B}_\infty(\mathfrak{H}), \quad \text{and} \quad V^{1/2}(A - zI_{\mathfrak{H}})^{-1/2} \in \mathcal{B}_\infty(\mathfrak{H}), \quad z \in \rho(A). \tag{2.6}$$

It follows that $(A - zI_{\mathfrak{H}})^{-1}V^{1/2}$ and $(A - zI_{\mathfrak{H}})^{-1/2}V^{1/2}$, $z \in \rho(A)$, are densely defined bounded operators, whose closures coincide with the adjoints of the operators in (2.6) for $\bar{z} \in \rho(A)$, and hence also belong to $\mathcal{B}_\infty(\mathfrak{H})$. Therefore, the Birman-Schwinger family $K(z)$, defined by

$$K(z) := \overline{V^{1/2}(A - zI_{\mathfrak{H}})^{-1}V^{1/2}}, \quad z \in \rho(A), \tag{2.7}$$

satisfies

$$K(z) = V^{1/2}(A - zI_{\mathfrak{H}})^{-1/2}\overline{(A - zI_{\mathfrak{H}})^{-1/2}V^{1/2}} \in \mathcal{B}_\infty(\mathfrak{H}), \quad z \in \rho(A). \tag{2.8}$$

Thus, if $z \in \rho(A)$ and α^{-1} is not an eigenvalue of the compact operator $K(z)$, then $K(z) + \alpha^{-1}I_{\mathfrak{H}}$ is boundedly invertible and one verifies in the same way as in [16, Proof of Theorem 2.3] that in the present case $z \in \rho(A+\alpha V)$ and the resolvent formula

$$\begin{aligned}(A+\alpha V - zI_{\mathfrak{H}})^{-1} &= (A - zI_{\mathfrak{H}})^{-1} \\ &\quad - \overline{(A - zI_{\mathfrak{H}})^{-1}V^{1/2}}\big[K(z)+\alpha^{-1}I_{\mathfrak{H}}\big]^{-1}V^{1/2}(A-zI_{\mathfrak{H}})^{-1}, \\ &\qquad z \in \rho(A+\alpha V)\cap\rho(A),\end{aligned} \tag{2.9}$$

holds.

The next theorem is our main abstract result; it provides a sufficient condition for spectral instability of the self-adjoint operator A in Hypothesis 2.1.

Theorem 2.2. *Let A and V be as in Hypothesis 2.1 and assume, in addition, that* $\ker(A) \not\subseteq \ker(V)$. *Then*

$$\sigma(A+\alpha V)\cap(-\infty,0) \neq \emptyset \ \text{ for } \textbf{any } \alpha<0. \tag{2.10}$$

Proof. By assumption there exists $k \in \ker(A)$, $\|k\|_{\mathfrak{H}} = 1$, such that $Vk \neq 0$ and hence also $V^{1/2}k \neq 0$. From this we conclude that there exists $f \in \operatorname{dom}\big(V^{1/2}\big)$ such that $h = V^{1/2}f$ satisfies $(h,k)_{\mathfrak{H}} \neq 0$ as otherwise $k \in \big(\operatorname{ran}\big(V^{1/2}\big)\big)^{\perp} = \ker\big(V^{1/2}\big)$. We shall now make use of the orthogonal direct sum decomposition

$$\mathfrak{H} = \text{lin.span}\{k\} \,\oplus\, \big(\text{lin.span}\{k\}\big)^{\perp} \tag{2.11}$$

and denote the orthogonal projection in $\mathfrak{H}$ onto $(\text{lin.span}\{k\})^{\perp}$ by P. Then

$$V^{1/2}f = h = (h,k)_{\mathfrak{H}}k + Ph \tag{2.12}$$

and for $\nu<0$ it follows from $(A-\nu I_{\mathfrak{H}})^{-1}k = -\frac{1}{\nu}k$ and (2.11) that

