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ORIGINAL ARTICLE

Approximation of Dirac operators with δ-shell potentials in the norm resolvent sense, II: Quantitative results

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Abstract
This paper is devoted to the approximation of two- and three-dimensional Dirac operators $H_{\widetilde{V}\delta_\Sigma}$ with combinations of electrostatic and Lorentz scalar δ-shell interactions in the norm resolvent sense. Relying on results from Behrndt, Holzmann, and Stelzer-Landauer [Math. Nachr. **298** (2025), 2499–2546], an explicit smallness condition on the coupling parameters is derived so that $H_{\widetilde{V}\delta_\Sigma}$ is the limit of Dirac operators with scaled electrostatic and Lorentz scalar potentials. Via counterexamples it is shown that this condition is sharp. The approximation of $H_{\widetilde{V}\delta_\Sigma}$ for larger coupling constants is achieved by adding an additional scaled magnetic term.

1 | INTRODUCTION

Dirac operators describe relativistic spin 1/2 particles in a quantum mechanical framework [39] and they play an important role in the mathematical description of graphene [1, 26]. In order to model interactions that are supported in a small neighborhood of a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$, Dirac operators with δ-shell potentials are used. These operators have been studied intensively in the recent years, see for instance [2, 3, 5–7, 9–13, 31], and are formally given by

$$H_{\widetilde{V}\delta_\Sigma} = -i(\alpha \cdot \nabla) + m\beta + \widetilde{V}\delta_\Sigma, \tag{1.1}$$

where $\alpha \cdot \nabla, \beta$ are as in Section 1.1 (vi), $m \in \mathbb{R}$ is the particle mass, δ_Σ is the δ-distribution supported on a C^2-smooth curve Σ in $\mathbb{R}^2$ or surface Σ in $\mathbb{R}^3$, and $\widetilde{V}$ is a symmetric matrix-valued function which models the interaction on Σ.

The main objective of the present paper-which is a continuation of our investigations in [8]-is the approximation of $H_{\widetilde{V}\delta_\Sigma}$ by Dirac operators with strongly localized potentials; here we complement the qualitative analysis in [8] with sharp quantitative results. We also refer the reader to [8] for a detailed introduction, overview, and references on the topic. The approximation of $H_{\widetilde{V}\delta_\Sigma}$ serves to justify the viewpoint that the expression in (1.1) is an idealized replacement for more realistic Dirac operators describing strongly localized interactions and was first considered in one dimension in [35], see also [14, 15, 17–19, 40]. Furthermore, in the multidimensional setting, strong resolvent convergence was shown in [10, 13, 24, 42] under various assumptions on $\widetilde{V}$ and Σ. Finally, in our recent paper [8], it was shown that $H_{\widetilde{V}\delta_\Sigma}$ can be approximated in the norm resolvent sense, if the matrix function $\widetilde{V}$ satisfies an implicit smallness condition. In this paper, we replace this implicit condition by an explicit and sharp condition for a specific class of interaction matrices, namely those that model electrostatic and Lorentz scalar interactions, see (1.9) and (1.11) below.

Let us recall the problem setting and the main result from [8]. The space dimension is denoted by $\theta \in \{2, 3\}$ and we set $N = 2$ for $\theta = 2$ and $N = 4$ for $\theta = 3$. Assume that Σ is the boundary of a C^2-domain $\Omega_+ \subset \mathbb{R}^\theta$ as in Section 1.1 (vii) and that ν is the unit normal vector field on Σ which points outwards of Ω_+. Note that Ω_+ can be any bounded C^2-domain, but also a wide class of unbounded domains is allowed in our assumptions. Then, define

$$\iota : \Sigma \times \mathbb{R} \to \mathbb{R}^\theta, \quad \iota(x_\Sigma, t) := x_\Sigma + t\nu(x_\Sigma), \quad (x_\Sigma, t) \in \Sigma \times \mathbb{R}, \tag{1.2}$$

and for $\varepsilon \in (0, \infty)$ set $\Omega_\varepsilon := \iota(\Sigma \times (-\varepsilon, \varepsilon))$, which is the so-called *tubular neighborhood* of Σ. It follows from the assumptions on Σ that one can fix an $\varepsilon_1 > 0$ such that $\iota|_{\Sigma\times(-\varepsilon_1,\varepsilon_1)}$ is injective; cf. [8, Proposition 2.4]. Next, we fix a symmetric matrix-valued function

$$V = V^* \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N}), \tag{1.3}$$

where W^1_∞ is the L^∞-based Sobolev space of once weakly differentiable functions, and

$$q \in L^\infty((-1,1); [0,\infty)) \ \text{ such that } \ \int_{-1}^{1} q(s)\,ds = 1, \tag{1.4}$$

and we introduce for $\varepsilon \in (0, \varepsilon_1)$ the strongly localized potentials

$$V_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon} V(x_\Sigma) q\left(\frac{t}{\varepsilon}\right), & \text{if } x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ 0, & \text{if } x \notin \Omega_\varepsilon. \end{cases} \tag{1.5}$$

For $m \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_1)$, consider the operator

$$H_{V_\varepsilon} u := -i(\alpha\cdot\nabla)u + m\beta u + V_\varepsilon u, \quad \operatorname{dom} H_{V_\varepsilon} := H^1(\mathbb{R}^\theta; \mathbb{C}^N), \tag{1.6}$$

where H^k denotes the L^2-based Sobolev space of k-times weakly differentiable functions. Note that $H_{V_\varepsilon} = H + V_\varepsilon$ is self-adjoint in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ as $V_\varepsilon \in L^\infty(\mathbb{R}^\theta; \mathbb{C}^{N\times N})$ is symmetric and bounded, and the free Dirac operator $H = -i(\alpha\cdot\nabla) + m\beta$ is self-adjoint in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$; cf. Section 4.1.

Next, the rigorous mathematical definition of $H_{\widetilde{V}\delta_\Sigma}$ as an operator in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ is recalled. We set $\Omega_- = \mathbb{R}^\theta \setminus \overline{\Omega_+}$, write $u_\pm := u|_{\Omega_\pm}$ for $u : \mathbb{R}^\theta \to \mathbb{C}^N$, and the Dirichlet trace operator is denoted by $\boldsymbol{t}_\Sigma^\pm : H^1(\Omega_\pm; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$. Then, for $\widetilde{V} = \widetilde{V}^* \in L^\infty(\Sigma; \mathbb{C}^{N\times N})$ the differential operator $H_{\widetilde{V}\delta_\Sigma}$ in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ is defined by

$$\begin{aligned} H_{\widetilde{V}\delta_\Sigma} u &:= (-i(\alpha\cdot\nabla) + m\beta)u_+ \oplus (-i(\alpha\cdot\nabla) + m\beta)u_-, \\ \operatorname{dom} H_{\widetilde{V}\delta_\Sigma} &:= \Big\{ u \in H^1(\Omega_+; \mathbb{C}^N) \oplus H^1(\Omega_-; \mathbb{C}^N) : \\ &\qquad i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma^+ u_+ - \boldsymbol{t}_\Sigma^- u_-) + \frac{\widetilde{V}}{2}(\boldsymbol{t}_\Sigma^+ u_+ + \boldsymbol{t}_\Sigma^- u_-) = 0 \Big\}. \end{aligned} \tag{1.7}$$

Eventually, the application of $\cos(x)$ and $\operatorname{sinc}(x) = \frac{1}{x}\sin(x)$ to matrices is understood via power series. Now, we are ready to recall the main result from [8] in a compressed form; note that H_{V_ε} and $H_{\widetilde{V}\delta_\Sigma}$ are denoted in [8] by H_ε and $H_{\widetilde{V}}$, respectively.

Theorem 1.1 [8, Theorem 1.1]. *Let V and q be as in (1.3)–(1.4). Furthermore, assume that*

$$\cos\left(\tfrac{1}{2}(\alpha\cdot\nu)V\right)^{-1} \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N}) \tag{1.8}$$

and set $\widetilde{V} = V \operatorname{sinc}(\frac{1}{2}(\alpha\cdot\nu)V) \cos(\frac{1}{2}(\alpha\cdot\nu)V)^{-1}$. If

$$\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \|q\|_{L^\infty(\mathbb{R})}$$

is sufficiently small, then $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ and H_{V_ε} converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

Similar to previous approximation results, a nonlinear rescaling of V to $\widetilde{V}$ is observed in the limit, which is often related to Klein's paradox; cf. [13, 24, 35].

A direct application of Theorem 1.1 to check the convergence of H_{V_ε} is difficult, since it contains a nonexplicit smallness assumption on V. The main motivation in this paper is to make this smallness condition explicit, if V is of the specific form

$$V = \eta I_N + \tau\beta, \qquad \eta, \tau \in C_b^1(\Sigma; \mathbb{R}); \tag{1.9}$$

here $C_b^1(\Sigma; \mathbb{R})$ is the space of continuously differentiable functions with bounded derivatives (see Section 1.1 (viii)). In fact, such potentials are of particular physical interest since ηI_N and $\tau\beta$ model electrostatic and Lorentz scalar interactions, respectively, and they are, in the context of δ-shell potentials, studied in [2, 3, 6, 7, 9, 11, 12, 31]. Moreover, for V as in (1.9) the condition (1.8) and the rescaling $V \mapsto \widetilde{V}$ simplify substantially, as then the anti-commutation rules for the Dirac matrices yield $((\alpha\cdot\nu)V)^2 = dI_N$, with $d = \eta^2 - \tau^2$, and thus, the power series representations of cos and sinc lead to

$$\cos\left(\tfrac{1}{2}(\alpha\cdot\nu)V\right) = \cos\left(\frac{\sqrt{d}}{2}\right) I_N \quad \text{and} \quad \operatorname{sinc}\left(\tfrac{1}{2}(\alpha\cdot\nu)V\right) = \operatorname{sinc}\left(\frac{\sqrt{d}}{2}\right) I_N.$$

Hence, (1.8) simplifies to

$$\inf_{x_\Sigma\in\Sigma, k\in\mathbb{N}_0} |d(x_\Sigma) - (2k+1)^2\pi^2| > 0 \tag{1.10}$$

and we have

$$\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta, \qquad (\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\left(\frac{\sqrt{d}}{2}\right)(\eta, \tau), \tag{1.11}$$

where $\operatorname{tanc}(x) = \frac{\tan(x)}{x}$, see also Section 1.1 (iii). Moreover, in the present situation, $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ if

$$\inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma) - 4| > 0, \qquad \widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2, \tag{1.12}$$

see [31, Section 6], which is called the *noncritical* case.

In the main result of this paper, Theorem 2.1, we prove that if V is chosen as in (1.9), then the explicit smallness condition

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2, \tag{1.13}$$

already guarantees norm resolvent convergence of H_{V_ε} to $H_{\widetilde{V}\delta_\Sigma}$. Note that (1.13) via (1.11) implies

$$\sup_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| < 4 \tag{1.14}$$

and hence (1.12) is satisfied. This is already a first indication that it is not possible to relax the condition (1.13), as otherwise Dirac operators with critical δ-shell potentials could be approximated, which have very different spectral properties compared to the noncritical case. However, to approximate $H_{\widetilde{V}\delta_\Sigma}$ also for $\widetilde{V}$ with

$$\inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| > 4, \tag{1.15}$$

we make use of a well-known unitary transformation. Its approximation results, in a similar way to that described in [13, Section 8], in an additional strongly localized magnetic potential $V_{m,\varepsilon}$ that is added to H_{V_ε}. In Theorem 2.5, we then show that $H_{V_\varepsilon} + V_{m,\varepsilon}$ also converges in the norm resolvent sense to a Dirac operator with a combination of electrostatic and Lorentz scalar δ-shell potentials, where the rescaling of η and τ is different. In fact, as a consequence of this result and Theorem 2.1, we obtain Corollary 2.7, which states that every Dirac operator with a given δ-shell potential of the form

$\widetilde{V}\delta_\Sigma$, $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma; \mathbb{R})$, and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ fulfilling (1.14) or (1.15) can be approximated by a sequence of Dirac operators with strongly localized potentials in the norm resolvent sense.

In Section 3, we show that the condition (1.13) is sharp, that is, that H_{V_ε} does, in general, not converge to $H_{\widetilde{V}\delta_\Sigma}$, if (1.13) is not satisfied. To do so, we provide suitable counterexamples under the assumption of constant $\eta, \tau \in \mathbb{R}$. In Theorem 3.1, we consider the case $\widetilde{d} = 4$ for a compact hypersurface $\Sigma \subset \mathbb{R}^\theta$ and in Theorem 3.2 the situation $\widetilde{d} > 4$ for $\Sigma = \{0\} \times \mathbb{R}$. In both cases, we are able to show that if (1.13) is not fulfilled, then norm resolvent convergence would yield contradictory spectral implications for the limit operator $H_{\widetilde{V}\delta_\Sigma}$.

Finally, let us describe the structure of the paper. We start by introducing various notations and conventions in Section 1.1. Then, in Section 2, we prove the main results of this paper on the approximation of $H_{\widetilde{V}\delta_\Sigma}$ by Dirac operators with squeezed potentials in the norm resolvent sense, and we provide explicit conditions on the coefficients such that the associated convergence results hold. The proof of Theorem 2.1 relies on the same statement for the special case where Σ is a rotated graph and the technically complicated long proof of this special case is outsourced to Section 4. After Section 2, we complement in Section 3 the results of Section 2 by providing counterexamples which show that (1.13) is sharp.

1.1 | Notations and assumptions

In this section we introduce frequently used notations and assumptions. Let us start with a few general conventions.

(i) The letter $C > 0$ always denotes a generic constant, which may change in between lines.
(ii) The branch of the square root is fixed by $\operatorname{Im} \sqrt{w} > 0$ for $w \in \mathbb{C} \setminus [0, \infty)$.
(iii) For $w \in \mathbb{C} \setminus \{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}$ we define the function

$$\operatorname{tanc}(w) := \begin{cases} \frac{\tan(w)}{w}, & w \in \mathbb{C} \setminus \left(\{0\} \cup \{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}\right), \\ 1, & w = 0. \end{cases}$$

For $x \in \mathbb{R} \setminus \{0\}$, the equation $\operatorname{tanc}(ix) = \frac{\tanh(x)}{x}$ is valid, which can be extended by continuity to $x = 0$.
(iv) The symbol $|\cdot|$ is used for the absolute value, the Euclidean vector norm, or the Frobenius norm of a number, vector, or matrix, respectively. We write $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product in $\mathbb{C}^n$, $n \in \mathbb{N}$, which is anti-linear in the second argument.
(v) Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces and A be a linear operator from $\mathcal{H}$ to $\mathcal{G}$. The domain, kernel, and range of A are denoted by $\operatorname{dom} A$, $\ker A$, and $\operatorname{ran} A$, respectively. If A is bounded and everywhere defined, then we write $\|A\|_{\mathcal{H}\to\mathcal{G}}$ for its operator norm. The expression $[\cdot,\cdot]$ denotes the commutator of two operators. If $\mathcal{H} = \mathcal{G}$ and A is a closed operator, then the resolvent set, the spectrum, and the point spectrum of A are denoted by $\rho(A)$, $\sigma(A)$, and $\sigma_{\mathrm{p}}(A)$, respectively. In the case that A is self-adjoint, we denote the essential and discrete spectrum of A by $\sigma_{\mathrm{ess}}(A)$ and $\sigma_{\mathrm{disc}}(A)$, respectively.

Next, we fix the space dimension, introduce Dirac matrices, and define related notations.

(vi) By $\theta \in \{2, 3\}$ we denote the space dimension and we set $N = 2$ for $\theta = 2$ and $N = 4$ for $\theta = 3$. With the help of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we define the Dirac matrices $\alpha_1, \ldots, \alpha_\theta, \beta \in \mathbb{C}^{N\times N}$ for $\theta = 2$ by

$$\alpha_1 := \sigma_1, \quad \alpha_2 := \sigma_2, \quad \text{and} \quad \beta := \sigma_3, \tag{1.16}$$

and for $\theta = 3$ by

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3 \quad \text{and} \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{1.17}$$

where I_n is the $n \times n$-identity matrix, $n \in \mathbb{N}$. The Dirac matrices satisfy

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2I_N \delta_{jk} \quad \text{and} \quad \alpha_j \beta + \beta \alpha_j = 0 \quad j,k \in \{1,\dots,\theta\}, \tag{1.18}$$

where δ_{jk} denotes the Kronecker delta. Vectors in $\mathbb{C}^\theta$ are denoted by $x = (x_1,\dots,x_\theta)$ and we will often make use of the notations

$$\alpha \cdot \nabla := \sum_{j=1}^{\theta} \alpha_j \partial_j \quad \text{and} \quad \alpha \cdot x := \sum_{j=1}^{\theta} \alpha_j x_j, \quad x = (x_1,\dots,x_\theta) \in \mathbb{C}^\theta.$$

Finally, we use the notation $x = (x', x_\theta)$ with $x' \in \mathbb{C}^{\theta-1}$ and $x_\theta \in \mathbb{C}$.

In the upcoming item we introduce a class of C^2-hypersurfaces, which is convenient for the definition of trace and extension operators, as well as tubular neighborhoods. We remark that the C^2-domain Ω_+ appearing below is not assumed to be connected.

