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An estimate on the non-real spectrum of a singular indefinite Sturm-Liouville operator

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It will be shown with the help of the Birman-Schwinger principle that the non-real spectrum of the singular indefinite Sturm-Liouville operator $\operatorname{sgn}(\cdot)(-\mathrm{d}^2/\mathrm{d}x^2 + q)$ with a real potential $q \in L^1 \cap L^2$ is contained in a circle around the origin with radius $\|q\|_{L^1}^2$.

1 Introduction and main result

Consider the operators

$$A_0 f = \operatorname{sgn}(\cdot)(-f'') \quad \text{and} \quad Af := A_0 f + \operatorname{sgn}(\cdot) q f = \operatorname{sgn}(\cdot)\big(-f'' + qf\big), \qquad f \in H^2(\mathbb{R}), \tag{1}$$

in $L^2(\mathbb{R})$, where $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a real function with $\lim_{x\to\pm\infty} q(x) = 0$. Note that q is a relatively compact perturbation of A_0 (cf. Theorem 11.2.11 in [10]). The operator A (and A_0) is neither symmetric nor self-adjoint with respect to the usual scalar product in $L^2(\mathbb{R})$, but symmetric and self-adjoint with respect to the indefinite inner product

$$[f, g] := \int_{\mathbb{R}} \operatorname{sgn}(x) f(x)\overline{g(x)}\,\mathrm{d}x, \qquad f, g \in L^2(\mathbb{R}),$$

and the essential spectrum is given by $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_0) = \sigma(A_0) = \mathbb{R}$; cf. [9] and Corollary 4.4 in [2]. It is well known that the operator A may have non-real spectrum, see e.g. [5]. The main objective of this note is to prove the following theorem.

Theorem 1.1 *Let $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\lim_{x\to\pm\infty} q(x) = 0$. Then the non-real spectrum of A consists only of isolated eigenvalues and every non-real eigenvalue λ of A satisfies $|\lambda| \le \|q\|_{L^1}^2$.*

This result improves the bounds in [6] for certain potentials and is based on the techniques in [1]. For further bounds on the non-real spectrum of indefinite Sturm-Liouville operators we refer to [4] for the case of a bounded potential q and [3, 7, 8, 11–13] for the regular case.

2 Proof of Theorem 1.1

Lemma 2.1 *For every $\lambda \in \mathbb{C}^+$ the resolvent of A_0 is an integral operator of the form*

$$\big[(A_0 - \lambda)^{-1} g\big](x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y)\,\mathrm{d}y, \quad g \in L^2(\mathbb{R}),$$

with a kernel function K_λ which is bounded by $|K_\lambda(x, y)| \le |\lambda|^{-\frac{1}{2}}$.

Proof. For $\lambda \in \mathbb{C}^+$ consider the solutions u, v of the differential equation $-\operatorname{sgn}(\cdot)f'' = \lambda f$ defined by

$$u(x) = \begin{cases} e^{i\sqrt{\lambda}x}, & x \ge 0, \\ \overline{\alpha} e^{\sqrt{\lambda}x} + \alpha e^{-\sqrt{\lambda}x}, & x < 0, \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \alpha e^{i\sqrt{\lambda}x} + \overline{\alpha} e^{-i\sqrt{\lambda}x}, & x \ge 0, \\ e^{\sqrt{\lambda}x}, & x < 0, \end{cases}$$

where $\alpha = \frac{1-i}{2}$. For a non-real λ we define $\sqrt{\lambda}$ as the principle value of the square root, so that, $\operatorname{Re}\sqrt{\lambda} > 0$ and $\operatorname{Im}\sqrt{\lambda} > 0$ for $\lambda \in \mathbb{C}^+$. As the Wronskian determinant equals $2\alpha\sqrt{\lambda}$ these two solutions are linearly independent. Moreover, for all $x \in \mathbb{R}$ the restrictions $u|_{(x,\infty)}$ and $v|_{(-\infty,x)}$ are square integrable functions. One verifies that for $g \in L^2(\mathbb{R})$

$$(T_\lambda g)(x) := \frac{1}{2\alpha\sqrt{\lambda}}\left(u(x)\int_{-\infty}^{x} v(y)\operatorname{sgn}(y) g(y)\,\mathrm{d}y + v(x)\int_{x}^{\infty} u(y)\operatorname{sgn}(y) g(y)\,\mathrm{d}y\right) \tag{2}$$