$$\begin{aligned}\big(V^{1/2}(A-\nu I_{\mathfrak{H}})^{-1}V^{1/2}f,f\big)_{\mathfrak{H}} &= \big((A-\nu I_{\mathfrak{H}})^{-1}h,h\big)_{\mathfrak{H}} \\ &= \big((A-\nu I_{\mathfrak{H}})^{-1}((h,k)_{\mathfrak{H}}k+Ph),(h,k)_{\mathfrak{H}}k+Ph\big)_{\mathfrak{H}} \\ &= -\frac{|(h,k)_{\mathfrak{H}}|^2}{\nu}(k,k)_{\mathfrak{H}} + \big((A-\nu I_{\mathfrak{H}})^{-1}Ph,Ph\big)_{\mathfrak{H}} \\ &= -\frac{|(h,k)_{\mathfrak{H}}|^2}{\nu} + \int_0^\infty \frac{1}{\lambda-\nu}d(E_A(\lambda)Ph,Ph)_{\mathfrak{H}},\end{aligned}$$

where $E_A(\lambda)$, $\lambda\in\mathbb{R}$, denotes the family of spectral projections of the self-adjoint operator A. Since $(h,k)_{\mathfrak{H}} \neq 0$ the first term tends to $+\infty$ as $\nu\uparrow 0$ and by monotone convergence the spectral integral converges in $[0,+\infty]$ as $\nu \uparrow 0$. Hence, we conclude

$$\lim_{\nu\uparrow 0}\big(V^{1/2}(A-\nu I_{\mathfrak{H}})^{-1}V^{1/2}f,f\big)_{\mathfrak{H}} = +\infty. \tag{2.13}$$

We note that for $\nu<0$ the Birman–Schwinger operator $K(\nu)$ in (2.7)–(2.8) is nonnegative and compact. Furthermore, from (2.13) we conclude that

$$\lim_{\nu\uparrow 0}\big\|K(\nu)\big\|_{\mathcal{B}(\mathfrak{H})} = +\infty \tag{2.14}$$

and since the operator norm of the nonnegative compact operator $K(\nu)$, $\nu<0$, coincides with its largest eigenvalue we conclude that for any $\alpha<0$ there exist $\nu_\alpha<0$ and $k_\alpha\in\mathfrak{H}$, $k_\alpha\neq 0$, such that

$$K(\nu_\alpha)k_\alpha = -\frac{1}{\alpha}k_\alpha. \tag{2.15}$$

Now consider $f_\alpha = \overline{(A - \nu_\alpha I_{\mathfrak{H}})^{-1} V^{1/2}} k_\alpha$ (see also [31] or [16, Proof of Theorem 3.2] for the following arguments) and observe first that

$$\begin{aligned} k_\alpha &= \big[K(z) + \alpha^{-1} I_{\mathfrak{H}}\big]^{-1} \big[K(z) - K(\nu_\alpha)\big] k_\alpha \\ &= (z - \nu_\alpha) \big[K(z) + \alpha^{-1} I_{\mathfrak{H}}\big]^{-1} V^{1/2} (A - z I_{\mathfrak{H}})^{-1} \overline{(A - \nu_\alpha I_{\mathfrak{H}})^{-1} V^{1/2}} k_\alpha \\ &= (z - \nu_\alpha) \big[K(z) + \alpha^{-1} I_{\mathfrak{H}}\big]^{-1} V^{1/2} (A - z I_{\mathfrak{H}})^{-1} f_\alpha, \quad z \in \rho(A), \end{aligned} \tag{2.16}$$

and hence, in particular, $f_\alpha \neq 0$ as otherwise $k_\alpha = 0$. Using (2.16) we see on the one hand

$$\begin{aligned} &\overline{(A - z I_{\mathfrak{H}})^{-1} V^{1/2}} k_\alpha \\ &\quad = (z - \nu_\alpha) \overline{(A - z I_{\mathfrak{H}})^{-1} V^{1/2}} \big[K(z) + \alpha^{-1} I_{\mathfrak{H}}\big]^{-1} V^{1/2} (A - z I_{\mathfrak{H}})^{-1} f_\alpha \\ &\quad = (z - \nu_\alpha) \big[(A - z I_{\mathfrak{H}})^{-1} - (A + \alpha V - z I_{\mathfrak{H}})^{-1}\big] f_\alpha, \end{aligned} \tag{2.17}$$

where (2.9) was used in the last equality. On the other hand, by the resolvent identity one obtains

$$\begin{aligned} &\overline{(A - z I_{\mathfrak{H}})^{-1} V^{1/2}} k_\alpha \\ &\quad = \overline{(A - \nu_\alpha I_{\mathfrak{H}})^{-1} V^{1/2}} k_\alpha + (z - \nu_\alpha)(A - z I_{\mathfrak{H}})^{-1} \overline{(A - \nu_\alpha I_{\mathfrak{H}})^{-1} V^{1/2}} k_\alpha \\ &\quad = f_\alpha + (z - \nu_\alpha)(A - z I_{\mathfrak{H}})^{-1} f_\alpha. \end{aligned} \tag{2.18}$$