(vii) We assume that Σ is the boundary of an open set $\Omega_+ \subset \mathbb{R}^\theta$ which satisfies the following: There exist open sets $W_1,\dots,W_p \subset \mathbb{R}^\theta$, mappings $\zeta_1,\dots,\zeta_p \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ (see (viii) below), rotation matrices $\kappa_1,\dots,\kappa_p \in \mathbb{R}^{\theta\times\theta}$, and $\varepsilon_0 > 0$ such that
  (i) $\Sigma \subset \bigcup_{l=1}^p W_l$;
  (ii) if $x \in \partial\Omega_+ = \Sigma$, then there exists $l \in \{1,\dots,p\}$ such that $B(x,\varepsilon_0) \subset W_l$;
  (iii) $W_l \cap \Omega_+ = W_l \cap \Omega_l$, where $\Omega_l = \{\kappa_l(x', x_\theta) : x_\theta < \zeta_l(x'),\ (x', x_\theta) \in \mathbb{R}^\theta\}$, for $l \in \{1,\dots,p\}$.
  Furthermore, we set $\Sigma_l := \partial\Omega_l = \{\kappa_l(x', \zeta_l(x')) : x' \in \mathbb{R}^{\theta-1}\}$, $\Omega_- := \mathbb{R}^\theta \setminus \overline{\Omega_+}$, and denote the unit normal vector field at Σ that is pointing outwards of Ω_+ by ν. Moreover, for $u : \mathbb{R}^\theta \to \mathbb{C}^N$, we define $u_\pm := u|_{\Omega_\pm}$.

Now, we turn to the introduction of various function spaces.

(viii) If $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ is open, then $H^r(U)$ and $W^r_\infty(U)$ denote the L^2- and L^∞-based Sobolev spaces of order r, respectively; cf. [25, Chapter 3]. Moreover, if $k \in \mathbb{N} \cup \{\infty\}$, then we write $C_b^k(U)$ for the space that contains all $f \in C^k(U)$ such that f and all partial derivatives of f up to order k are bounded. Vector- or matrix-valued function spaces are defined in the natural way, that is, component-wise. Moreover, for function spaces with $\mathbb{C}^{n\times l}$-valued functions, $n, l \in \mathbb{N}$, we use the notations $H^r(U;\mathbb{C}^{n\times l})$, $W^r_\infty(U;\mathbb{C}^{n\times l})$, and $C_b^k(U;\mathbb{C}^{n\times l})$.

(ix) The trace spaces $H^r(\Sigma)$, $r \in [-2,2]$, $W^1_\infty(\Sigma)$, and $C_b^k(\Sigma)$, $k \in \{1,2\}$, are defined via local coordinates, a partition of unity, and the corresponding function spaces on open sets; cf. [8, Section 2]. Moreover, the well-defined and bounded Dirichlet trace operator is denoted by

$$\boldsymbol{t}_\Sigma^\pm : H^r(\Omega_\pm) \to H^{r-1/2}(\Sigma) \quad \text{and} \quad \boldsymbol{t}_\Sigma : H^r(\mathbb{R}^\theta) \to H^{r-1/2}(\Sigma),$$

$r \in (1/2, 5/2)$; cf. [22, Theorem 2], where we use for

$$u = u_+ \oplus u_- \in H^r(\mathbb{R}^\theta \setminus \Sigma) = H^r(\Omega_+) \oplus H^r(\Omega_-)$$

the shortened notation $\boldsymbol{t}_\Sigma^\pm u$ for $\boldsymbol{t}_\Sigma^\pm u_\pm$. Vector- or matrix-valued trace spaces and trace operators are defined component-wise.

(x) For a Hilbert space $\mathcal{H}$ the usual $L^2((-1,1))$-based Bochner Lebesgue space of $\mathcal{H}$-valued functions is denoted by $L^2((-1,1);\mathcal{H})$; cf. [8, Section 2.2]. In the case $\mathcal{H} = H^r(S;\mathbb{C}^N)$ with $S \in \{\Sigma, \mathbb{R}^{\theta-1}\}$, we write $\mathcal{B}^r(S)$ instead of $L^2((-1,1);H^r(S;\mathbb{C}^N))$, respectively. We also write $\|\cdot\|_r$ for the norm in $\mathcal{B}^r(S)$. In a similar way, we define

$$\|\cdot\|_{r\to r'} := \|\cdot\|_{\mathcal{B}^r(S)\to\mathcal{B}^{r'}(S)},$$

$$\|\cdot\|_{r\to\mathcal{H}} := \|\cdot\|_{\mathcal{B}^r(S)\to\mathcal{H}},$$

$$\|\cdot\|_{\mathcal{H}\to r'} := \|\cdot\|_{\mathcal{H}\to\mathcal{B}^{r'}(S)}.$$

We will use the bounded embedding

$$\mathfrak{J} : H^r(S;\mathbb{C}^N) \to \mathcal{B}^r(S), \quad \mathfrak{J}\varphi(t) := \varphi,$$

and its adjoint

$$\mathfrak{J}^* : \mathcal{B}^r(S) \to H^r(S;\mathbb{C}^N), \quad \mathfrak{J}^* f = \int_{-1}^{1} f(t)\,dt.$$

We also use the following convenient identification: Let $\mathcal{Q} \in L^\infty((-1,1))$ and $\mathcal{A}$ be bounded operator in $H^r(S;\mathbb{C}^N)$, $r \in [-2,2]$. Then, we identify

$$\mathcal{M}_\mathcal{Q} : \mathcal{B}^r(S) \to \mathcal{B}^r(S), \quad (\mathcal{M}_\mathcal{Q} f)(t) := \mathcal{Q}(t) f(t),$$

and

$$\mathcal{M}_\mathcal{A} : \mathcal{B}^r(S) \to \mathcal{B}^r(S), \quad (\mathcal{M}_\mathcal{A} f)(t) := \mathcal{A}(f(t)),$$

with $\mathcal{Q}$ and $\mathcal{A}$, respectively. Note that the norms $\|\mathcal{M}_\mathcal{Q}\|_{r\to r}$ and $\|\mathcal{M}_\mathcal{A}\|_{r\to r}$ are equal to $\|\mathcal{Q}\|_{L^\infty((-1,1))}$ and $\|\mathcal{A}\|_{H^r(S;\mathbb{C}^N)\to H^r(S;\mathbb{C}^N)}$, respectively.

Finally, we fix notations and state simple properties regarding Fourier transforms.

(xi) The expression $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^{\theta-1}$. Moreover, $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the partial Fourier transforms in $\mathbb{R}^\theta$ with respect to the first $\theta-1$ variables and the θth variable, respectively. These transforms are given for $\psi \in \mathcal{S}(\mathbb{R}^{\theta-1})$ and $u \in \mathcal{S}(\mathbb{R}^\theta)$ by

$$\begin{aligned}
\mathcal{F}\psi(\xi') &= \frac{1}{\sqrt{(2\pi)^{\theta-1}}} \int_{\mathbb{R}^{\theta-1}} \psi(x') e^{-i\langle x',\xi'\rangle} dx', && \xi' \in \mathbb{R}^{\theta-1},\\
\mathcal{F}_1 u(\xi) &= \frac{1}{\sqrt{(2\pi)^{\theta-1}}} \int_{\mathbb{R}^{\theta-1}} u(x',\xi_\theta) e^{-i\langle x',\xi'\rangle} dx', && \xi = (\xi',\xi_\theta) \in \mathbb{R}^\theta,\\
\mathcal{F}_2 u(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\xi',x_\theta) e^{-ix_\theta\xi_\theta} dx_\theta, && \xi = (\xi',\xi_\theta) \in \mathbb{R}^\theta,
\end{aligned}$$

and can be uniquely extended to bounded operators in $\mathcal{S}'(\mathbb{R}^{\theta-1})$ and $\mathcal{S}'(\mathbb{R}^\theta)$, where $\mathcal{S}'$ denotes the space of tempered distributions; cf. [33, Chapter IX]. Moreover, the application of the Fourier transform to vector- and matrix-valued functions or distributions is defined component-wise. The (usual) Fourier transform in $\mathbb{R}^\theta$ with respect to all variables is given by $\mathcal{F}_{1,2} := \mathcal{F}_1\mathcal{F}_2 = \mathcal{F}_2\mathcal{F}_1$.

2 | EXPLICIT CONDITIONS FOR THE APPROXIMATION OF DIRAC OPERATORS WITH δ-SHELL POTENTIALS IN THE NORM RESOLVENT SENSE

In the first main theorem of this paper, we replace the nonexplicit smallness condition from Theorem 1.1 by the simple and explicit condition (2.1) below. This condition will also turn out to be optimal later.

Theorem 2.1. *Let $q \in L^\infty((-1,1);[0,\infty))$ with $\int_{-1}^{1} q(s)\,ds = 1$ and assume that $\eta,\tau \in C_b^1(\Sigma;\mathbb{R})$ satisfy the condition*

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2. \tag{2.1}$$

Let V and V_ε be as in (1.9) and (1.5), and define $\widetilde{V}$ by (1.11). Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $r \in (0, 1/2)$, there exist $C > 0$ and $\varepsilon' \in (0, \varepsilon_1)$ such that

$$\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}, \quad \varepsilon \in (0, \varepsilon').$$

In particular, H_{V_ε} converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

The strategy to prove Theorem 2.1 is to reduce it to the case where Σ is a rotated C^2_b-graph, which is shown as a separate result in Section 4. The actual proof of Theorem 2.1 makes use of three preparatory results, which relate the resolvents of $H_{\widetilde{V}\delta_\Sigma}$ and H_{V_ε} to similar operators with potentials supported on the C^2_b-graphs Σ_l and tubular neighborhoods of Σ_l (see (vii) in Section 1.1), respectively, via a suitable partition of unity. We also remark that the condition (2.1) implies

$$\sup_{x_\Sigma \in \Sigma} \widetilde{d}(x_\Sigma) = \sup_{x_\Sigma \in \Sigma} (\widetilde{\eta}^2(x_\Sigma) - \widetilde{\tau}^2(x_\Sigma)) = \sup_{x_\Sigma \in \Sigma} 4\tan^2\left(\frac{\sqrt{d(x_\Sigma)}}{2}\right) < 4,$$

so that $\widetilde{V}$ in (1.11) is noncritical; cf. (1.12).

The first preparatory lemma is a slight variation of [8, Lemma B.2] and allows us to construct a suitable partition of unity for Σ.

Lemma 2.2. *Let $\Sigma \subset \mathbb{R}^\theta$, $\theta \in \{2, 3\}$, be a set as described in Section 1.1 (vii). Then, there exists a partition of unity $\varphi_1, \dots, \varphi_p \in C^\infty_b(\mathbb{R}^\theta; \mathbb{R})$ for Σ subordinate to the open cover $W_1, \dots, W_p$ of Σ. This partition of unity can be chosen such that $\varphi_1, \dots, \varphi_p$ is also a partition of unity for $\Sigma + B(0, \delta)$ for some $\delta > 0$ and $\operatorname{supp} \varphi_l + B(0, \delta) \subset W_l$ for all $l \in \{1, \dots, p\}$.*

Proof. According to [36, Appendix A, Lemmas 1.2 and 1.3], there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^\theta$, $M \in \mathbb{N}$, $0 < \delta < \frac{\varepsilon_0}{4}$, with ε_0 from Section 1.1 (vii), and a sequence of real-valued C^∞-functions $(\phi_n)_{n\in\mathbb{N}}$ such that $(B(x_n, \delta))_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}^\theta$, $(\phi_n)_{n\in\mathbb{N}}$ is a partition of unity for $\mathbb{R}^\theta$, $\operatorname{supp} \phi_n \subset B(x_n, \delta)$ for all $n \in \mathbb{N}$, every point $x \in \mathbb{R}^\theta$ is contained in at most M of the sets $B(x_n, \delta)$, and the derivatives (of any order) of the functions ϕ_n are uniformly bounded. Next, we define the set

$$Y := \{x_n : B(x_n, 2\delta) \cap \Sigma \neq \emptyset\}.$$

By construction, for all $x_n \notin Y$ one has $B(x_n, \delta) \cap (\Sigma + B(0, \delta)) = \emptyset$. Moreover, note that for all $x_n \in Y$, there exists an $l \in \{1, \dots, p\}$ such that $B(x_n, 2\delta) \subset W_l$. In fact, as $B(x_n, 2\delta) \cap \Sigma \neq \emptyset$, there exists a $y_\Sigma \in B(x_n, 2\delta) \cap \Sigma$ and thus item (vii) in Section 1.1 implies $B(y_\Sigma, \varepsilon_0) \subset W_l$ for an $l \in \{1, \dots, p\}$. Hence, for any $y \in B(x_n, 2\delta)$, one has

$$|y - y_\Sigma| \leq |y - x_n| + |x_n - y_\Sigma| < 4\delta < \varepsilon_0,$$

which shows $B(x_n, 2\delta) \subset W_l$. Define the sets

$$I_1 := \{n : x_n \in Y, B(x_n, 2\delta) \subset W_1\}$$

and for $l \in \{2, \dots, p\}$

$$I_l := \{n : x_n \in Y, B(x_n, 2\delta) \subset W_l, B(x_n, 2\delta) \not\subset W_k, k \in \{1, \dots, l-1\}\}.$$

Then, it is not difficult to see that

$$\varphi_l = \sum_{n \in I_l} \phi_n$$

is a partition of unity having the claimed properties. Moreover, the construction of φ_l, $l \in \{1, \dots, p\}$, also implies $\operatorname{supp} \varphi_l + B(0, \delta) \subset W_l$. □

In the following, we use a partition of unity as in Lemma 2.2 to connect Dirac operators with strongly localized potentials supported in Ω_ε with Dirac operators with strongly localized potentials supported in the tubular neighborhoods $\Omega_{\varepsilon,l}$ of the rotated C_b^2-graphs

$$\Sigma_l = \{\kappa(x', \zeta_l(x')) : x' \in \mathbb{R}^{\theta-1}\}, \qquad l \in \{1, \dots, p\};$$

cf. Section 1.1 (vii). Note that V is only defined on Σ and hence V is a priori only defined on a subset of Σ_l. Thus, to be able to define a strongly localized potential $V_{\varepsilon,l}$ in $\Omega_{\varepsilon,l}$ in the same way as V_ε in (1.5), we first construct suitable extensions V_l of V to Σ_l. To do so, we choose $\rho_\nu \in C^1(\mathbb{R};\mathbb{R})$ with $0 \le \rho_\nu \le 1$, $\rho_\nu(0) = 1$ and compact support in $(-\varepsilon_1, \varepsilon_1)$, where ε_1 is the number specified below (1.2). Since Σ is assumed to satisfy (vii) in Section 1.1, it is not difficult to show that for $\omega \in \{V, \eta, \tau\}$ the function

$$\omega_{\text{ext}}(x) = \begin{cases} \omega(x_\Sigma)\rho_\nu(t), & \text{if } x = x_\Sigma + t\nu(x_\Sigma) \in \Omega_{\varepsilon_1}, \\ 0, & \text{if } x \notin \Omega_{\varepsilon_1}, \end{cases} \tag{2.2}$$

is a C_b^1-extension of ω to $\mathbb{R}^\theta$, which is supported in Ω_{ε_1}. We then define the functions

$$V_l := V_{\text{ext}}|_{\Sigma_l} \in C_b^1(\Sigma; \mathbb{C}^{N\times N}), \;\; \eta_l := \eta_{\text{ext}}|_{\Sigma_l}, \;\; \tau_l := \tau_{\text{ext}}|_{\Sigma_l} \in C_b^1(\Sigma_l;\mathbb{R}), \tag{2.3}$$

and note that $V_l = \eta_l I_N + \tau_l \beta$ for $l \in \{1, \dots, p\}$; cf. (1.9). Clearly,

$$V_l|_{\Sigma_l\cap\Sigma} = V|_{\Sigma_l\cap\Sigma}.$$

Moreover, we mention that $d_l = \eta_l^2 - \tau_l^2$ satisfies by construction

$$\sup_{x_{\Sigma_l}\in\Sigma_l} d_l(x_{\Sigma_l}) < \frac{\pi^2}{4}, \qquad l \in \{1, \dots, p\}, \tag{2.4}$$

as $d = \eta^2 - \tau^2$ satisfies (2.1). With this extension at hand, we define $V_{\varepsilon,l}$ as V_ε in (1.5) (with Σ and V replaced by Σ_l and V_l) and in the same way as $H_{V_\varepsilon} = H + V_\varepsilon$ in (1.6) we define

$$H_{V_{\varepsilon,l}} = H + V_{\varepsilon,l}, \quad \text{dom}\, H_{V_{\varepsilon,l}} := H^1(\mathbb{R}^\theta;\mathbb{C}^N), \qquad l \in \{1, \dots, p\},$$

where H is the free Dirac operator (see (4.3)). Note that the operators $H_{V_{\varepsilon,l}}$ are self-adjoint in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$.