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is a solution of $-\operatorname{sgn}(\cdot)f'' - \lambda f = g$. It remains to show that T_λ is a bounded operator in $L^2(\mathbb{R})$. Rearranging the terms in (2) one sees that $(T_\lambda g)(x) = (2\alpha\sqrt{\lambda})^{-1}\int_{\mathbb{R}} (k_1(x,y) + k_2(x,y))\, g(y)\,\mathrm{d}y$ for $g \in L^2(\mathbb{R})$ with

$$k_1(x,y) := \begin{cases} \alpha e^{i\sqrt{\lambda}(x+y)}, & x>0,\ y>0,\\ -e^{\sqrt{\lambda}(ix+y)}, & x>0,\ y<0,\\ e^{\sqrt{\lambda}(x+iy)}, & x<0,\ y>0,\\ -\overline{\alpha}e^{\sqrt{\lambda}(x+y)}, & x<0,\ y<0, \end{cases} \quad \text{and} \quad k_2(x,y) := \begin{cases} \overline{\alpha} e^{i\sqrt{\lambda}|x-y|}, & x>0,\ y>0,\\ 0, & x>0,\ y<0,\\ 0, & x<0,\ y>0,\\ -\alpha e^{-\sqrt{\lambda}|x-y|}, & x<0,\ y<0. \end{cases}$$

We have $k_1 \in L^2(\mathbb{R}^2)$. Calculating the resolvents of the self-adjoint operator $-\mathrm{d}^2/\mathrm{d}x^2$ at the points $\pm\lambda$ (cf. Satz 11.26 in [14]) yields

$$\int_{\mathbb{R}} k_2(x,y)g(y)\,\mathrm{d}y = \pm 2\alpha\sqrt{\lambda}\left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mp \lambda\right)^{-1}(\mathbf{1}_{\mathbb{R}^\pm} g)\right](x), \quad g \in L^2(\mathbb{R}), \quad x \in \mathbb{R}^\pm,$$

where $\mathbf{1}_{\mathbb{R}^+}$ and $\mathbf{1}_{\mathbb{R}^-}$ denote the characteristic functions of the positive and negative half-lines, respectively. Hence, T_λ is a bounded operator in $L^2(\mathbb{R})$ and $(A_0 - \lambda)^{-1} = T_\lambda$. It is easy to see that the sum $k_1 + k_2$ is bounded by $2|\alpha| = \sqrt{2}$. Defining $K_\lambda(x,y) := \frac{1}{2\alpha\sqrt{\lambda}}\big(k_1(x,y) + k_2(x,y)\big)$ completes the proof. □

Proof of Theorem 1.1. We assume $\|q\|_{L^1} \neq 0$ as otherwise there are no non-real eigenvalues of A. Since the operator A is a self-adjoint operator with respect to $[\cdot,\cdot]$ the point spectrum of A is symmetric with respect to the real line and hence it suffices to consider an eigenvalue $\lambda \in \mathbb{C}^+$ with corresponding eigenfunction $f \in \operatorname{dom}(A) = H^2(\mathbb{R})$. Note, that f is bounded, since $f \in H^2(\mathbb{R})$. As $Af = \lambda f$ we have in terms of the unperturbed operator A_0

$$(A_0 - \lambda)f = -\operatorname{sgn}(\cdot)qf \in L^2(\mathbb{R}). \tag{3}$$

Setting $q^{\frac{1}{2}}(x) := \operatorname{sgn}\big(q(x)\big)|q(x)|^{\frac{1}{2}}$ we have $|q|^{\frac{1}{2}}q^{\frac{1}{2}} = q$, and hence (3) and $\lambda \in \rho(A_0)$ yield

$$g := q^{\frac{1}{2}}f = -q^{\frac{1}{2}}(A_0-\lambda)^{-1}\Big(\operatorname{sgn}(\cdot)|q|^{\frac{1}{2}}q^{\frac{1}{2}}f\Big) = -q^{\frac{1}{2}}(A_0-\lambda)^{-1}\Big(\operatorname{sgn}(\cdot)|q|^{\frac{1}{2}}g\Big).$$

Here the boundedness of f implies $g \in L^2(\mathbb{R})$. Now with Lemma 2.1 we estimate

$$\begin{aligned} \|g\|_{L^2}^2 &= \int_{\mathbb{R}} |g(x)| \cdot \left|\Big(-q^{\frac{1}{2}}(A_0-\lambda)^{-1}\Big(\operatorname{sgn}(\cdot)|q|^{\frac{1}{2}}g\Big)\Big)(x)\right| \,\mathrm{d}x \\ &\le \int_{\mathbb{R}} \left|q^{\frac{1}{2}}(x)g(x)\right| \int_{\mathbb{R}} |K_\lambda(x,y)| \left|q^{\frac{1}{2}}(y)g(y)\right| \,\mathrm{d}y\,\mathrm{d}x \\ &\le |\lambda|^{-\frac{1}{2}}\left(\int_{\mathbb{R}}\left|q^{\frac{1}{2}}(x)g(x)\right|\,\mathrm{d}x\right)^2 \le |\lambda|^{-\frac{1}{2}}\|g\|_{L^2}^2 \int_{\mathbb{R}}\left|q^{\frac{1}{2}}(x)\right|^2\mathrm{d}x = |\lambda|^{-\frac{1}{2}}\|g\|_{L^2}^2\,\|q\|_{L^1}. \end{aligned}$$

Since g is non-trivial the estimate $|\lambda| \le \|q\|_{L^1}^2$ follows. □

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