It follows from (2.17) and (2.18) that $(\nu_\alpha - z)(A + \alpha V - z I_{\mathfrak{H}})^{-1} f_\alpha = f_\alpha$ which implies $f_\alpha \in \operatorname{dom}(A + \alpha V)$ and $(A + \alpha V) f_\alpha = \nu_\alpha f_\alpha$. Hence ν_α is an eigenvalue of $A + \alpha V$, thus $\sigma(A + \alpha V) \cap (-\infty, 0) \neq \emptyset$ for any $\alpha < 0$. □

Remark 2.3. We note that for the unperturbed nonnegative self-adjoint operator A in Hypothesis 2.1 it is only assumed that $0 \in \sigma_p(A)$, but no further restrictions on the spectrum of A are required; for example, in general 0 may be an eigenvalue of infinite multiplicity or an accumulation point of positive spectrum of A. In the special case where 0 is an isolated eigenvalue of finite multiplicity of A, the spectral instability of A in Theorem 2.2 would already follow from well-known results in analytic perturbation theory, see, for instance, [24, Sect. VII.3], [39, Theorems XII.8, XII.9], [40, Ch. II] and monotonicity of eigenvalues. ◇

3. Spectral instability of the Neumann Laplacian

In this section we shall show that Theorem 2.2 applies to the Neumann Laplacian on bounded Lipschitz domains, (arbitrary) intervals, and graphs, and conclude spectral instability for certain classes of potentials V that are relatively compact.

In the following let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and let ν be the unit normal vector field pointing outwards on $\partial\Omega$. We shall use the notation

$$H^{3/2}_\Delta(\Omega) = \big\{ f \in H^{3/2}(\Omega) \,\big|\, \Delta f \in L^2(\Omega) \big\}, \tag{3.1}$$

where $H^{3/2}(\Omega)$ is the L^2-based Sobolev space on Ω of fractional order $3/2$. We recall from [2, 19] that the Dirichlet trace mapping $C^\infty(\overline{\Omega}) \ni f \mapsto f|_{\partial\Omega}$ and the Neumann trace mapping $C^\infty(\overline{\Omega}) \ni f \mapsto \nu \cdot \nabla f|_{\partial\Omega}$ extend by continuity to continuous surjective mappings

$$\tau_D : H^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega) \quad \text{and} \quad \tau_N : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega), \tag{3.2}$$

respectively, where $H^1(\partial\Omega)$ denotes the first-order L^2-based Sobolev space on $\partial\Omega$. In the next corollary we study the weak coupling behaviour of the Neumann Laplacian

$$A_N f = -\Delta f, \quad f \in \operatorname{dom}(A_N) = \big\{ g \in H^{3/2}_\Delta(\Omega) \,\big|\, \tau_N g = 0 \big\}, \tag{3.3}$$

which is self-adjoint in $L^2(\Omega)$, see, for instance, [2, Theorem 6.10] or [18, Theorem 2.6 and Lemma 4.8] and also [22].

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \in \mathbb{N}$, $n \geq 2$, suppose that A_N is the self-adjoint Neumann Laplacian in $L^2(\Omega)$, and assume that $V \neq 0$ is a nonnegative function such that $V \in L^p(\Omega)$ with $p \geq 2$ if $n = 2$ and $p > 2n/3$ if $n \geq 3$. Then*

$$(A_N + \alpha V) f = -\Delta f + \alpha V f, \quad f \in \operatorname{dom}(A_N + \alpha V) = \operatorname{dom}(A_N), \tag{3.4}$$

is self-adjoint in $L^2(\Omega)$,

$$V(A_N - zI)^{-1} \in \mathcal{B}_\infty\big(L^2(\Omega)\big), \quad z \in \rho(A_N), \tag{3.5}$$

and

$$\sigma(A_N + \alpha V) \cap (-\infty, 0) \neq \emptyset \ \text{ for } \textbf{any } \alpha < 0. \tag{3.6}$$

Moreover, for $0 < -\alpha$ sufficiently small, the unique eigenvalue $\nu(\alpha) \in (-\infty, 0)$ of $A_N + \alpha V$ satisfies

$$\nu(\alpha) \underset{\alpha \uparrow 0}{=} \frac{\alpha}{|\Omega|} \int_\Omega V(x)\, d^n x + O\big(\alpha^2\big), \tag{3.7}$$

where $|\Omega|$ abbreviates the volume of Ω.