In the next lemma, we express the resolvent of H_{V_ε} in terms of the resolvents of $H_{V_{\varepsilon,l}}$. Here we fix the partition of unity $\varphi_1, \dots, \varphi_p \in C_b^\infty(\mathbb{R}^\theta;\mathbb{R})$ from Lemma 2.2 and we set $\varphi_{p+1} := 1 - \sum_{l=1}^p \varphi_l$.

Lemma 2.3. *Let the functions $V_\varepsilon, V_{\varepsilon,l}$ and the self-adjoint operators $H_{V_\varepsilon} = H + V_\varepsilon$, $H_{V_{\varepsilon,l}} = H + V_{\varepsilon,l}$ for $l \in \{1, \dots, p\}$ be as above, and set $H_{V_{\varepsilon,p+1}} := H$. Then, for $z \in \mathbb{C}$ such that $|\text{Im}\, z| > \sum_{l=1}^{p+1} \|\alpha\cdot\nabla\varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$, the operator*

$$I + \sum_{l=1}^{p+1} i(H_{V_{\varepsilon,l}} - z)^{-1}(\alpha\cdot\nabla\varphi_l)$$

is continuously invertible in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and the resolvent formula

$$(H_{V_\varepsilon} - z)^{-1} = \left(I + \sum_{l=1}^{p+1} i(H_{V_{\varepsilon,l}} - z)^{-1}(\alpha\cdot\nabla\varphi_l)\right)^{-1}\left(\sum_{l=1}^{p+1}(H_{V_{\varepsilon,l}} - z)^{-1}\varphi_l\right)$$

is valid for all $\varepsilon \in (0, \min\{\varepsilon_1, \delta\})$, where $\varepsilon_1 > 0$ and $\delta > 0$ are chosen as below (1.2) and as in Lemma 2.2, respectively.

Proof. Our first goal is to verify that for $\varphi_l \in C_b^\infty(\mathbb{R}^\theta;\mathbb{R})$ from the fixed partition of unity the identity

$$H_{V_\varepsilon}\varphi_l u = H_{V_{\varepsilon,l}}\varphi_l u, \qquad l \in \{1,\dots,p\}, \tag{2.5}$$

holds for $u \in \operatorname{dom} H_{V_\varepsilon} = \operatorname{dom} H_{V_{\varepsilon,l}} = H^1(\mathbb{R}^\theta;\mathbb{C}^N)$ and $\varepsilon \in (0,\min\{\delta,\varepsilon_1\})$. For this it suffices to show

$$V_\varepsilon(x)\varphi_l(x) = V_{\varepsilon,l}(x)\varphi_l(x), \qquad x \in \mathbb{R}^\theta. \tag{2.6}$$

In fact, it is obvious that $V_\varepsilon(x)\varphi_l(x) = V_{\varepsilon,l}(x)\varphi_l(x)$ for $x \notin \operatorname{supp}\varphi_l$. Next, consider $x \in \Omega_\varepsilon \cap \operatorname{supp}\varphi_l$. Then, there exists $(x_\Sigma, t) \in \Sigma \times (-\varepsilon,\varepsilon)$ such that $x = x_\Sigma + t\nu(x_\Sigma)$. The inclusion $\operatorname{supp}\varphi_l + B(0,\delta) \subset W_l$ implies $x_\Sigma \in W_l \cap \Sigma$. Hence, by Section 1.1 (vii) $x_\Sigma \in \Sigma_l$ and therefore $x = x_\Sigma + \nu_l(x_\Sigma) \in \Omega_{\varepsilon,l}$, where ν_l is the unit normal vector field at Σ_l, which has the same orientation on $\Sigma_l \cap \Sigma$ as ν. In turn, we have

$$V_\varepsilon(x)\varphi_l(x) = V(x_\Sigma)\frac{q\left(\frac{t}{\varepsilon}\right)}{\varepsilon}\varphi_l(x) = V_l(x_\Sigma)\frac{q\left(\frac{t}{\varepsilon}\right)}{\varepsilon}\varphi_l(x) = V_{\varepsilon,l}(x)\varphi_l(x).$$

If $x \in \Omega_{\varepsilon,l} \cap \operatorname{supp}\varphi_l$, one shows (2.6) in the same way. It remains to treat the case $x \in \Omega_\varepsilon^c \cap \Omega_{\varepsilon,l}^c \cap \operatorname{supp}\varphi_l$. However, then both sides of (2.6) are zero. Therefore, (2.6) and hence (2.5) are true.

Applying the product rule and using (2.5) yields for $u \in \operatorname{dom} H_{V_\varepsilon} = \operatorname{dom} H_{V_{\varepsilon,l}}$, $l \in \{1,\dots,p\}$, and $\varepsilon \in (0,\min\{\delta,\varepsilon_1\})$

$$\varphi_l H_{V_\varepsilon} u = H_{V_\varepsilon}\varphi_l u + i(\alpha\cdot\nabla\varphi_l)u = H_{V_{\varepsilon,l}}\varphi_l u + i(\alpha\cdot\nabla\varphi_l)u. \tag{2.7}$$

As $\varphi_1,\dots,\varphi_p$ also form a partition of unity for $\Sigma + B(0,\delta) \supset \Omega_\varepsilon$, $\varepsilon \in (0,\delta)$, we have $V_\varepsilon\varphi_{p+1} = 0$ for all $\varepsilon \in (0,\min\{\varepsilon_1,\delta\})$. Hence, the product rule and the convention $H = H_{V_{\varepsilon,p+1}}$ show that (2.7) remains valid for $l = p+1$ and $\varepsilon \in (0,\min\{\varepsilon_1,\delta\})$. For $z \in \mathbb{C}\setminus\mathbb{R}$ and $u \in \operatorname{dom} H_{V_\varepsilon} = H^1(\mathbb{R}^\theta;\mathbb{C}^N)$, it follows from (2.7) that

$$\begin{aligned}
&\left(\sum_{l=1}^{p+1}(H_{V_{\varepsilon,l}}-z)^{-1}\varphi_l\right)(H_{V_\varepsilon}-z)u \\
&= \sum_{l=1}^{p+1}\Big((H_{V_{\varepsilon,l}}-z)^{-1}(H_{V_{\varepsilon,l}}-z)\varphi_l u + i(H_{V_{\varepsilon,l}}-z)^{-1}(\alpha\cdot\nabla\varphi_l)u\Big) \\
&= \sum_{l=1}^{p+1}\Big(\varphi_l u + i(H_{V_{\varepsilon,l}}-z)^{-1}(\alpha\cdot\nabla\varphi_l)u\Big) \\
&= \left(I + \sum_{l=1}^{p+1} i(H_{V_{\varepsilon,l}}-z)^{-1}(\alpha\cdot\nabla\varphi_l)\right)u.
\end{aligned} \tag{2.8}$$

In particular, if $|\operatorname{Im} z| > \sum_{l=1}^{p+1}\|\alpha\cdot\nabla\varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$ holds, then it is clear that $I + \sum_{l=1}^{p+1} i(H_{V_{\varepsilon,l}}-z)^{-1}(\alpha\cdot\nabla\varphi_l)$ is continuously invertible in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and therefore

$$(H_{V_\varepsilon}-z)^{-1} = \left(I + \sum_{l=1}^{p+1} i(H_{V_{\varepsilon,l}}-z)^{-1}(\alpha\cdot\nabla\varphi_l)\right)^{-1}\left(\sum_{l=1}^{p+1}(H_{V_{\varepsilon,l}}-z)^{-1}\varphi_l\right).$$

□

Next, we provide an analogous formula as in the previous lemma for the resolvents of Dirac operators with δ-shell potentials. For this, recall first from (1.9) that $V = \eta I_N + \tau\beta$ with $\eta,\tau \in C_b^1(\Sigma;\mathbb{R})$, assume that $d = \eta^2 - \tau^2$ satisfies (2.1), and let $\widetilde{V} = \operatorname{tanc}\left(\frac{\sqrt{d}}{2}\right)V$ be as in (1.11). Furthermore, with $V_l = \eta_l I_N + \tau_l\beta$ and $\eta_l,\tau_l \in C_b^1(\Sigma;\mathbb{R})$, $l \in \{1,\dots,p\}$, as described below (2.2) we set (as in (1.11))

$$\widetilde{V}_l = \widetilde{\eta}_l I_N + \widetilde{\tau}_l\beta, \qquad (\widetilde{\eta}_l,\widetilde{\tau}_l) = \operatorname{tanc}\left(\frac{\sqrt{d_l}}{2}\right)(\eta_l,\tau_l), \qquad d_l = \eta_l^2 - \tau_l^2.$$

In the third preparatory lemma we shall again make use of the partition of unity $\varphi_1, \dots, \varphi_p \in C_b^\infty(\mathbb{R}^\theta; \mathbb{R})$ from Lemma 2.2 and as before we set $\varphi_{p+1} = 1 - \sum_{l=1}^p \varphi_l$.

Lemma 2.4. *Let the functions $\widetilde{V}$ and $\widetilde{V}_l$ be as above, let the self-adjoint operator $H_{\widetilde{V}\delta_\Sigma}$ be defined as in (1.7), and define in the same way (with $\widetilde{V}$ and Σ replaced by $\widetilde{V}_l$ and Σ_l) the self-adjoint operators*

$$H_{\widetilde{V}_l\delta_{\Sigma_l}}, \quad l \in \{1, \dots, p\}, \qquad \text{and} \qquad H_{\widetilde{V}_{p+1}\delta_{\Sigma_{p+1}}} := H.$$

Then, for $z \in \mathbb{C}$ such that $|\mathrm{Im}\, z| > \sum_{l=1}^{p+1} \|\alpha \cdot \nabla\varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$, the operator

$$I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha \cdot \nabla\varphi_l)$$

is continuously invertible in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ and the resolvent formula

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = \left(I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha \cdot \nabla\varphi_l)\right)^{-1} \left(\sum_{l=1}^{p+1} (H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}\varphi_l\right)$$

is valid.

Proof. Similar as in the proof of the previous lemma, we start by showing for $u \in \mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma}$ and $l \in \{1, \dots, p\}$ that

$$H_{\widetilde{V}\delta_\Sigma}\varphi_l u = H_{\widetilde{V}_l\delta_{\Sigma_l}}\varphi_l u. \tag{2.9}$$

First we argue that for $u \in \mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma}$ we have

$$\varphi_l u \in \mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma} \cap \mathrm{dom}\, H_{\widetilde{V}_l\delta_{\Sigma_l}}. \tag{2.10}$$

In fact, since $\mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma} \subset H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ and $\varphi_l \in C_b^\infty(\mathbb{R}^\theta; \mathbb{R})$, we have $\varphi_l u \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ for $u \in \mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma}$ as well as

$$\left(i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-) + \frac{\widetilde{V}}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)\right)\varphi_l u = (\varphi_l|_\Sigma)\left(i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-) + \frac{\widetilde{V}}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)\right)u = 0,$$

that is, $\varphi_l u \in \mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma}$. For (2.10) it remains to verify $\varphi_l u \in \mathrm{dom}\, H_{\widetilde{V}_l\delta_{\Sigma_l}}$. It is convenient to introduce $(\Omega_l)_+ := \Omega_l$ and $(\Omega_l)_- := \mathbb{R}^\theta \setminus \overline{\Omega_l}$. Since we already have $\varphi_l u \in \mathrm{dom}\, H_{\widetilde{V}\delta_\Sigma} \subset H^1(\mathbb{R}^\theta \setminus \Sigma)$, it is clear that $(\varphi_l u)|_{\Omega_\pm} \in H^1(\Omega_\pm; \mathbb{C}^N)$. Furthermore, as $\mathrm{supp}\, \varphi_l \subset W_l$ and $\Omega_\pm \cap W_l = (\Omega_l)_\pm \cap W_l$, see Section 1.1 (vii), it follows that $(\varphi_l u)|_{(\Omega_l)_\pm} \in H^1((\Omega_l)_\pm; \mathbb{C}^N)$. Thus, we can apply the trace operator to $(\varphi_l u)|_{(\Omega_l)_\pm}$ and obtain

$$\begin{aligned}
&\left(i(\alpha \cdot \nu)(\boldsymbol{t}_{\Sigma_l}^+ - \boldsymbol{t}_{\Sigma_l}^-) + \frac{\widetilde{V}_l}{2}(\boldsymbol{t}_{\Sigma_l}^+ + \boldsymbol{t}_{\Sigma_l}^-)\right)\varphi_l u \\
&= \begin{cases} \left(i(\alpha \cdot \nu)(\boldsymbol{t}_{\Sigma_l}^+ - \boldsymbol{t}_{\Sigma_l}^-) + \frac{\widetilde{V}_l}{2}(\boldsymbol{t}_{\Sigma_l}^+ + \boldsymbol{t}_{\Sigma_l}^-)\right)\varphi_l u, & \text{on } \Sigma_l \cap W_l, \\ 0, & \text{on } \Sigma_l \setminus W_l, \end{cases} \\
&= \begin{cases} \left(i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-) + \frac{\widetilde{V}}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)\right)\varphi_l u, & \text{on } \Sigma \cap W_l, \\ 0, & \text{on } \Sigma \setminus W_l, \end{cases} \\
&= 0,
\end{aligned}$$

where we used $\varphi_l u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$, $\operatorname{supp}\varphi_l \subset W_l$, and $\widetilde{V}_l = \widetilde{V}$ on $\Sigma_l \cap W_l = \Sigma \cap W_l$. Hence, $\varphi_l u \in H^1(\mathbb{R}^\theta \setminus \Sigma_l)$ and $\varphi_l u$ fulfils the boundary condition

$$\left(i(\alpha\cdot\nu)(\boldsymbol{t}^+_{\Sigma_l} - \boldsymbol{t}^-_{\Sigma_l}) + \frac{\widetilde{V}_l}{2}(\boldsymbol{t}^+_{\Sigma_l} + \boldsymbol{t}^-_{\Sigma_l})\right)\varphi_l u = 0,$$

that is, $\varphi_l u \in \operatorname{dom} H_{\widetilde{V}_l\delta_{\Sigma_l}}$ and hence we have (2.10). Moreover,

$$\begin{aligned} H_{\widetilde{V}\delta_\Sigma}\varphi_l u &= \begin{cases} (-i(\alpha\cdot\nabla) + m\beta)(\varphi_l u)|_{\Omega_\pm\cap W_l}, & \text{in } \Omega_\pm \cap W_l,\\ 0, & \text{else,}\end{cases}\\ &= \begin{cases} (-i(\alpha\cdot\nabla) + m\beta)(\varphi_l u)|_{(\Omega_l)_\pm\cap W_l}, & \text{in } (\Omega_l)_\pm \cap W_l,\\ 0, & \text{else,}\end{cases}\\ &= H_{\widetilde{V}_l\delta_{\Sigma_l}}\varphi_l u, \end{aligned}$$

and hence also (2.9) is valid.

Applying the product rule and using (2.9) yields for $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ and any $l \in \{1,\dots,p\}$

$$\varphi_l H_{\widetilde{V}\delta_\Sigma} u = H_{\widetilde{V}\delta_\Sigma}\varphi_l u + i(\alpha\cdot\nabla\varphi_l)u = H_{\widetilde{V}_l\delta_{\Sigma_l}}\varphi_l u + i(\alpha\cdot\nabla\varphi_l)u. \tag{2.11}$$

Since $\varphi_{p+1} = 0$ on $\Sigma + B(0,\delta)$, we get for $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma} \subset H^1(\mathbb{R}^\theta\setminus\Sigma;\mathbb{C}^N)$ that $\varphi_{p+1}u \in H^1(\mathbb{R}^\theta;\mathbb{C}^N)$ and

$$H_{\widetilde{V}\delta_\Sigma}\varphi_{p+1}u = H\varphi_{p+1}u = H_{\widetilde{V}_{p+1}\delta_{\Sigma_{p+1}}}\varphi_{p+1}u.$$

Thus, the product rule implies that (2.11) is also valid for $l = p+1$. Next, since $d = \eta^2 - \tau^2$ and $d_l = \eta_l^2 - \tau_l^2$, $l \in \{1,\dots,p\}$, satisfy (2.1) and (2.4), respectively, $\widetilde{d} = 4\tan^2\left(\frac{\sqrt{d}}{2}\right)$ and $\widetilde{d_l} = 4\tan^2\left(\frac{\sqrt{d_l}}{2}\right)$, $l \in \{1,\dots,p\}$, fulfil (1.12). Hence, $H_{\widetilde{V}\delta_\Sigma}$ and $H_{\widetilde{V}_l\delta_{\Sigma_l}}$ are self-adjoint operators in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$. Now, one can show in the same way as in (2.8) for $z \in \mathbb{C}\setminus\mathbb{R}$ and $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ that

$$\left(\sum_{l=1}^{p+1}(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}\varphi_l\right)(H_{\widetilde{V}\delta_\Sigma} - z)u = \left(I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha\cdot\nabla\varphi_l)\right)u.$$

In particular, if $|\operatorname{Im} z| > \sum_{l=1}^{p+1}\|\alpha\cdot\nabla\varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$, then it is clear that the operator $I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha\cdot\nabla\varphi_l)$ is continuously invertible in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and therefore

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = \left(I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha\cdot\nabla\varphi_l)\right)^{-1}\left(\sum_{l=1}^{p+1}(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}\varphi_l\right).$$

□

With the help of the previous lemmas, we are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $V_l = \eta_l I_N + \tau_l\beta$, $l \in \{1,\dots,p\}$, be as in (2.3) and $z \in \mathbb{C}$ such that $|\operatorname{Im} z| > \sum_{l=1}^{p+1}\|\alpha\cdot\nabla\varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$ and $r \in (0,1/2)$. Since $d_l = \eta_l^2 - \tau_l^2$ satisfies (2.4), we can apply Theorem 4.1, which implies that there exists an $\varepsilon' \in (0, \min\{\varepsilon_1, \delta, \varepsilon_{3,1},\dots,\varepsilon_{3,p}\})$, where $\varepsilon_{3,l}$, $l \in \{1,\dots,p\}$, corresponds to ε_3 from Theorem 4.1 for Σ_l, such that

$$\|(H_{V_{\varepsilon,l}} - z)^{-1} - (H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r} \tag{2.12}$$

for $l \in \{1,\dots,p\}$ and $\varepsilon \in (0,\varepsilon')$. Our conventions from Lemma 2.3 and Lemma 2.4 ensure that (2.12) is also true for $l = p+1$.