Proof. Consider the densely defined closed symmetric operator

$$S f = -\Delta f, \quad f \in \operatorname{dom}(S) = H^2_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{H^2(\Omega)}}, \tag{3.8}$$

in $L^2(\Omega)$ and note that S is semibounded from below by $\kappa > 0$, where κ is the smallest eigenvalue of the Friedrichs (or Dirichlet) extension

$$A_F f = -\Delta f, \quad f \in \operatorname{dom}(A_F) = \big\{ g \in H^{3/2}_\Delta(\Omega) \,\big|\, \tau_D g = 0 \big\}; \tag{3.9}$$

cf. [2, Theorem 6.9 and Lemma 6.11] or [18, Theorem 2.10 and Lemma 3.4] and also [23, Theorem B.2]. The Neumann Laplacian A_N in (3.3) is a self-adjoint extension of S and one has $\ker(A_N) = \text{lin.span}\{1\}$. One notes that the condition $\ker(A_N) \not\subseteq \ker(V)$ in Theorem 2.2 is satisfied for the multiplication operator V as otherwise the constant function would be in $\ker(V)$, which is only possible if $V = 0$.

It remains to show that V is relatively compact with respect to A_N as then Hypothesis 2.1 is satisfied and the statement follows from Theorem 2.2. In order to see that V is relatively compact with respect to A_N we shall use that for $0 < \delta < 1$ one has

$$\|f\|_{L^{2q}(\Omega)} \leq C_q \|f\|_{H^{3/2-\delta/2}(\Omega)} \ \text{ for } \ q \in \begin{cases} [1, \infty] & \text{if } n = 2, \\ [1, n/(n-3+\delta)] & \text{if } n \in \mathbb{N},\ n \geq 3, \end{cases} \tag{3.10}$$

by [5, Theorem 8.12.6.I]. Let us consider the case $n \geq 3$ first. As Ω is bounded we have $L^{p_2}(\Omega) \subseteq L^{p_1}(\Omega)$, $1 \leq p_1 \leq p_2 \leq \infty$, and hence under our assumptions there exists $0 < \delta < 1$ such that $V \in L^p(\Omega)$, where $p = 2n/(3-\delta)$. This yields $V \in L^{2r}(\Omega)$, where $r = n/(3-\delta)$. For $s = n/(n-3+\delta)$ we have $1/r + 1/s = 1$ and the Hölder inequality together with (3.10) leads to

$$\|V f\|_{L^2(\Omega)} \leq \|V\|_{L^{2r}(\Omega)} \|f\|_{L^{2s}(\Omega)} \leq C_s \|V\|_{L^{2r}(\Omega)} \|f\|_{H^{3/2-\delta/2}(\Omega)}, \tag{3.11}$$

so that

$$V : H^{3/2-\delta/2}(\Omega) \to L^2(\Omega) \tag{3.12}$$

is bounded. In the case $n = 2$ it follows in the same way with $V \in L^{2r}(\Omega)$, $r = 1$, and $s = \infty$ that the mapping V in (3.12) is bounded.

Next, one observes that $(A_N + I)^{-1} : L^2(\Omega) \to H^{3/2}(\Omega)$ is bounded; this follows, for instance, from the norm equivalences on $\operatorname{dom}(A_N)$ in [2, Theorem 6.10]. As Ω is bounded it is clear that the embedding $H^{3/2}(\Omega) \hookrightarrow H^{3/2-\delta/2}(\Omega)$ is compact (see, e.g., [5, Theorem 8.12.6.IV]) and hence $(A_N + I)^{-1} : L^2(\Omega) \to H^{3/2-\delta/2}(\Omega)$ is compact. Together with (3.12) we obtain that $V(A_N + I)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact, that is, V is relatively compact with respect to A_N.