Now, we define for $\varepsilon \in (0, \varepsilon')$,

$$\mathfrak{R}_\varepsilon = \sum_{l=1}^{p+1} (H_{V_{\varepsilon,l}} - z)^{-1} \varphi_l, \qquad \mathfrak{S}_\varepsilon = \sum_{l=1}^{p+1} i(H_{V_{\varepsilon,l}} - z)^{-1}(\alpha \cdot \nabla \varphi_l),$$

$$\mathfrak{R}_0 = \sum_{l=1}^{p+1} (H_{\widetilde{V}_l \delta_{\Sigma_l}} - z)^{-1} \varphi_l, \quad \mathfrak{S}_0 = \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l \delta_{\Sigma_l}} - z)^{-1}(\alpha \cdot \nabla \varphi_l).$$

Thus, (2.12) and $\varphi_l \in C_b^\infty(\mathbb{R}^\theta; \mathbb{R})$, $l \in \{1, \dots, p+1\}$, imply

$$\|\mathfrak{S}_\varepsilon - \mathfrak{S}_0\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}, \; \|\mathfrak{R}_\varepsilon - \mathfrak{R}_0\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}.$$

Furthermore, since

$$\max\left\{ \sup_{\varepsilon\in(0,\varepsilon')} \|\mathfrak{S}_\varepsilon\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}, \|\mathfrak{S}_0\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \right\}$$
$$\leq \frac{\sum_{l=1}^{p+1} \|\alpha \cdot \nabla \varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}}{|\mathrm{Im}\, z|} < 1,$$

we conclude with the help of Lemma 2.3 and Lemma 2.4

$$\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}$$
$$= \|(I + \mathfrak{S}_\varepsilon)^{-1}\mathfrak{R}_\varepsilon - (1 + \mathfrak{S}_0)^{-1}\mathfrak{R}_0\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r},$$

which proves the claim if $|\mathrm{Im}\, z| > \sum_{l=1}^{p+1} \|\alpha \cdot \nabla \varphi_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$. Using the identity

$$(H_{\widetilde{V}\delta_\Sigma} - w)^{-1} - (H_{V_\varepsilon} - w)^{-1}$$
$$= (I + (w - z)(H_{\widetilde{V}\delta_\Sigma} - w)^{-1}) \cdot \Big((H_{\widetilde{V}\delta_\Sigma} - z)^{-1} - (H_{V_\varepsilon} - z)^{-1} \Big) (I + (w - z)(H_{V_\varepsilon} - w)^{-1}), \quad w \in \mathbb{C} \setminus \mathbb{R},$$

it follows that the claim is true for all $w \in \mathbb{C} \setminus \mathbb{R}$. This completes the proof of Theorem 2.1. □

In order to obtain the operator $H_{\widetilde{V}\delta_\Sigma}$ as a limit operator of Dirac operators with squeezed potentials also in the case

$$\inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| > 4,$$

we use an approach that is inspired by the last paragraph of [13, Section 8] and now add a strongly localized magnetic potential $V_{m,\varepsilon}$ to H_{V_ε} with the interaction strength π. It turns out that in this case $H_{V_\varepsilon} + V_{m,\varepsilon}$ also converges in the norm resolvent sense, see the following Theorem 2.5. However, by the specific choice of π as the magnetic interaction strength, the magnetic term disappears in the limit. Hence, we end up with a limit operator $H_{\widetilde{V}\delta_\Sigma}$ which is again a Dirac operator with δ-shell potential and only electrostatic and Lorentz-scalar interactions; however, we emphasize that here a different rescaling than (1.11) in Theorem 2.1 appears.

Theorem 2.5. *Let $q \in L^\infty((-1,1);[0,\infty))$ with $\int_{-1}^1 q(s)\,ds = 1$, let $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$, and assume that $d = \eta^2 - \tau^2$ satisfies the condition (2.1) and $\inf_{x_\Sigma\in\Sigma} |d(x_\Sigma)| > 0$. Let V and V_ε be as in (1.9) and (1.5), let the strongly localized magnetic potential $V_{m,\varepsilon} \in L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})$, $\varepsilon \in (0,\varepsilon_1)$, be defined by*

$$V_{m,\varepsilon}(x) := \begin{cases} \pi(\alpha \cdot \nu(x_\Sigma))\frac{1}{\varepsilon} q\left(\frac{t}{\varepsilon}\right), & \text{for } x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ 0, & \text{else,} \end{cases} \tag{2.13}$$

and set

$$\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta, \qquad (\widetilde{\eta}, \widetilde{\tau}) = \frac{-2}{\sqrt{d}\tan\left(\frac{\sqrt{d}}{2}\right)}(\eta, \tau), \qquad d = \eta^2 - \tau^2. \tag{2.14}$$

Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ *and* $r \in (0, 1/2)$ *there exist* $C > 0$ *and* $\varepsilon' \in (0, \varepsilon_1)$ *such that*

$$\|(H_{V_\varepsilon} + V_{m,\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}, \ \ \varepsilon \in (0, \varepsilon').$$

In particular, $H_{V_\varepsilon} + V_{m,\varepsilon}$ *converges to* $H_{\widetilde{V}\delta_\Sigma}$ *in the norm resolvent sense as* $\varepsilon \to 0$.

We note that the function

$$(-\infty, \pi^2/4) \ni w \mapsto 4\tan^2\left(\frac{\sqrt{w}}{2}\right) = \begin{cases} \tan^2\left(\frac{\sqrt{w}}{2}\right) \in [0, 1), & w \in [0, \pi^2/4), \\ -\tanh^2\left(\frac{\sqrt{-w}}{2}\right) \in (-1, 0), & w \in (-\infty, 0), \end{cases}$$

is continuous, monotonically increasing, and has its only root in zero. Hence, the assumption $\inf_{x_\Sigma\in\Sigma} |d(x_\Sigma)| > 0$, (2.1), and the boundedness of η and τ imply

$$\inf_{x_\Sigma\in\Sigma}\left|\tan^2\left(\frac{\sqrt{d(x_\Sigma)}}{2}\right)\right| > 0 \quad \text{and} \quad \sup_{x_\Sigma\in\Sigma}\left|\tan^2\left(\frac{\sqrt{d(x_\Sigma)}}{2}\right)\right| < 1.$$

In particular, $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4/\tan^2\left(\frac{\sqrt{d}}{2}\right)$ is well-defined and

$$\inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| = \frac{4}{\sup_{x_\Sigma\in\Sigma}\left|\tan^2\left(\frac{\sqrt{d(x_\Sigma)}}{2}\right)\right|} > 4.$$

Thus, $\widetilde{V}$ in (2.14) is noncritical; cf. (1.12). We point out again that the rescaling (2.14) in Theorem 2.5 differs from the rescaling (1.11) in Theorem 2.1.

The following preparatory lemma is an essential ingredient in the proof of Theorem 2.5 and also of independent interest.

Lemma 2.6. *Let* ε_1 *be chosen as below (1.2),* $\varepsilon \in (0, \varepsilon_1)$*, and let* Ω_ε *be the tubular neighborhood of* Σ*. Then, for all* $s \in (1/2, \infty)$ *there exists* $C > 0$ *such that*

$$\int_{\Omega_\varepsilon} |u(x)|^2\, dx \leq C\varepsilon \|u\|^2_{H^s(\mathbb{R}^\theta\setminus\Sigma;\mathbb{C}^N)}, \qquad u \in H^s(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N).$$

Proof. Since $H^{s'}(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ is continuously embedded in $H^s(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ for $s < s'$, it is no restriction to assume that $s \in (1/2, 1]$. We start by considering $u = u_+ \oplus u_-$ with $u_\pm \in \mathcal{D}(\overline{\Omega_\pm}; \mathbb{C}^N)$, where

$$\mathcal{D}(\overline{\Omega_\pm}; \mathbb{C}^N) := \{u|_{\Omega_\pm} : u \in \mathcal{D}(\mathbb{R}^\theta; \mathbb{C}^N)\}.$$

Moreover, by [8, Proposition 2.4 and Corollary A.3] and our choice of $\varepsilon_1 > 0$, see the lines below (1.2), we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} |u(x)|^2\, dx &\leq C \int_{-\varepsilon}^{\varepsilon}\int_\Sigma |u(x_\Sigma + t\nu(x_\Sigma))|^2\, d\sigma(x_\Sigma)\, dt \\ &= C\varepsilon\left(\int_{-1}^{0}\int_\Sigma |u_+(x_\Sigma + t\varepsilon\nu(x_\Sigma))|^2\, d\sigma(x_\Sigma)\, dt + \int_0^1\int_\Sigma |u_-(x_\Sigma + t\varepsilon\nu(x_\Sigma))|^2\, d\sigma(x_\Sigma)\, dt\right). \end{aligned}$$

Next, we estimate the term $\int_{-1}^{0}\int_{\Sigma}|u_+(x_\Sigma+t\varepsilon\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt$. Note that the smoothness of u implies that for $t\in(-1,0)$ the function

$$\Sigma\ni x_\Sigma\mapsto u_+(x_\Sigma+t\varepsilon\nu(x_\Sigma))$$

coincides with the trace of the function

$$\mathbb{R}^\theta\ni x\mapsto\widetilde{u}_+(x+t\varepsilon\nu_{\mathrm{ext}}(x)),$$

where ν_{ext} is a C_b^1-extension of the unit normal vector ν to $\mathbb{R}^\theta$, which can be constructed in the same way as the extensions of η, τ, and V in (2.2), and $\widetilde{u}_+=Eu_+\in H^1(\mathbb{R}^\theta;\mathbb{C}^N)$ is the extension of u_+ to $\mathbb{R}^\theta$ defined by Stein's continuous extension operator

$$E:H^s(\Omega_+;\mathbb{C}^N)\mapsto H^s(\mathbb{R}^\theta;\mathbb{C}^N);$$

cf. [37, Chapter 6, Section 3, Theorem 5]. Then,

$$\int_{-1}^{0}\int_{\Sigma}|u_+(x_\Sigma+t\varepsilon\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt=\int_{-1}^{0}\int_{\Sigma}\left|\boldsymbol{t}_\Sigma(\widetilde{u}_+((\cdot)+t\varepsilon\nu_{\mathrm{ext}}))(x_\Sigma)\right|^2 d\sigma(x_\Sigma)\,dt$$

and we can estimate this term by

$$\begin{aligned}\int_{-1}^{0}\int_{\Sigma}|\boldsymbol{t}_\Sigma(\widetilde{u}_+((\cdot)+t\varepsilon\nu_{\mathrm{ext}}))(x_\Sigma)|^2\,d\sigma(x_\Sigma)\,dt&=\int_{-1}^{0}\|\boldsymbol{t}_\Sigma(Eu_+)((\cdot)+t\varepsilon\nu_{\mathrm{ext}})\|^2_{L^2(\Sigma;\mathbb{C}^N)}\,dt\\&\leq\int_{-1}^{0}\|\boldsymbol{t}_\Sigma(Eu_+)((\cdot)+t\varepsilon\nu_{\mathrm{ext}})\|^2_{H^{s-1/2}(\Sigma;\mathbb{C}^N)}\,dt\\&\leq C\int_{-1}^{0}\|(Eu_+)((\cdot)+t\varepsilon\nu_{\mathrm{ext}})\|^2_{H^s(\mathbb{R}^\theta;\mathbb{C}^N)}\,dt.\end{aligned}$$

According to [8, Proposition 3.3], the term $\|(Eu_+)((\cdot)+t\varepsilon\nu_{\mathrm{ext}})\|_{H^s(\mathbb{R}^\theta;\mathbb{C}^N)}$ is uniformly bounded with respect to $t\in(0,1)$ and $\varepsilon\in(0,\varepsilon_1)$ by $C\|Eu_+\|_{H^s(\mathbb{R}^\theta;\mathbb{C}^N)}$, which is in turn bounded by $C\|u_+\|_{H^s(\Omega_+;\mathbb{C}^N)}$. Therefore,

$$\int_{-1}^{0}\int_{\Sigma}|u_+(x_\Sigma+t\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt\leq C\|u_+\|^2_{H^s(\Omega_+;\mathbb{C}^N)};$$

in the same way one gets

$$\int_{0}^{1}\int_{\Sigma}|u_-(x_\Sigma+t\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt\leq C\|u_-\|^2_{H^s(\Omega_-;\mathbb{C}^N)}.$$

This implies the assertion for $u\in\mathcal{D}(\overline{\Omega_+};\mathbb{C}^N)\oplus\mathcal{D}(\overline{\Omega_-};\mathbb{C}^N)$, which is a dense subspace of $H^s(\mathbb{R}^\theta\setminus\Sigma;\mathbb{C}^N)=H^s(\Omega_+;\mathbb{C}^N)\oplus H^s(\Omega_-;\mathbb{C}^N)$, see, for example, [25, Chapter 3]. Therefore, the assertion of the lemma follows for all $u\in H^s(\mathbb{R}^\theta\setminus\Sigma;\mathbb{C}^N)$. ☐

Proof of Theorem 2.5. The proof is based on an idea that is also described below the proof of Theorem 2.6 in [13, Section 8] and the equality

$$H_{\widetilde{V}\delta_\Sigma}=UH_{(-4/\widetilde{d})\widetilde{V}\delta_\Sigma}U,\tag{2.15}$$

where U is the self-adjoint unitary multiplication operator induced by the function $w=\chi_{\Omega_+}-\chi_{\Omega_-}$ in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$. Recall that $\widetilde{V}=\widetilde{\eta}I_N+\widetilde{\tau}\beta$ is as in (2.14), $\widetilde{d}=\widetilde{\eta}^2-\widetilde{\tau}^2$, and by our assumptions, we have $\inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)|>4$, so that all terms in (2.15) are well-defined. Equation (2.15) can be proven as in, for example, [9, Lemma 5.11], [13, Section 4], or [23, Theorem 1.1]. Furthermore, according to (2.14), we have

$$\widetilde{V} = \frac{-2}{\sqrt{d}\tan\left(\frac{\sqrt{d}}{2}\right)}V \quad \text{and} \quad \widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = \frac{4}{\tan^2\left(\frac{\sqrt{d}}{2}\right)}.$$

Set $\widetilde{V}_{\mathrm{cl}} := \operatorname{tanc}\left(\frac{\sqrt{d}}{2}\right)V$ and observe

$$\frac{-4}{\widetilde{d}}\widetilde{V} = -\tan^2\left(\frac{\sqrt{d}}{2}\right)\frac{-2}{\sqrt{d}\tan\left(\frac{\sqrt{d}}{2}\right)}V = \frac{\tan\left(\frac{\sqrt{d}}{2}\right)}{\frac{\sqrt{d}}{2}}V = \widetilde{V}_{\mathrm{cl}}.$$

In particular,

$$H_{\widetilde{V}\delta_\Sigma} = UH_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma}U, \tag{2.16}$$

and since $\widetilde{V}_{\mathrm{cl}} = \operatorname{tanc}\left(\frac{\sqrt{d}}{2}\right)V$, which is the rescaling from Theorem 2.1, it follows from Theorem 2.1 that H_{V_ε} converges to $H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma}$ in the norm resolvent sense. We then proceed in a similar way as in [13], that is, we provide unitary multiplication operators W_ε such that

$$W_\varepsilon^* H_{V_\varepsilon} W_\varepsilon = H_{V_\varepsilon} + V_{m,\varepsilon}, \tag{2.17}$$

where $V_{m,\varepsilon}$ is the strongly localized magnetic potential introduced in (2.13), and $W_\varepsilon \to U$ for $\varepsilon \to 0$, so that the left-hand side of (2.17) converges in the norm resolvent sense to $UH_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma}U = H_{\widetilde{V}\delta_\Sigma}$.