Finally, (3.7) is a consequence of analytic first-order Rayleigh–Schrödinger perturbation theory (see, e.g., [24, eq. (II.2.36), Sect. VII.3], [39, p. 5, Theorems XII.8, XII.9], [40, Ch. II]), since $|\Omega|^{-1/2}$ is the normalized eigenfunction corresponding to the simple discrete eigenvalue 0 of A_N. □

For completeness we also discuss the one-dimensional case for a finite interval $\Omega = (a, b)$. In this context we recall that the self-adjoint Neumann Laplacian in $L^2((a, b))$ is given by

$$A_N f = -f'', \quad f \in \operatorname{dom}(A_N) = \big\{g \in H^2((a, b)) \,\big|\, g'(a) = g'(b) = 0\big\}. \tag{3.13}$$

Corollary 3.2. *Let (a, b) be a finite interval, let A_N be the self-adjoint Neumann Laplacian in $L^2((a, b))$, and assume that $V \neq 0$ is a nonnegative function such that $V \in L^p((a, b))$ with $p \geq 2$. Then*

$$(A_N + \alpha V)f = -f'' + \alpha V f, \quad f \in \operatorname{dom}(A_N + \alpha V) = \operatorname{dom}(A_N), \tag{3.14}$$

is self-adjoint in $L^2((a, b))$,

$$V(A_N - zI)^{-1} \in \mathcal{B}_\infty\big(L^2((a, b))\big), \quad z \in \rho(A_N), \tag{3.15}$$

and

$$\sigma(A_N + \alpha V) \cap (-\infty, 0) \neq \emptyset \ \textit{ for } \textbf{any } \alpha < 0. \tag{3.16}$$

Moreover, for $0 < -\alpha$ sufficiently small, the unique eigenvalue $\nu(\alpha) \in (-\infty, 0)$ of $A_N + \alpha V$ satisfies

$$\nu(\alpha) \underset{\alpha \uparrow 0}{=} \frac{\alpha}{b - a} \int_a^b V(x)\, dx + O\big(\alpha^2\big). \tag{3.17}$$

Proof. Consider the densely defined closed symmetric operator

$$Sf = -f'', \quad f \in \operatorname{dom}(S) = \big\{g \in H^2((a, b)) \,\big|\, g(a) = g(b) = g'(a) = g'(b) = 0\big\}, \tag{3.18}$$

in $L^2((a, b))$ and note that S is semibounded from below by $\kappa = (\pi/(b - a))^2 > 0$. The Neumann Laplacian A_N in (3.13) is a self-adjoint extension of S and one has $\ker(A_N) = \text{lin.span}\{1\}$. Note that the condition $\ker(A_N) \not\subseteq \ker(V)$ in Theorem 2.2 is satisfied for the multiplication operator V as otherwise the constant function would be in $\ker(V)$, which is only possible if $V = 0$. We claim that V is relatively compact with respect to A_N. In fact, using the inequality

$$\|g\|_{L^\infty((a,b))} \leq C \|g\|_{H^1((a,b))}, \quad g \in H^1((a, b)), \tag{3.19}$$

one has

$$\begin{aligned} \|Vg\|_{L^2((a,b))} \leq \|V\|_{L^2((a,b))} \|g\|_{L^\infty((a,b))} \leq C \|V\|_{L^2((a,b))} \|g\|_{H^1((a,b))}, \\ g \in H^1((a, b)), \end{aligned} \tag{3.20}$$

and hence $V : H^1((a,b)) \to L^2((a,b))$ is bounded. Therefore, as $(A_N + I)^{-1} : L^2((a,b)) \to H^2((a,b))$ is bounded and the embedding $H^2((a,b)) \hookrightarrow H^1((a,b))$ is compact we see that $(A_N + I)^{-1} : L^2((a,b)) \to H^1((a,b))$ is compact and thus also $V(A_N + I)^{-1} : L^2((a,b)) \to L^2((a,b))$ is compact.

Relation (3.17) is the special one-dimensional case of (3.7) in Corollary 3.1. □

Remark 3.3. The observations in Corollaries 3.1 and 3.2 remain valid for more general classes of self-adjoint Laplacians. More precisely, if $\alpha \in L^\infty(\partial\Omega)$ is real-valued, then the Robin Laplacian

$$A_\alpha f = -\Delta f, \quad f \in \operatorname{dom}(A_\alpha) = \big\{g \in H^{3/2}_\Delta(\Omega) \,\big|\, \tau_N g = \alpha \tau_D g\big\}, \tag{3.21}$$

is self-adjoint in $L^2(\Omega)$ and if, in addition, A_α is nonnegative and $\ker(A_\alpha) \neq \{0\}$, then