We now start the main part of the proof by defining for $\varepsilon \in (0, \varepsilon_1)$ the function

$$w_\varepsilon : \mathbb{R}^\theta \to \mathbb{C}, \qquad w_\varepsilon(x) := \begin{cases} 1, & x \in \Omega_+ \setminus \Omega_\varepsilon, \\ e^{i\pi\int_{-1}^{t/\varepsilon} q(s)\,ds}, & x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ -1, & x \in \Omega_- \setminus \Omega_\varepsilon. \end{cases}$$

This function is well-defined according to the text below (1.2). We define W_ε to be the unitary multiplication operator in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ induced by w_ε. Using $\int_{-1}^{1} q(s)\,ds = 1$ and tubular coordinates, one shows that $w_\varepsilon \in W^1_\infty(\mathbb{R}^\theta)$. Moreover, by [42, eq. (2.11)] and [13, eq. (3.10)], we have $\nabla(\iota^{-1}(x))_2 = \nu(x_\Sigma)$ for $x = \iota(x_\Sigma, t) \in \Omega_\varepsilon$ ($(\iota^{-1}(x))_2 = t$) and therefore the chain rule implies

$$\nabla w_\varepsilon(x) = \begin{cases} \nu(x_\Sigma)\frac{i\pi q\left(\frac{t}{\varepsilon}\right)}{\varepsilon}w_\varepsilon(x), & x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ 0, & \text{else}. \end{cases}$$

These considerations and the definition of $V_{m,\varepsilon}$ show

$$W_\varepsilon^* H_{V_\varepsilon} W_\varepsilon = H_{V_\varepsilon} - i\overline{w_\varepsilon}(\alpha \cdot \nabla w_\varepsilon) = H_{V_\varepsilon} + V_{m,\varepsilon}; \tag{2.18}$$

cf. [13, Section 8, below the proof of Theorem 2.6]. We note that w_ε converges pointwise to $w = \chi_{\Omega_+} - \chi_{\Omega_-}$ and therefore W_ε converges in the strong sense to the operator U. In addition, Lemma 2.6 shows that for $\varepsilon \in (0, \varepsilon_1)$, the estimate

$$\begin{aligned} \|(W_\varepsilon^* - U)u\|^2_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} &= \int_{\Omega_\varepsilon} |(\overline{w_\varepsilon(x)} - \chi_{\Omega_+}(x) + \chi_{\Omega_-}(x))u(x)|^2\,dx \\ &\leq 4\int_{\Omega_\varepsilon} |u(x)|^2\,dx \\ &\leq C\varepsilon\|u\|^2_{H^1(\mathbb{R}^\theta\setminus\Sigma;\mathbb{C}^N)}, \qquad u \in H^1(\mathbb{R}^\theta \setminus \Sigma), \end{aligned} \tag{2.19}$$

is also valid. Moreover, since $(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1}$ is closed as an operator from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta \setminus \Sigma;\mathbb{C}^N)$ and defined on $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$, it is also bounded from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta \setminus \Sigma;\mathbb{C}^N)$. Thus, (2.19) implies for $\varepsilon \in (0,\varepsilon_1)$

$$\|(W_\varepsilon^* - U)(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C\varepsilon^{1/2}. \tag{2.20}$$

We use (2.18) and (2.16) and estimate for $\varepsilon \in (0,\varepsilon')$ (with $\varepsilon' > 0$ from Theorem 2.1)

$$\begin{aligned}
&\|(H_{V_\varepsilon}+V_{m,\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad= \|(W_\varepsilon^* H_{V_\varepsilon} W_\varepsilon - z)^{-1} - (U H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} U - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad= \|W_\varepsilon^* (H_{V_\varepsilon} - z)^{-1} W_\varepsilon - U(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1} U\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad\le \|W_\varepsilon^*((H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1})W_\varepsilon\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\qquad+ \|(W_\varepsilon^* - U)(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1} W_\varepsilon\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\qquad+ \|U(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1}(W_\varepsilon - U)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}.
\end{aligned}$$

Furthermore, since U is unitary and self-adjoint and W_ε is unitary, we can continue the above estimate for $r \in (0,1/2)$ and obtain

$$\begin{aligned}
&\|(H_{V_\varepsilon}+V_{m,\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad\le \|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\qquad+ \|(W_\varepsilon^* - U)(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\qquad+ \|(W_\varepsilon^* - U)(H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma} - \overline{z})^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}.\\
&\quad\le C\varepsilon^{1/2-r} + C\varepsilon^{1/2}\\
&\quad\le C\varepsilon^{1/2-r},
\end{aligned}$$

where the norm resolvent convergence of H_{V_ε} to $H_{\widetilde{V}_{\mathrm{cl}}\delta_\Sigma}$ (see Theorem 2.1) and (2.20) were used in the penultimate estimate. □

In the next corollary, we observe that every Dirac operator with a given δ-shell potential $\widetilde{V}\delta_\Sigma$, $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, $\widetilde{\eta},\widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$, and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ such that

$$\sup_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| < 4 \quad \text{or} \quad \inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| > 4 \tag{2.21}$$

holds, can be approximated by a sequence of Dirac operators with strongly localized potentials.

Corollary 2.7. *Let $q \in L^\infty((-1,1);[0,\infty))$ with $\int_{-1}^1 q(s)\,ds = 1$, let $\widetilde{\eta},\widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$ and $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, and assume that $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ satisfies the condition (2.21). Define the interaction strengths $\eta,\tau \in C_b^1(\Sigma)$ by*

$$(\eta,\tau) = \begin{cases} \dfrac{2\arctan\left(\frac{\sqrt{\widetilde{d}}}{2}\right)}{\sqrt{\widetilde{d}}}(\widetilde{\eta},\widetilde{\tau}), & \textit{if } \sup_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| < 4,\\[2ex] \dfrac{-2\arctan\left(\frac{2}{\sqrt{\widetilde{d}}}\right)}{\sqrt{\widetilde{d}}}(\widetilde{\eta},\widetilde{\tau}), & \textit{if } \inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| > 4, \end{cases} \tag{2.22}$$

and let $V = \eta I_N + \tau\beta$ and V_ε be as in (1.5). Then, for all $r \in (0, 1/2)$ and $z \in \mathbb{C} \setminus \mathbb{R}$, there exist $C > 0$ and $\varepsilon' \in (0, \varepsilon_1)$ such that the following is true:

(i) If $\sup_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| < 4$, then

$$\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}$$

for all $\varepsilon \in (0, \varepsilon')$.

(ii) If $\inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| > 4$, then

$$\|(H_{V_\varepsilon} + V_{m,\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}$$

for all $\varepsilon \in (0, \varepsilon')$, where $V_{m,\varepsilon}$ is as in (2.13).

Note that in (2.22) the convention $\frac{\arctan(x)}{x}|_{x=0} = 1$ is used. Moreover, we would like to point out that for constant $\widetilde{\eta}, \widetilde{\tau} \in \mathbb{R}$, the previous corollary is particularly interesting, as it shows that every Dirac operator with a δ-potential and constant electrostatic and Lorentz-scalar interaction strengths satisfying $|\widetilde{d}| \neq 4$ can be approximated by Dirac operators with strongly localized potentials.

Proof of Corollary 2.7. Throughout this proof, we use $\arctan^2(w) < \frac{\pi^2}{16}$ for $w \in [0, 1]$ and $w \in i[0, 1]$.

(i) The assumption $\sup_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| < 4$ implies for $d = \eta^2 - \tau^2$

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) = \sup_{x_\Sigma \in \Sigma} 4 \arctan^2\Big(\frac{\sqrt{\widetilde{d}(x_\Sigma)}}{2}\Big) < \frac{\pi^2}{4}$$

and thus condition (2.1) is fulfilled. Hence, the assertion follows from Theorem 2.1.

(ii) The assumption $\inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| > 4$ implies

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) = \sup_{x_\Sigma \in \Sigma} 4 \arctan^2\Big(\frac{2}{\sqrt{\widetilde{d}(x_\Sigma)}}\Big) < \frac{\pi^2}{4}$$

and thus condition (2.1) is fulfilled. Furthermore, since $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 \in C_b^1(\Sigma; \mathbb{R})$, we also have $\inf_{x_\Sigma \in \Sigma} |d(x_\Sigma)| > 0$. Hence, the assertion follows from Theorem 2.5. □

3 | COUNTEREXAMPLES

We show in this section that the condition (2.1) for the norm resolvent convergence of H_{V_ε} is optimal by providing suitable counterexamples. Throughout this section, we assume that V has the form $V = \eta I_N + \tau\beta$ with $\eta, \tau \in \mathbb{R}$. In this situation, (2.1) simplifies to $d = \eta^2 - \tau^2 < \frac{\pi^2}{4}$ and by (1.11) we have

$$\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = \operatorname{tanc}^2\Big(\frac{\sqrt{d}}{2}\Big)d = 4\tan^2\Big(\frac{\sqrt{d}}{2}\Big). \tag{3.1}$$

In our first counterexample, we treat those cases $d \geq \frac{\pi^2}{4}$ that lead to the critical interaction strength $\widetilde{d} = 4$. These are exactly the cases $d = (2k+1)^2\frac{\pi^2}{4}$, $k \in \mathbb{N}_0$, and include, in particular, the important borderline case $d = \frac{\pi^2}{4}$. In this situation, the operator $H_{\widetilde{V}\delta_\Sigma}$ is only essentially self-adjoint and hence H_{V_ε} cannot converge in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$. However, if the interaction support Σ is a compact C^∞-hypersurface it turns out in Theorem 3.1 that H_{V_ε} does not even converge to the closure of $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense. In our second counterexample discussed in Theorem 3.2, we treat the case $d > \frac{\pi^2}{4}$ and assume $d \neq (2k+1)^2\frac{\pi^2}{4}$, $k \in \mathbb{N}_0$, so that $\widetilde{V}$ is noncritical (i.e., $\widetilde{d} \neq 4$). If $\theta = 2$ and the interaction support Σ is the y-axis, then we show that H_{V_ε} does not converge to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense.

Theorem 3.1. *Let $\Sigma \subset \mathbb{R}^\theta$ be a compact C^∞-hypersurface, $q \in L^\infty((-1,1);[0,\infty))$ with $\int_{-1}^1 q(s)\,ds = 1$, and $\eta, \tau \in \mathbb{R}$ such that $d = \eta^2 - \tau^2 = (2k+1)^2\frac{\pi^2}{4}$, $k \in \mathbb{N}_0$. Let $V = \eta I_N + \tau\beta$ and V_ε be as in (1.5), and define $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ by (1.11). Then, $\widetilde{V}$ is critical (i.e., $\widetilde{d} = 4$), $H_{\widetilde{V}\delta_\Sigma}$ is essentially self-adjoint but not self-adjoint, and H_{V_ε} does not converge in the norm resolvent sense to the closure of $H_{\widetilde{V}\delta_\Sigma}$.*

Proof. Since the convergence in norm resolvent sense is invariant with respect to bounded perturbations it is no restriction to assume $m > 0$. It is clear from (3.1) that $d = (2k+1)^2\frac{\pi^2}{4}$ leads to $\widetilde{d} = 4$ and hence the interaction strength $\widetilde{V}$ is critical. The claims regarding the (essential) self-adjointness follow from [7, Theorem 4.11] for $\theta = 2$ and from [11, Theorem 3.1 (ii)] for $\theta = 3$. Thus, it only remains to prove that H_{V_ε} does not converge in the norm resolvent sense to the closure of $H_{\widetilde{V}\delta_\Sigma}$. If Σ is compact, then supp $V_\varepsilon \subset \Omega_\varepsilon$ is compact and hence by [25, Theorem 3.27 (ii)] V_ε induces a compact operator from $H^1(\mathbb{R}^\theta;\mathbb{C}^N)$ to $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$. In turn the resolvent difference

$$(H - z)^{-1} - (H_{V_\varepsilon} - z)^{-1} = (H_{V_\varepsilon} - z)^{-1} V_\varepsilon (H - z)^{-1}$$

is compact in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$, which shows

$$\sigma_{\mathrm{ess}}(H_{V_\varepsilon}) = \sigma_{\mathrm{ess}}(H) = (-\infty, -m] \cup [m, \infty).$$

Consequently, [41, Satz 9.24 a)] implies that if H_{V_ε} converged in norm resolvent sense to the closure of $H_{\widetilde{V}\delta_\Sigma}$, then also

$$\sigma_{\mathrm{ess}}(\overline{H_{\widetilde{V}\delta_\Sigma}}) = \sigma_{\mathrm{ess}}(H) = (-\infty, -m] \cup [m, \infty).$$

However, since $\widetilde{\eta}^2 - \widetilde{\tau}^2 = 4$, one has $|\widetilde{\tau}| < |\widetilde{\eta}|$ and

$$-\frac{\widetilde{\tau}}{\widetilde{\eta}} m \in \sigma_{\mathrm{ess}}(\overline{H_{\widetilde{V}\delta_\Sigma}}) \cap (-m, m)$$

according to [7, Theorem 1.2 and Theorem 1.3] and [12]; a contradiction. □

Our second counterexample concerns the noncritical case and the special interaction support $\Sigma = \{0\} \times \mathbb{R}$ in $\mathbb{R}^2$. The idea of the proof is to use the direct integral method and to verify that $0 \in \sigma(H_{V_\varepsilon})$ for all $\varepsilon > 0$ sufficiently small, while $0 \notin \sigma(H_{\widetilde{V}\delta_\Sigma})$; cf. Remark 3.3 below for a further discussion.

Theorem 3.2. *Let $\Sigma = \{0\} \times \mathbb{R} \subset \mathbb{R}^2$, let $q = \frac{1}{2}\chi_{(-1,1)}$, and let $\eta, \tau \in \mathbb{R}$ be such that $d = \eta^2 - \tau^2 > \frac{\pi^2}{4}$ and $d \neq (2k+1)^2\pi^2$, $k \in \mathbb{N}_0$. Let $V = \eta I_N + \tau\beta$ and V_ε be as in (1.9) and (1.5), and define $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ by (1.11). Moreover, assume that $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 \neq 4$. Then, the operator H_{V_ε} does not converge in norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$.*

Proof. The main idea of the proof is to use the direct integral method and to show that if $\varepsilon > 0$ is sufficiently small, then $0 \in \sigma(H_{V_\varepsilon})$. However, then norm resolvent convergence would imply $0 \in \sigma(H_{\widetilde{V}\delta_\Sigma})$, see, for example, [21, Chapter IV, Theorem 3.1], [32, Theorem VIII.23 (a)], or [41, Satz 9.24]. In the case that Σ is the straight line the spectrum has been calculated explicitly in [10, eqs. (5.7), (6.7)]. In particular, if $m \neq 0$ and $(\widetilde{d} - 4)m\widetilde{\tau} \leq 0$, then $0 \notin \sigma(H_{\widetilde{V}\delta_\Sigma})$, which yields a contradiction. Note that it is no restriction to assume $m \neq 0$ and $(\widetilde{d} - 4)m\widetilde{\tau} \leq 0$, since the bounded perturbation $m\beta$ does not influence the norm resolvent convergence.

We show $0 \in \sigma(H_{V_\varepsilon})$ by applying the direct integral method. For this observe first that in the special case of Theorem 3.2 the operator H_{V_ε} can be represented by

$$H_{V_\varepsilon} = \sigma_1(-i\partial_1) + \sigma_2(-i\partial_2) + m\sigma_3 + (\eta I_2 + \tau\sigma_3)\frac{\chi_{(-\varepsilon,\varepsilon)\times\mathbb{R}}}{2\varepsilon},$$

$$\mathrm{dom}\, H_{V_\varepsilon} = H^1(\mathbb{R}^2;\mathbb{C}^2) \subset L^2(\mathbb{R}^2;\mathbb{C}^2);$$

cf. (1.5), (1.6), and Section 1.1 (vi). By identifying $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with $\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R};\mathbb{C}^2)\,d\xi$, we get similar to [10, eq. (2.3) and the text below]

$$\mathcal{F}_2 H_{V_\varepsilon} \mathcal{F}_2^{-1} = \int_{\mathbb{R}}^{\oplus} H_{V_\varepsilon}[\xi]\,d\xi$$

with the fiber operators

$$H_{V_\varepsilon}[\xi] = -i\sigma_1 \frac{d}{dx} + \xi\sigma_2 + m\sigma_3 + (\eta I_2 + \tau\sigma_3)\frac{\chi_{(-\varepsilon,\varepsilon)}}{2\varepsilon},$$
$$\operatorname{dom} H_{V_\varepsilon}[\xi] = H^1(\mathbb{R};\mathbb{C}^2),$$

for $\xi \in \mathbb{R}$. We split the proof of $0 \in \sigma(H_{V_\varepsilon})$ for $\varepsilon > 0$ sufficiently small in 4 steps. In *Step 1* we find a condition for 0 being in the point spectrum of $H_{V_\varepsilon}[\xi]$. *Step 2* is an intermediate step in which we consider the inverse of the function $[\frac{\pi}{2}, \pi) \ni u \mapsto -u\cot(u)$. Using this function, we verify in *Step 3* that if $\varepsilon > 0$ is sufficiently small, then there always exists an $\xi_\varepsilon > 0$ such that $0 \in \sigma_{\mathrm{p}}(H_{V_\varepsilon}[\xi_\varepsilon])$. Finally, we prove in *Step 4* that this implies $0 \in \sigma(H_{V_\varepsilon})$.