$$\sigma(A_\alpha + \alpha V) \cap (-\infty, 0) \neq \emptyset \text{ for } \textbf{any } \alpha < 0 \tag{3.22}$$

by Theorem 2.2 under the same integrability assumptions on V as in Corollary 3.1 if $\ker(A_\alpha) \not\subseteq \ker(V)$ holds. The latter condition is satisfied, for instance, if $V(x) > 0$ for a.e. $x \in \Omega$. Similarly, in the case of a finite interval the Neumann realization A_N of $-d^2/dx^2$ in Corollary 3.2 can be replaced by any nonnegative self-adjoint realization A of $-d^2/dx^2$ in $L^2((a,b))$ such that $\ker(A) \neq \{0\}$. As $\ker(A) \subseteq \operatorname{lin.span}\{1, x\}$ in this case, it is clear that $\ker(A) \not\subseteq \ker(V)$ holds. ⋄

Next, we consider the case of the Neumann (or Kirchhoff) Laplacian on a compact finite (not necessarily connected) graph Γ, which consists of $e < \infty$ edges (finite intervals) $\mathcal{E}_n$, $n = 1, \dots, e$, and $v < \infty$ vertices $\mathcal{V}_m$, $m = 1, \dots, v$. One recalls from [4, 32] that the self-adjoint Neumann Laplacian in $L^2(\Gamma) = \oplus_{n=1}^e L^2(\mathcal{E}_n)$ is given by

$$\begin{aligned} &A_N f = (-f_n'')_{n=1}^e, \\ &f \in \operatorname{dom}(A_N) = \left\{ g = (g_n)_{n=1}^e \,\middle|\, \begin{matrix} g_n \in H^2(\mathcal{E}_n),\, g(x_i) = g(x_j),\, x_i, x_j \in \mathcal{V}_m, \\ \sum_{x_j \in \mathcal{V}_m} \partial g(x_j) = 0, m = 1, \dots, v, \end{matrix} \right\}, \end{aligned} \tag{3.23}$$

and that the multiplicity of $0 \in \sigma_p(A_N)$ equals the number of connected components of the metric graph Γ.

Corollary 3.4. *Let Γ be a compact finite graph, let A_N be the self-adjoint Neumann Laplacian in $L^2(\Gamma)$, and assume that $V \neq 0$ is a nonnegative function such that $V \in L^p(\Gamma)$ with $p \geq 2$. Then*

$$(A_N + \alpha V)f = A_N f + \alpha V f, \quad f \in \operatorname{dom}(A_N + \alpha V) = \operatorname{dom}(A_N), \tag{3.24}$$

is self-adjoint in $L^2(\Gamma)$,

$$V(A_N - zI)^{-1} \in \mathcal{B}_\infty\big(L^2(\Gamma)\big), \quad z \in \rho(A_N), \tag{3.25}$$

and

$$\sigma(A_N + \alpha V) \cap (-\infty, 0) \neq \emptyset \text{ for } \textbf{any } \alpha < 0. \tag{3.26}$$

The proof of Corollary 3.4 is similar to that of Corollary 3.2 and hence is not repeated here.

In the next corollary we consider a perturbed Neumann Laplacian in $L^2((0,\infty))$, where $0 \in \sigma_p(A)$ is an embedded eigenvalue.

Corollary 3.5. *Let*

$$A_N(q)f = -f'' + qf, \quad f \in \operatorname{dom}(A_N(q)) = \{g \in H^2((0,\infty)) \,|\, g'(0) = 0\}, \tag{3.27}$$

where

$$q(x) = -\frac{2}{x^2+1} + \frac{8x^2}{(x^2+1)^2} = \frac{6x^2-2}{(x^2+1)^2}, \quad x \geq 0, \tag{3.28}$$

and assume that $V \neq 0$ is a nonnegative function such that $V \in L^2((0,\infty))$. Then

$$(A_N(q) + \alpha V)f = -f'' + qf + \alpha V f, \quad f \in \operatorname{dom}(A_N(q) + \alpha V) = \operatorname{dom}(A_N(q)), \tag{3.29}$$

is self-adjoint in $L^2((0,\infty))$,

$$V(A_N(q) - zI)^{-1} \in \mathcal{B}_\infty\big(L^2((0,\infty))\big), \quad z \in \rho(A_N(q)), \tag{3.30}$$

$$\sigma_{ess}(A_N(q) + \alpha V) = \sigma_{ess}(A_N(q)) = [0,\infty), \tag{3.31}$$

and

$$\sigma(A_N(q) + \alpha V) \cap (-\infty, 0) \neq \emptyset \ \textit{ for } \textbf{any } \alpha < 0. \tag{3.32}$$