Step 1. In this step, we prove for $\xi \in \mathbb{R}$ that $0 \in \sigma_{\mathrm{p}}(H_{V_\varepsilon}[\xi])$ is equivalent to the condition

$$\cos(\mu_{\xi,\varepsilon}) + \frac{d - \mu_{\xi,\varepsilon}^2 - 2\varepsilon\tau m}{\sqrt{d - \mu_{\xi,\varepsilon}^2 - 4\varepsilon\tau m}} \operatorname{sinc}(\mu_{\xi,\varepsilon}) = 0, \tag{3.2}$$

where $\mu_{\xi,\varepsilon} := \sqrt{d - 4\varepsilon^2 v_\xi^2 - 4\varepsilon\tau m}$ and $v_\xi := \sqrt{\xi^2 + m^2}$.

Let us assume that there exists a nonzero function $u \in H^1(\mathbb{R};\mathbb{C}^2)$ such that $H_{V_\varepsilon}[\xi]u = 0$. Then,

$$\frac{d}{dx}u = -i\sigma_1\left(\xi\sigma_2 + m\sigma_3 + (\eta I_2 + \tau\sigma_3)\frac{\chi_{(-\varepsilon,\varepsilon)}}{2\varepsilon}\right)u$$

a.e. on $\mathbb{R}$. Thus, there exist $w_1, w_2, w_3 \in \mathbb{C}^2$ such that

$$u(x) = \begin{cases} \exp(Ax)w_1, & x \in (-\infty, -\varepsilon), \\ \exp(Bx)w_2, & x \in (-\varepsilon, \varepsilon), \\ \exp(Ax)w_3, & x \in (\varepsilon, \infty), \end{cases} \tag{3.3}$$

where

$$A = -i\sigma_1(\xi\sigma_2 + m\sigma_3) = \begin{pmatrix} \xi & im \\ -im & -\xi \end{pmatrix} \quad \text{and} \quad B = A - i\sigma_1(\eta I_2 + \tau\sigma_3)\frac{1}{2\varepsilon}.$$

Note that A has the distinct eigenvalues $\pm v_\xi = \pm\sqrt{\xi^2 + m^2}$ and the corresponding orthogonal eigenvectors are given by $a_+ = (-im, \xi - v_\xi)$ and $a_- = (\xi - v_\xi, -im)$. Since $u \in H^1(\mathbb{R};\mathbb{C}^2)$, $u(x)$ has to converge to zero for $x \to \pm\infty$. Hence, $w_1 = c_1 a_+$ and $w_3 = c_3 a_-$ with $c_1, c_3 \in \mathbb{C}$. Moreover, $u \in H^1(\mathbb{R};\mathbb{C}^2) \subset C(\mathbb{R};\mathbb{C}^2)$ yields the conditions

$$\begin{aligned} \exp(-A\varepsilon)w_1 &= c_1 e^{-\varepsilon v_\xi} a_+ = \exp(-B\varepsilon)w_2 \\ \exp(A\varepsilon)w_3 &= c_3 e^{-\varepsilon v_\xi} a_- = \exp(B\varepsilon)w_2; \end{aligned} \tag{3.4}$$

this implies

$$c_1 a_+ - c_3 \exp(-2B\varepsilon)a_- = 0. \tag{3.5}$$

Next, we write

$$\exp(-2B\varepsilon) = \cos(i2B\varepsilon) - 2B\varepsilon\,\mathrm{sinc}(i2B\varepsilon).$$

Note that $(i2B\varepsilon)^2 = \mu_{\xi,\varepsilon}^2 I_2$ and hence $\cos(i2B\varepsilon) = \cos(\mu_{\xi,\varepsilon})I_2$ as well as $\mathrm{sinc}(i2B\varepsilon) = \mathrm{sinc}(\mu_{\xi,\varepsilon})I_2$. These considerations and $2B\varepsilon = 2A\varepsilon - i\sigma_1(\eta I_2 + \tau\sigma_3)$ yield

$$\begin{aligned}\exp(-2B\varepsilon)a_- &= \cos(\mu_{\xi,\varepsilon})a_- - 2B\varepsilon\,\mathrm{sinc}(\mu_{\xi,\varepsilon})a_- \\ &= \cos(\mu_{\xi,\varepsilon})a_- - (2A\varepsilon - i\sigma_1(\eta I_2 + \tau\sigma_3))\mathrm{sinc}(\mu_{\xi,\varepsilon})a_- \\ &= (\cos(\mu_{\xi,\varepsilon}) + 2\varepsilon v_\xi \mathrm{sinc}(\mu_{\xi,\varepsilon}))a_- + i\sigma_1(\eta I_2 + \tau\sigma_3)\mathrm{sinc}(\mu_{\xi,\varepsilon})a_-.\end{aligned}$$

Since a_+ and a_- are an orthogonal basis of $\mathbb{C}^2$, (3.5) is fulfilled if the scalar product of (3.5) with a_+ and a_- is zero. This yields the system

$$\begin{aligned}0 &= c_1|a_+|^2 - c_3\big\langle i\sigma_1(\eta I_2 + \tau\sigma_3)\mathrm{sinc}(\mu_{\xi,\varepsilon})a_-, a_+\big\rangle \\ 0 &= c_3\big((\cos(\mu_{\xi,\varepsilon}) + 2\varepsilon v_\xi \mathrm{sinc}(\mu_{\xi,\varepsilon}))|a_-|^2 + \big\langle i\sigma_1(\eta I_2 + \tau\sigma_3)\mathrm{sinc}(\mu_{\xi,\varepsilon})a_-, a_-\big\rangle\big).\end{aligned} \tag{3.6}$$

Note that $c_3 \neq 0$, as otherwise $c_1 = 0$ and (3.4) would imply $w_2 = 0$, and thus $w_1 = w_2 = w_3 = 0$, which in turn would lead to $u = 0$. However, $u \neq 0$ by assumption. Thus, the second line of the above system implies

$$0 = (\cos(\mu_{\xi,\varepsilon}) + 2\varepsilon v_\xi \mathrm{sinc}(\mu_{\xi,\varepsilon}))|a_-|^2 + \big\langle i\sigma_1(\eta I_2 + \tau\sigma_3)\mathrm{sinc}(\mu_{\xi,\varepsilon})a_-, a_-\big\rangle.$$

Using the relations $\sigma_1^* = \sigma_1, \sigma_1 a_- = a_+, a_+ \perp a_-, \langle\sigma_3 a_-, a_+\rangle = 2im(\xi - v_\xi)$, and $|a_-|^2 = m^2 + (\xi - v_\xi)^2$, we can simplify this equation to

$$0 = \cos(\mu_{\xi,\varepsilon}) + \left(2\varepsilon v_\xi - 2\frac{(\xi - v_\xi)\tau m}{m^2 + (\xi - v_\xi)^2}\right)\mathrm{sinc}(\mu_{\xi,\varepsilon}). \tag{3.7}$$

Next, we use $m^2 = v_\xi^2 - \xi^2 = -(\xi + v_\xi)(\xi - v_\xi)$ and $2\varepsilon v_\xi = \sqrt{d - \mu_{\xi,\varepsilon}^2 - 4\varepsilon\tau m}$ to rewrite

$$\begin{aligned}2\varepsilon v_\xi - \frac{2(\xi - v_\xi)\tau m}{m^2 + (\xi - v_\xi)^2} &= 2\varepsilon v_\xi - 2\frac{(\xi - v_\xi)\tau m}{(-(\xi + v_\xi) + (\xi - v_\xi))(\xi - v_\xi)} \\ &= 2\varepsilon v_\xi + \frac{\tau m}{v_\xi} \\ &= \frac{1}{2\varepsilon v_\xi}\big((2\varepsilon v_\xi)^2 + 2\varepsilon\tau m\big) \\ &= \frac{d - \mu_{\xi,\varepsilon}^2 - 2\varepsilon\tau m}{2\varepsilon v_\xi} \\ &= \frac{d - \mu_{\xi,\varepsilon}^2 - 2\varepsilon\tau m}{\sqrt{d - \mu_{\xi,\varepsilon}^2 - 4\varepsilon\tau m}}.\end{aligned} \tag{3.8}$$

Plugging (3.8) into (3.7) gives us (3.2).

Now, we argue that the reverse direction is also true. If (3.2) is fulfilled, then using (3.8) implies that (3.7) is valid. Then, we fix a arbitrary $c_3 \in \mathbb{C} \setminus \{0\}$. For this c_3 there exists exactly one $c_1 \in \mathbb{C}$ such that (3.6) is satisfied. Furthermore, (3.5) is also true. The choices $w_1 = c_1 a_+, w_2 = \exp(B\varepsilon)\exp(-A\varepsilon)w_1$, and $w_3 = c_3 a_-$ yield with (3.4) that u defined by (3.3) is in $H^1(\mathbb{R}; \mathbb{C}^N) = \mathrm{dom}\, H_{V_\varepsilon}[\xi]$. Moreover, by construction $u \in (\ker H_{V_\varepsilon}[\xi]) \setminus \{0\}$ which implies $0 \in \sigma_{\mathrm{p}}(H_{V_\varepsilon}[\xi])$.

Step 2. Consider the function $a_0 : [\pi/2, \pi) \mapsto [0, \infty)$, $a_0(u) = -u\cot(u)$. It easy to see that a_0 is continuous and $a_0([\pi/2, \pi)) = [0, \infty)$. Furthermore,

$$a_0'(u) = -\cot(u) + \frac{u}{\sin^2(u)} = \frac{-\cos(u)\sin(u) + u}{\sin^2(u)}$$
$$= \frac{-\frac{1}{2}\sin(2u) + u}{\sin^2(u)} \geq \frac{-\frac{1}{2} + \frac{\pi}{2}}{\sin^2(u)} > 0, \qquad u \in [\pi/2, \pi),$$

and hence a_0 is monotonically increasing and bijective. Thus, the same holds for its inverse function

$$u_0 := a_0^{-1} : [0, \infty) \mapsto [\pi/2, \pi).$$

For $a \in [0, \infty)$, we have $a = -u_0(a)\cot(u_0(a))$ and therefore multiplying with $\operatorname{sinc}(u_0(a))$ shows that u_0 fulfils the relation

$$\cos(u_0(a)) + a\operatorname{sinc}(u_0(a)) = 0, \qquad a \in [0, \infty). \tag{3.9}$$

Step 3. Now we use the function u_0 from *Step 2* to find for sufficiently small $\varepsilon > 0$ a ξ_ε such that $0 \in \sigma_p(H_{V_\varepsilon}[\xi_\varepsilon])$. We start by claiming that there exists an $a_\varepsilon > 0$ which fulfils

$$a_\varepsilon = \frac{d - u_0^2(a_\varepsilon) - 2\varepsilon\tau m}{\sqrt{d - u_0^2(a_\varepsilon) - 4\varepsilon\tau m}}. \tag{3.10}$$

In fact, if we set

$$b_\varepsilon := \sqrt{\min\{d, \pi^2\} - 4\varepsilon(|\tau m| + 1)}$$

and choose $\varepsilon > 0$ sufficiently small we can guarantee $b_\varepsilon \in (\frac{\pi}{2}, \pi)$, and hence $u_0^{-1}(b_\varepsilon) \in (0, \infty)$. Since u_0 is monotonically increasing and continuous, the function

$$F : [0, u_0^{-1}(b_\varepsilon)] \to \mathbb{R}, \quad F(a) = a - \frac{d - u_0^2(a) - 2\varepsilon\tau m}{\sqrt{d - u_0^2(a) - 4\varepsilon\tau m}}$$

is continuous and well-defined. The properties of the function u_0 and the assumption $d > \frac{\pi^2}{4}$ show that

$$F(0) = -\frac{d - \frac{\pi^2}{4} - 2\varepsilon\tau m}{\sqrt{d - \frac{\pi^2}{4} - 4\varepsilon\tau m}}$$

is smaller than zero for sufficiently small $\varepsilon > 0$. Moreover, since u_0^{-1} is continuous and $\lim_{b\to\pi} u_0^{-1}(b) = \infty$, we get

$$\lim_{\varepsilon\to 0} F(u_0^{-1}(b_\varepsilon)) = \lim_{\varepsilon\to 0}\left(u_0^{-1}(b_\varepsilon) - \frac{d - \min\{d, \pi^2\} + 4\varepsilon(|\tau m| + 1) - 2\varepsilon\tau m}{\sqrt{d - \min\{d, \pi^2\} + 4\varepsilon(|\tau m| + 1) - 4\varepsilon\tau m}}\right)$$
$$= \begin{cases} u_0^{-1}(\sqrt{d}), & d < \pi^2, \\ \infty, & d \geq \pi^2, \end{cases}$$
$$> 0.$$

Thus, $F(u_0^{-1}(b_\varepsilon)) > 0$ if $\varepsilon > 0$ is sufficiently small. Hence, there exists an $a_\varepsilon \in (0, u_0^{-1}(b_\varepsilon))$ such that (3.10) is fulfilled.

For this a_ε we also have

$$\begin{aligned} d - u_0^2(a_\varepsilon) - 4\varepsilon^2 m^2 - 4\varepsilon\tau m &> d - u_0^2(u_0^{-1}(b_\varepsilon)) - 4\varepsilon^2 m^2 - 4\varepsilon\tau m \\ &= d - \min\{d, \pi^2\} + 4\varepsilon(|\tau m| + 1) - 4\varepsilon^2 m^2 - 4\varepsilon\tau m \\ &> 0 \end{aligned}$$

for sufficiently small $\varepsilon > 0$. Thus,

$$\xi_\varepsilon := \frac{1}{2\varepsilon}\sqrt{d - u_0^2(a_\varepsilon) - 4\varepsilon^2 m^2 - 4\varepsilon\tau m} > 0 \tag{3.11}$$

is well-defined. From (3.11) and the definition of $\mu_{\xi,\varepsilon}$ below (3.2) it follows that

$$u_0(a_\varepsilon) = \sqrt{d - 4\varepsilon^2 v_{\xi_\varepsilon}^2 - 4\varepsilon\tau m} = \mu_{\xi_\varepsilon,\varepsilon},$$

and plugging this expression into (3.10) yields

$$a_\varepsilon = \frac{d - \mu_{\xi_\varepsilon,\varepsilon}^2 - 2\varepsilon\tau m}{\sqrt{d - \mu_{\xi_\varepsilon,\varepsilon}^2 - 4\varepsilon\tau m}}.$$

Combining these relations with (3.9) shows that (3.2) is fulfilled for $\xi = \xi_\varepsilon$. Thus, $0 \in \sigma_{\mathrm{p}}(H_{V_\varepsilon}[\xi_\varepsilon])$ by *Step 1*.

Step 4. Finally, we show in this step $0 \in \sigma(H_{V_\varepsilon})$ for $\varepsilon > 0$ chosen sufficiently small. This follows from [34, Theorem XIII.85 (d)] if we can show that for all $\delta > 0$ there is a $\gamma_\delta > 0$ such that $(-\delta, \delta) \cap \sigma(H_{V_\varepsilon}[\xi]) \neq \emptyset$ for all $\xi \in (\xi_\varepsilon - \gamma_\delta, \xi_\varepsilon + \gamma_\delta)$. We assume that our claim is not true. In this case, there exists a $\delta' > 0$ and a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to \xi_\varepsilon$ for $n \to \infty$ and $(-\delta', \delta') \cap \sigma(H_{V_\varepsilon}[\xi_n]) = \emptyset$ for all $n \in \mathbb{N}$. Note that

$$\begin{aligned} &\|(H_{V_\varepsilon}[\xi_\varepsilon] - z)^{-1} - (H_{V_\varepsilon}[\xi_n] - z)^{-1}\|_{L^2(\mathbb{R};\mathbb{C}^2)\to L^2(\mathbb{R};\mathbb{C}^2)} \\ &= \|(H_{V_\varepsilon}[\xi_\varepsilon] - z)^{-1}\sigma_2(\xi_n - \xi_\varepsilon)(H_{V_\varepsilon}[\xi_n] - z)^{-1}\|_{L^2(\mathbb{R};\mathbb{C}^2)\to L^2(\mathbb{R};\mathbb{C}^2)} \\ &\leq \frac{1}{(\mathrm{Im}\, z)^2}|\xi_n - \xi_\varepsilon| \to 0, \quad n \to \infty, \end{aligned}$$

holds for $z \in \mathbb{C} \setminus \mathbb{R}$, i.e., $H_{V_\varepsilon}[\xi_n]$ converges in norm resolvent sense to $H_{V_\varepsilon}[\xi_\varepsilon]$. Moreover, $(-\delta', \delta') \cap \sigma(H_{V_\varepsilon}[\xi_n]) = \emptyset$, $n \in \mathbb{N}$, and [32, Theorem VIII.24(a)] imply the contradiction $(-\delta', \delta') \cap \sigma(H_{V_\varepsilon}[\xi_\varepsilon]) = \emptyset$. □

Remark 3.3. We note that in the situation of Theorem 3.2 one can even show $\sigma(H_{V_\varepsilon}) = \mathbb{R}$ for all $\varepsilon > 0$ which are sufficiently small; cf. [38, Theorem 6.5]. Furthermore, the statement of Theorem 3.2, that H_{V_ε} does not converge in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$, remains true in more general situations, for example, if $\Sigma \subset \mathbb{R}^\theta$, $\theta \in \{2, 3\}$, is as in Section 1.1 (vii) and contains a flat part, see [38, Theorem 6.7] for more details.

4 | APPROXIMATION OF DIRAC OPERATORS WITH δ-SHELL POTENTIALS SUPPORTED ON ROTATED C_b^2-GRAPHS

The main aim of this section is to prove Theorem 2.1 for the case that Σ is a rotated C_b^2-graph. In this case, there exist $\zeta \in C_b^2(\mathbb{R}^{\theta-1}; \mathbb{R})$ and $\kappa \in \mathrm{SO}(\theta)$ such that

$$\begin{aligned} \Omega_+ &= \{\kappa(x', x_\theta) : (x', x_\theta) \in \mathbb{R}^\theta \text{ and } x_\theta < \zeta(x')\}, \\ \Omega_- &= \mathbb{R}^\theta \setminus \overline{\Omega_+} = \{\kappa(x', x_\theta) : (x', x_\theta) \in \mathbb{R}^\theta \text{ and } x_\theta > \zeta(x')\}, \\ \Sigma &= \partial\Omega_+ = \Sigma_{\zeta,\kappa} := \{\kappa(x', \zeta(x')) : x' \in \mathbb{R}^{\theta-1}\}. \end{aligned} \tag{4.1}$$

To emphasize the importance of this result, let us rephrase Theorem 2.1 for the special case of a rotated C_b^2-graph.

Theorem 4.1. *Let Σ be a rotated C_b^2-graph, let $q \in L^\infty((-1,1);[0,\infty))$ with $\int_{-1}^1 q(s)\,ds = 1$ and assume that $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ satisfy the condition*

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2. \tag{4.2}$$

Let V and V_ε be as in (1.9) and (1.5), and define $\widetilde{V}$ by (1.11). Then, for all $z \in \mathbb{C}\setminus\mathbb{R}$ and $r \in (0,1/2)$ there exist $C>0$ and $\varepsilon_3 \in (0,\varepsilon_1)$ such that

$$\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}, \quad \varepsilon \in (0,\varepsilon_3).$$

In particular, H_{V_ε} converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

This theorem is a direct consequence of Proposition 4.7, Proposition 4.9, and Proposition 4.28.

Section 4 is devoted to proving these propositions. We proceed as follows: In Section 4.1, we introduce various integral operators associated to the free Dirac operator. Then, in Section 4.2, we recall comparable resolvent formulas for $H_{\widetilde{V}\delta_\Sigma}$ and H_{V_ε} from [8]. Moreover, relying again on results from [8] we provide convergence results for the operators involved in these resolvent formulas and end the section by stating Proposition 4.7 which gives abstract conditions for the norm resolvent convergence of H_{V_ε}. Afterwards, we show in Section 4.3 and Section 4.4 (in Proposition 4.9 and Proposition 4.28, respectively) that these conditions are met if (4.2) is fulfilled.

Finally, let us mention that the restriction to rotated C_b^2-graphs is only necessary in Section 4.4 and all results from the Sections 4.1–4.3 remain valid for the general class of hypersurfaces described in Section 1.1 (vii).

4.1 | The free Dirac operator and associated integral operators

Let $m \in \mathbb{R}$ and recall that the Dirac matrices $\alpha_1, \dots, \alpha_\theta, \beta \in \mathbb{C}^{N\times N}$ are given by (1.16)–(1.17). The free Dirac operator H is the differential operator in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ given by

$$H := -i(\alpha\cdot\nabla) + m\beta, \qquad \operatorname{dom} H := H^1(\mathbb{R}^\theta;\mathbb{C}^N). \tag{4.3}$$

With the help of the Fourier transform one gets that H is self-adjoint in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and $\sigma(H) = (-\infty,-|m|]\cup[|m|,\infty)$, see for instance [10, Section 2] for $\theta = 2$ and [39, Theorem 1.1] for $\theta = 3$. For $z \in \rho(H) = \mathbb{C}\setminus\sigma(H)$ we have

$$R_z u(x) := (H-z)^{-1}u(x) = \int_{\mathbb{R}^\theta} G_z(x-y)u(y)\,dy, \qquad u \in L^2(\mathbb{R}^\theta;\mathbb{C}^N),\ x \in \mathbb{R}^\theta,$$

where G_z is given for $\theta = 2$ and $x \in \mathbb{R}^2\setminus\{0\}$ by

$$G_z(x) = \frac{\sqrt{z^2-m^2}}{2\pi}K_1\Big(-i\sqrt{z^2-m^2}|x|\Big)\frac{\alpha\cdot x}{|x|} + \frac{1}{2\pi}K_0\Big(-i\sqrt{z^2-m^2}|x|\Big)(m\beta + zI_2) \tag{4.4}$$

and for $\theta = 3$ and $x \in \mathbb{R}^3\setminus\{0\}$ by

$$G_z(x) = \left(zI_4 + m\beta + i\Big(1 - i\sqrt{z^2-m^2}|x|\Big)\frac{\alpha\cdot x}{|x|^2}\right)\frac{e^{i\sqrt{z^2-m^2}|x|}}{4\pi|x|}; \tag{4.5}$$

cf., for example, [7, 9, 39]. Here, K_0 and K_1 denote the modified Bessel functions of the second kind of order zero and one, respectively. Note that R_z is bounded in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and it can also be viewed as a bounded operator from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta;\mathbb{C}^N)$.

We move on to the discussion of potential and boundary integral operators associated with the free Dirac operator. In the following, let $z \in \rho(H) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$ be fixed and let $\Omega_\pm$ and $\Sigma \subset \mathbb{R}^\theta$ be as in (4.1). First, we introduce the potential operator $\Phi_z : L^2(\Sigma; \mathbb{C}^N) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ by

$$\Phi_z\varphi(x) := \int_\Sigma G_z(x - y_\Sigma)\varphi(y_\Sigma)\, d\sigma(y_\Sigma), \qquad \varphi \in L^2(\Sigma; \mathbb{C}^N),\ x \in \mathbb{R}^\theta. \tag{4.6}$$

We note that Φ_z is indeed well-defined and bounded, see [2, Lemma 2.1]. Further properties of Φ_z are summarized in the following proposition. These results are well-known and a proof can be found in [8, Appendix C].

Proposition 4.2. *Let $z \in \rho(H) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$ and let Φ_z be given by (4.6). Then, the following is true:*

(i) For any $r \in [0, 1/2]$ the operator Φ_z gives rise to a bounded operator

$$\Phi_z : H^r(\Sigma; \mathbb{C}^N) \to H^{r+1/2}(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N).$$

(ii) For $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$ one has $(-i(\alpha \cdot \nabla) + m\beta - zI_N)(\Phi_z\varphi)_\pm = 0$.

(iii) The adjoint $\Phi_z^* : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to L^2(\Sigma; \mathbb{C}^N)$ of Φ_z acts on $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ as

$$\Phi_z^* u(x_\Sigma) = \int_{\mathbb{R}^\theta} G_{\bar z}(x_\Sigma - y)u(y)\, dy = \boldsymbol{t}_\Sigma R_{\bar z} u(x_\Sigma), \qquad x_\Sigma \in \Sigma,$$

and Φ_z^* gives rise to a bounded operator $\Phi_z^* : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$.

Finally, we introduce a family of boundary integral operators. Let $z \in \rho(H) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$. Then, we define the map $C_z : H^{1/2}(\Sigma; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$ by

$$C_z\varphi := \frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)\Phi_z\varphi, \qquad \varphi \in H^{1/2}(\Sigma; \mathbb{C}^N). \tag{4.7}$$

We remark that the operator C_z can be represented as a strongly singular boundary integral operator, see for instance [9, eq. (4.5) and Proposition 4.4 (ii)] for the case that Ω_+ is bounded. However, for our purposes the representation in (4.7) is more convenient. The basic properties of C_z are stated in the following proposition. Again, a proof can be found in [8, Appendix C].

Proposition 4.3. *Let $z \in \rho(H) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$ and let C_z be given by (4.7). Then, the following is true:*

(i) For any $r \in [-1/2, 1/2]$ the map C_z has a bounded extension $C_z : H^r(\Sigma; \mathbb{C}^N) \to H^r(\Sigma; \mathbb{C}^N)$.

(ii) For any $r \in (0, 1/2]$ and $\varphi \in H^r(\Sigma; \mathbb{C}^N)$ one has

$$C_z\varphi = \pm\frac{i}{2}(\alpha \cdot \nu)\varphi + \boldsymbol{t}_\Sigma^\pm(\Phi_z\varphi)_\pm.$$

4.2 | Formulas and convergence estimates for the resolvents of H_{V_ε} and $H_{\widetilde V\delta_\Sigma}$

In this section, we introduce resolvent formulas for H_{V_ε} and for $H_{\widetilde V\delta_\Sigma}$ in terms of (Bochner) integral operators and study their convergence properties.

Recall that $\mathcal{B}^0(\Sigma) = L^2((-1, 1); L^2(\Sigma; \mathbb{C}^N))$ and that W is the Weingarten map associated to Σ; cf. [8, Definition 2.3] and [4, Definition 2.2]. We (formally) define for $\varepsilon > 0$ and $z \in \rho(H)$ the following integral operators:

$$\begin{aligned} &A_\varepsilon(z) : \mathcal{B}^0(\Sigma) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N), \\ &A_\varepsilon(z)f(x) := \int_{-1}^1 \int_\Sigma G_z(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) f(s)(y_\Sigma) \det(I - \varepsilon s W(y_\Sigma))\, d\sigma(y_\Sigma)\, ds, \end{aligned} \tag{4.8a}$$

$$B_\varepsilon(z) : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma),$$
$$B_\varepsilon(z)f(t)(x_\Sigma) := \int_{-1}^{1} \int_\Sigma G_z(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) f(s)(y_\Sigma) \det(I - \varepsilon s W(y_\Sigma))\, d\sigma(y_\Sigma)\, ds, \tag{4.8b}$$

$$C_\varepsilon(z) : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to \mathcal{B}^0(\Sigma),$$
$$C_\varepsilon(z)u(t)(x_\Sigma) := \int_{\mathbb{R}^\theta} G_z(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y)u(y)\, dy. \tag{4.8c}$$

In the next proposition, the well-definedness of these operators and also their connection to the resolvent of H_{V_ε} are discussed.

Proposition 4.4. *Let $z \in \rho(H)$, q be as in (1.4), V be as in (1.9), H_{V_ε} be defined as in (1.6), and $\varepsilon_1 > 0$ be as below (1.2). Then, for all $\varepsilon \in (0, \varepsilon_1)$ the operators given in (4.8a)–(4.8c) are well-defined and if $-1 \in \rho(B_\varepsilon(z)Vq)$, then $z \in \rho(H_{V_\varepsilon})$ and the resolvent identity*

$$(H_{V_\varepsilon} - z)^{-1} = (H - z)^{-1} - A_\varepsilon(z)Vq(I + B_\varepsilon(z)Vq)^{-1}C_\varepsilon(z)$$

holds.

Proof. Note that $\varepsilon_1 > 0$ is chosen in exactly the same way as in [8, Proposition 2.4]. Thus, the assertions follow from [8, (3.1a)–(3.1c), Proposition 3.1, and Proposition 3.2]. □

Next, we introduce the operators $A_0(z)$, $B_0(z)$, and $C_0(z)$ which will turn out to be the limit operators of $A_\varepsilon(z)$, $B_\varepsilon(z)$, and $C_\varepsilon(z)$, respectively. For $z \in \rho(H)$, they are defined by

$$A_0(z) : \mathcal{B}^0(\Sigma) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N),$$
$$A_0(z)f := \Phi_z \int_{-1}^{1} f(t)\, dt,$$
$$B_0(z) : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma),$$
$$B_0(z)f(t) := \frac{i}{2}(\alpha \cdot \nu) \int_{-1}^{1} \operatorname{sign}(t-s) f(s)\, ds + \mathcal{C}_z \int_{-1}^{1} f(s)\, ds,$$
$$C_0(z) : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to \mathcal{B}^0(\Sigma),$$
$$C_0(z)u(t) := \Phi_{\bar{z}}^* u,$$

where Φ_z is the operator defined in (4.6) and $\mathcal{C}_z$ is the extension of the operator defined in (4.7) to $L^2(\Sigma; \mathbb{C}^N)$, see also Proposition 4.3 (i). Using the identifications described in Section 1.1 (x), one sees that the operators $A_0(z)$, $B_0(z)$, and $C_0(z)$ are well-defined and bounded since by Proposition 4.2 (i) Φ_z is a bounded operator from $L^2(\Sigma; \mathbb{C}^N)$ to $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ and by Proposition 4.3 (i) $\mathcal{C}_z$ is a bounded operator in $L^2(\Sigma; \mathbb{C}^N)$. The operator $B_0(z)$ can also be represented by

$$B_0(z) = T(\alpha \cdot \nu) + \mathfrak{I}\mathcal{C}_z\mathfrak{I}^*, \tag{4.9}$$

where $\mathfrak{I}$ is defined in Section 1.1 (x) and

$$T : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma), \qquad Tf(t) := \frac{i}{2} \int_{-1}^{1} \operatorname{sign}(t-s) f(s)\, ds.$$

Moreover, since C_z also acts as a bounded operator in $H^r(\Sigma;\mathbb{C}^N)$ for $r\in[-1/2,1/2]$, see Proposition 4.3 (i), $B_0(z)$ acts also as a bounded operator in $\mathcal{B}^r(\Sigma)$ for $r\in[-1/2,1/2]$.

Our next goal is to show a resolvent formula for $H_{\widetilde{V}\delta_\Sigma}$, which is defined by (1.7), in terms of $A_0(z)$, $B_0(z)$, and $C_0(z)$.

Proposition 4.5. *Let $z\in\rho(H)$, q and $V=\eta I_N+\tau\beta$ be as described in (1.4) and (1.9), respectively, and $d=\eta^2-\tau^2$ such that (1.10) is fulfilled. Moreover, let $(\widetilde{\eta},\widetilde{\tau})=\operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta,\tau)$, $\widetilde{V}=\widetilde{\eta}I_N+\widetilde{\tau}\beta$ and $\widetilde{d}=\widetilde{\eta}^2-\widetilde{\tau}^2$ fulfill (1.12). If $-1\in\rho(B_0(z)Vq)$, then for the self-adjoint operator $H_{\widetilde{V}\delta_\Sigma}$ the resolvent identity*

$$(H_{\widetilde{V}\delta_\Sigma}-z)^{-1}=(H-z)^{-1}-A_0(z)Vq(I+B_0(z)Vq)^{-1}C_0(z)$$

is valid.

Proof. Since $\widetilde{d}$ fulfils (1.12), $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint by the text above (1.12). Moreover, in the same way as in [8, below eq. (4.15)] one can show that if $-1\in\rho(B_0(z)Vq)$, then for $u\in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ holds

$$\big((H-z)^{-1}-A_0(z)Vq(I+B_0(z)Vq)^{-1}C_0(z)\big)u\in\operatorname{dom}H_{\widetilde{V}\delta_\Sigma}$$

and

$$(H_{\widetilde{V}\delta_\Sigma}-z)\big((H-z)^{-1}-A_0(z)Vq(I+B_0(z)Vq)^{-1}C_0(z)\big)u=u.$$

This shows that the resolvent identity is true. □

In the next proposition, we summarize the convergence properties of $A_\varepsilon(z)$, $B_\varepsilon(z)$, and $C_\varepsilon(z)$; cf. [8, Proposition 3.7, Proposition 3.8, and Proposition 3.10] for the proof.

Proposition 4.6. *Let $z\in\rho(H)$ and $\varepsilon_1>0$ be as below (1.2). Then, there exists an $\varepsilon_2\in(0,\varepsilon_1)$ such that $A_\varepsilon(z)$, $B_\varepsilon(z)$, and $C_\varepsilon(z)$ are uniformly bounded operators with respect to $\varepsilon\in(0,\varepsilon_2)$. Moreover, $C_\varepsilon(z)$ and $C_0(z)$ act also as uniformly bounded operators from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $\mathcal{B}^{1/2}(\Sigma)$ and for $r\in(0,1/2)$ there exists a $C>0$ such that one has for $\varepsilon\in(0,\varepsilon_2)$*

$$\|A_\varepsilon(z)-A_0(z)\|_{0\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\le C\varepsilon^{1/2-r},$$

$$\|B_\varepsilon(z)-B_0(z)\|_{1/2\to 0}\le C\varepsilon^{1/2-r},$$

$$\|C_\varepsilon(z)-C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0}\le C\varepsilon^{1/2-r}.$$

After summarizing the convergence properties of $A_\varepsilon(z)$, $B_\varepsilon(z)$, and $C_\varepsilon(z)$, we state in Proposition 4.7 conditions for the norm resolvent convergence of H_{V_ε}.