Proof. Since $q \in L^\infty((0,\infty))$, $A_N(q)$ is self-adjoint in $L^2((0,\infty))$ as the same is true for the unperturbed Neumann operator $A_N f = -f''$, $\operatorname{dom}(A_N) = \operatorname{dom}(A_N(q))$. It is also clear that ∞ is in the limit point case for the differential expression $-(d^2/dx^2) + q(x)$, $x \in [0,\infty)$, and since $q \in L^1((0,\infty))$ it follows from [3, Proposition 6.13.7] that $\sigma_{ess}(A_N(q)) = \sigma_{ess}(A_N) = [0,\infty)$. Alternatively, one can argue that the resolvent difference of the full-line Schrödinger operator associated with $-(d^2/dx^2) + q(x)$, $x \in \mathbb{R}$, in $L^2(\mathbb{R})$ and the direct sum of the corresponding two half-line Neumann operators in $L^2((-\infty,0)) \oplus L^2((0,\infty))$ is a rank-one operator and combine this with the fact that $q(x) = q(-x)$, $x \in [0,\infty)$, and the full-line Schrödinger operator has essential spectrum equal to $[0,\infty)$ as $\lim_{x\to\pm\infty} q(x) = 0$. Moreover, it is easy to see that 0 is a simple eigenvalue of $A_N(q)$ with corresponding normalized eigenfunction

$$f_0(x) = \frac{2}{\pi^{1/2}} \frac{1}{x^2+1}, \quad x \in [0,\infty), \quad \|f_0\|_{L^2((0,\infty))} = 1, \tag{3.33}$$

and it follows from

$$A_N(q) = BB^* \geq 0, \tag{3.34}$$

that $A_N(q)$ is nonnegative. Here,

$$\begin{aligned} &Bf = f' + \phi f, \quad f \in \operatorname{dom}(B) = H_0^1([0,\infty)), \\ &B^*g = -g' + \phi g, \quad g \in \operatorname{dom}(B^*) = H^1([0,\infty)), \end{aligned} \tag{3.35}$$

where

$$\phi(x) = f_0'(x)/f_0(x) = -\frac{2x}{x^2+1}, \quad x \in [0,\infty). \tag{3.36}$$

We also note that the condition $\ker(A_N(q)) \not\subseteq \ker(V)$ in Theorem 2.2 is satisfied for the multiplication operator V as otherwise $V = 0$. We claim that V is relatively compact with respect to $A_N(q)$. In fact, for $z \in \mathbb{C} \setminus [0,\infty)$ we have the identity

$$(A_N(q) - zI)^{-1} = (A_N - zI)^{-1} - (A_N - zI)^{-1} q (A_N(q) - zI)^{-1} \tag{3.37}$$

and $V(A_N - zI)^{-1} \in \mathcal{B}_\infty(L^2((0,\infty)))$ by [39, Problem 41] (for the half-line), and thus also $V(A_N(q) - zI)^{-1} \in \mathcal{B}_\infty(L^2((0,\infty)))$. This implies $\sigma_{ess}(A_N(q) + \alpha V) = \sigma_{ess}(A_N(q)) = [0,\infty)$ and hence (3.32) follows from Theorem 2.2. $\square$

Remark 3.6. Without going into more details we note that Corollary 3.5 permits the analog of (3.7) and (3.17) in the following form: For $0 < -\alpha$ sufficiently small, the unique eigenvalue $\nu(\alpha) \in (-\infty, 0)$ of $A_N(q) + \alpha V$ satisfies

$$\nu(\alpha) \underset{\alpha\uparrow 0}{=} 4\alpha\pi^{-1} \int_0^\infty \big(x^2+1\big)^{-2} V(x)\,dx + O\big(\alpha^2\big). \tag{3.38}$$

While (3.38) is not a result of analytic first-order Rayleigh–Schrödinger perturbation theory as 0 is not a discrete eigenvalue of $A_N(q)$, one can apply the Fredholm determinant approach developed by Simon [41] to arrive at (3.38). ⋄

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