Proposition 4.7 [8, Proposition 3.12 and Remark 4.5]. *Let the assumptions of Proposition 4.5 hold. Moreover, let $r\in(0,1/2)$ and $\varepsilon_2>0$ be as in Proposition 4.6. Assume that the following conditions hold:*

(i) There exists an $\varepsilon_3\in(0,\varepsilon_2)$ such that $(I+B_\varepsilon(z)Vq)^{-1}$ exists for $\varepsilon\in(0,\varepsilon_3)$ and is uniformly bounded in $\mathcal{B}^0(\Sigma)$.
(ii) $I+B_0(z)Vq$ is bijective in $\mathcal{B}^{1/2}(\Sigma)$.

Then,

$$\|(H_{V_\varepsilon}-z)^{-1}-(H_{\widetilde{V}\delta_\Sigma}-z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\le C\varepsilon^{1/2-r}$$

for $\varepsilon\in(0,\varepsilon_3)$. In particular, H_{V_ε} converges for $\varepsilon\to 0$ to $H_{\widetilde{V}\delta_\Sigma}$ in norm resolvent sense.

According to Proposition 4.7, it is essential to study the operators $I + B_0(z)Vq$ and $I + B_\varepsilon(z)Vq$ in order to find explicit conditions for the norm resolvent convergence of H_{V_ε}. In Section 4.3 and Section 4.4, we show that (4.2) guarantees that the requirements (i) and (ii) of Proposition 4.7 are met.

4.3 | Analysis of $I + B_0(z)Vq$

We study the operator $I + B_0(z)Vq$ and are particularly interested under which conditions (ii) of Proposition 4.7 is fulfilled, that is, under which conditions $I + B_0(z)Vq$ is bijective in $\mathcal{B}^{1/2}(\Sigma)$. First, we state an auxiliary result about the invertibility of $I + C_z\widetilde{V}$ with C_z defined by (4.7), see also Proposition 4.3 (i).

Lemma 4.8. *Let $z \in \mathbb{C} \setminus \mathbb{R}$, $r \in [0, 1/2]$, $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma; \mathbb{R})$, and $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ such that $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ fulfils (1.12). Then, the operator $I + C_z\widetilde{V}$ is continuously invertible in $H^r(\Sigma; \mathbb{C}^N)$.*

Proof. We split the proof into three steps. In *Step 1*, we show that $\widetilde{V}$ and C_z act as bounded operators in $H^r(\Sigma; \mathbb{C}^N)$ for $r \in [0, 1/2]$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Afterwards, we show in *Step 2* that for $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $I + \widetilde{V}C_z$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$. In *Step 3*, we use *Steps 1 & 2* to prove the assertion.

Step 1. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $r \in [0, 1/2]$. The assumption $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma; \mathbb{R})$ implies

$$\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta \in C_b^1(\Sigma; \mathbb{C}^{N\times N}) \subset W^1_\infty(\Sigma; \mathbb{C}^{N\times N}).$$

Hence, $\widetilde{V}$ induces a bounded multiplication operator in $H^r(\Sigma; \mathbb{C}^N)$ which is bounded by $\|\widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$. Moreover, C_z acts as a bounded operator in $H^r(\Sigma; \mathbb{C}^N)$ by Proposition 4.3 (i).

Step 2. We already know from *Step 1* that C_z and $\widetilde{V}$ act as bounded operators in $H^{1/2}(\Sigma; \mathbb{C}^N)$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Hence, in order to prove that $I + \widetilde{V}C_z$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$, it suffices to show that $I + \widetilde{V}C_z$ is bijective in $H^{1/2}(\Sigma; \mathbb{C}^N)$. We begin with the injectivity. Let $\psi \in H^{1/2}(\Sigma; \mathbb{C}^N)$ such that $(I + \widetilde{V}C_z)\psi = 0$. We set $u = \Phi_z\psi$ with Φ_z defined by (4.6). Then, Proposition 4.2 (i) implies $u \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$. Next Proposition 4.2 (ii) yields $(-i(\alpha \cdot \nabla) + m\beta - zI_N)u_\pm = 0$ and Proposition 4.3 (ii) gives us

$$i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)u = \psi \quad \text{and} \quad \frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)u = C_z\psi. \tag{4.10}$$

Thus, (4.10) leads to

$$i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)u + \widetilde{V}\frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)u = \psi + \widetilde{V}C_z\psi = 0 \tag{4.11}$$

and hence $u \in \ker(H_{\widetilde{V}\delta_\Sigma} - z)$. Since $\widetilde{d}$ fulfills (1.12), the operator $H_{\widetilde{V}\delta_\Sigma}$ is by the text above (1.12) self-adjoint and therefore $\ker(H_{\widetilde{V}\delta_\Sigma} - z) = \{0\}$; implying $u = 0$ and therefore (4.10) shows $\psi = 0$. Now we turn to the surjectivity. Let $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$. Then, according to [22, Theorem 2], there exist $w_\pm \in H^1(\Omega_\pm; \mathbb{C}^N)$ such that $\boldsymbol{t}_\Sigma^\pm w_\pm = \frac{\mp i(\alpha\cdot\nu)}{2}\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$. Next, we set $w = w_+ \oplus w_- \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ and see

$$i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)w = \varphi \quad \text{as well as} \quad \frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)w = 0. \tag{4.12}$$

Moreover, let

$$v := (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}[(-i(\alpha \cdot \nabla) + m\beta - zI_N)w_+ \oplus (-i(\alpha \cdot \nabla) + m\beta - zI_N)w_-].$$

Then, $v \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ and $(-i(\alpha \cdot \nabla) + m\beta - zI_N)(v - w)_\pm = 0$, and thus due to [8, eq. (C.4) and the text below] there exists a $\psi \in H^{1/2}(\Sigma; \mathbb{C}^N)$ such that $\Phi_z\psi = w - v$. Hence, we use the relations (4.10) (for $u = \Phi_z\psi$) as in (4.11), (4.12) and

$v \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ to obtain

$$\begin{aligned}(I + \widetilde{V}C_z)\psi &= i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)\Phi_z\psi + \widetilde{V}\frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)\Phi_z\psi \\ &= i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)(w - v) + \widetilde{V}\frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)(w - v) \\ &= i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)w + \widetilde{V}\frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)w = \varphi.\end{aligned}$$

This proves the surjectivity and completes *Step 2*.

Step 3. First, let $r = 1/2$. In this case we already know from *Step 2* that $I + \widetilde{V}C_z$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Hence, elementary calculations show that then $I - C_z(I + \widetilde{V}C_z)^{-1}\widetilde{V}$ is the inverse of $I + C_z\widetilde{V}$. Moreover, it follows from *Step 1 & Step 2* that $I - C_z(I + \widetilde{V}C_z)^{-1}\widetilde{V}$ is also bounded as an operator in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Consequently, the assertion is true for $r = 1/2$. Next, let us consider the case $r \in [0, 1/2)$. From the proof of [8, Proposition 2.9] we obtain that the map $(C_{\overline{z}} \upharpoonright H^{1/2}(\Sigma;\mathbb{C}^N))'$ (where $'$ is used to denote the anti-dual operator) is a continuous extension of C_z to $H^{-1/2}(\Sigma;\mathbb{C}^N)$. Moreover, using the symmetry of $\widetilde{V}$ and the fact that $\widetilde{V}$ induces a bounded multiplication operator in $H^{1/2}(\Sigma;\mathbb{C}^N)$ shows that $\widetilde{V}$ can also be extended to a bounded multiplication operator in $H^{-1/2}(\Sigma;\mathbb{C}^N)$. Therefore,

$$((I + \widetilde{V}C_{\overline{z}}) \upharpoonright H^{1/2}(\Sigma;\mathbb{C}^N))' = I + (C_{\overline{z}} \upharpoonright H^{1/2}(\Sigma;\mathbb{C}^N))'(\widetilde{V} \upharpoonright H^{1/2}(\Sigma;\mathbb{C}^N))' \tag{4.13}$$

is a continuous extension of $I + C_z\widetilde{V}$ in $H^{-1/2}(\Sigma;\mathbb{C}^N)$. *Step 2* shows that $I + \widetilde{V}C_{\overline{z}}$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$ and therefore the operator in (4.13) has the bounded inverse $((I + \widetilde{V}C_{\overline{z}})^{-1} \upharpoonright H^{1/2}(\Sigma;\mathbb{C}^N))'$. Hence, one can use interpolation to show the assertion for $r \in [0, 1/2]$; cf. [8, eq. (2.2)] and [25, Theorem B.11]. □

Proposition 4.9. *Let $z \in \mathbb{C}\setminus\mathbb{R}$, $r \in [0, 1/2]$, q and $V = \eta I_N + \tau\beta$ be as described in (1.4) and (1.9), respectively, and $d = \eta^2 - \tau^2$ such that (1.10) is fulfilled. Moreover, let $(\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}(\frac{\sqrt{d}}{2})(\eta, \tau)$, $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ fulfill (1.12). Then, the operator $I + B_0(z)Vq$ is continuously invertible in the space $\mathcal{B}^r(\Sigma)$. In particular, assumption (ii) of Proposition 4.7 is fulfilled in this case.*

Proof. We start by arguing that $I + B_0(z)Vq$ is a bounded operator in $\mathcal{B}^r(\Sigma)$. Due to the representation of $B_0(z)$ in (4.9) and the text below, $B_0(z)$ acts as a bounded operator in $\mathcal{B}^r(\Sigma)$. Moreover, $V \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ and $q \in L^\infty((-1,1);[0,\infty))$ imply that Vq induces a bounded operator in $\mathcal{B}^r(\Sigma)$, see Section 1.1 (x). Hence, it remains to prove the bijectivity of $I + B_0(z)Vq$ in $\mathcal{B}^r(\Sigma)$.

Let us start with the injectivity. To do so, we use the representation of $B_0(z)$ given by (4.9) and assume for $f \in \mathcal{B}^r(\Sigma)$

$$(I + B_0(z)Vq)f = (I + T(\alpha\cdot\nu)Vq)f + \mathfrak{I}C_zV\mathfrak{I}^*qf = 0. \tag{4.14}$$

By [8, Lemma 4.2 (i)], $I + T(\alpha\cdot\nu)Vq$ is continuously invertible in in the space $\mathcal{B}^r(\Sigma)$ if $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. Note that the structure of $V = \eta I_N + \tau\beta$ and the rules for the Dirac matrices from (1.18) imply $((\alpha\cdot\nu)V)^2 = (\eta^2 - \tau^2)I_N = dI_N$ and therefore

$$\cos\Big(\frac{(\alpha\cdot\nu)V}{2}\Big) = \sum_{j=0}^{\infty}(-1)^j\frac{((\alpha\cdot\nu)V/2)^{2j}}{(2j)!} = \sum_{j=0}^{\infty}(-1)^j\frac{(\sqrt{d}/2)^{2j}}{(2j)!}I_N = \cos\Big(\frac{\sqrt{d}}{2}\Big)I_N.$$

Consequently, $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ since (1.10) is satisfied. Hence, we can apply $(I + T(\alpha\cdot\nu)Vq)^{-1}$ to (4.14). This yields

$$f + (I + T(\alpha\cdot\nu)Vq)^{-1}\mathfrak{I}C_zV\mathfrak{I}^*qf = 0.$$

Using [8, Lemma 4.2 (ii)] and defining $Q(t) = \int_{-1}^{t} q(s)\,ds - \frac{1}{2}$, $t \in [-1, 1]$, gives us

$$(I + T(\alpha \cdot \nu)Vq)^{-1}\mathfrak{I} = \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \exp(-i(\alpha \cdot \nu)VQ)\mathfrak{I}$$

and therefore

$$f + \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \exp(-i(\alpha \cdot \nu)VQ)\mathfrak{I}C_z V\mathfrak{I}^* qf = 0. \tag{4.15}$$

By applying $\mathfrak{I}^* q$ and [8, Lemma 4.3 (ii)] we obtain

$$\mathfrak{I}^* qf + \mathfrak{I}^* q \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \exp(-i(\alpha \cdot \nu)VQ)\mathfrak{I}C_z V\mathfrak{I}^* qf = (I + SC_z V)\mathfrak{I}^* qf = 0 \tag{4.16}$$

with $S = \operatorname{sinc}\big(\frac{(\alpha\cdot\nu)V}{2}\big)\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1}$. Similar as we showed $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big) = \cos\big(\frac{\sqrt{d}}{2}\big)I_N$ one can show the equality $\operatorname{sinc}\big(\frac{(\alpha\cdot\nu)V}{2}\big) = \operatorname{sinc}\big(\frac{\sqrt{d}}{2}\big)I_N$ and therefore

$$VS = V\operatorname{tanc}\big(\tfrac{\sqrt{d}}{2}\big) = \widetilde{V}. \tag{4.17}$$

Moreover, Lemma 4.8 shows that $-1 \in \rho(C_z\widetilde{V})$. By [28, Proposition 2.1.8], there holds $\rho(C_z\widetilde{V}) \setminus \{0\} = \rho(SC_z V) \setminus \{0\}$. Consequently, $-1 \in \rho(SC_z V)$ and hence (4.16) implies $\mathfrak{I}^* qf = 0$. This, in turn, implies according to (4.15) $f = 0$.

Next, we show the surjectivity of the operator $I + B_0(z)Vq$. Let $g \in \mathcal{B}^r(\Sigma)$. We set $f_g = (I + T(\alpha \cdot \nu)Vq)^{-1}(g + \mathfrak{I}\psi)$, where

$$\psi = -(I + C_z\widetilde{V})^{-1}C_z\mathfrak{I}^* Vq(I + T(\alpha \cdot \nu)Vq)^{-1}g.$$

Item (iii) from [8, Lemma 4.3] yields together with (4.17)

$$(I + B_0(z)Vq)(I + T(\alpha \cdot \nu)Vq)^{-1}\mathfrak{I} = \mathfrak{I}(I + C_z\widetilde{V}).$$

Thus, by applying $I + B_0(z)Vq$ to f_g and using $B_0(z) = T(\alpha \cdot \nu) + \mathfrak{I}C_z\mathfrak{I}^*$, see (4.9), we obtain

$$\begin{aligned}(I + B_0(z)Vq)f_g &= (I + B_0(z)Vq)(I + T(\alpha \cdot \nu)Vq)^{-1}g + (I + B_0(z)Vq)(I + T(\alpha \cdot \nu)Vq)^{-1}\mathfrak{I}\psi \\ &= (I + T(\alpha \cdot \nu)Vq + \mathfrak{I}C_z\mathfrak{I}^* Vq)(I + T(\alpha \cdot \nu)Vq)^{-1}g + \mathfrak{I}(I + C_z\widetilde{V})\psi \\ &= g + \mathfrak{I}C_z\mathfrak{I}^* Vq(I + T(\alpha \cdot \nu)Vq)^{-1}g + \mathfrak{I}(I + C_z\widetilde{V})\psi \\ &= g,\end{aligned}$$

which completes the proof. □

4.4 | Analysis of $I + B_\varepsilon(z)Vq$

In this section, we show that if

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2,$$

then (i) of Proposition 4.7 is fulfilled, that is, $(I + B_\varepsilon(z)Vq)^{-1}$ is uniformly bounded in $\mathcal{B}^0(\Sigma)$. Combining this result with Proposition 4.7 and Proposition 4.9 proves the norm resolvent convergence of H_{V_ε}. To prove the uniform boundedness of $(I + B_\varepsilon(z)Vq)^{-1}$ a careful and deep analysis of this operator is necessary. We do this by studying $(I + B_\varepsilon(z)Vq)^{-1}$ in the case that Σ is a hyperplane and η and τ are constant in detail in Section 4.4.1. Then, we use a parameter-dependent partition of unity in Section 4.4.2 to transfer the results to the case where $\Sigma = \Sigma_{\zeta,\kappa}$, $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$, $\kappa \in \mathrm{SO}(\theta)$, is a rotated C_b^2-graph as in (4.1) and $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$. We start by introducing for $\varepsilon \in (0,\varepsilon_2)$ with ε_2 as in Proposition 4.6 the auxiliary operator

$$\begin{aligned} \overline{B}_\varepsilon(z) &: \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma), \\ \overline{B}_\varepsilon(z) f(t)(x_\Sigma) &:= \int_{-1}^{1} \int_\Sigma G_z(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) - y_\Sigma) f(s)(y_\Sigma)\, d\sigma(y_\Sigma)\, ds; \end{aligned} \tag{4.18}$$

cf. [8, eq. (3.17), eq. (3.23), and Appendix B]. According to [8, eq. (3.29)],

$$\|\overline{B}_\varepsilon(z) - B_\varepsilon(z)\|_{0\to 0} \leq C\varepsilon^{1/2}(1 + |\log(\varepsilon)|)^{1/2}, \qquad \varepsilon \in (0,\varepsilon_2). \tag{4.19}$$

Next, we transfer this operator to $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. To do so, we introduce the isomorphism

$$\iota_{\zeta,\kappa} : L^2(\Sigma;\mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N), \quad (\iota_{\zeta,\kappa} f)(x') := f(\kappa(x', \zeta(x'))). \tag{4.20}$$

By transforming the integral on Σ to an integral on $\mathbb{R}^{\theta-1}$ one obtains that the norm of $\iota_{\zeta,\kappa}$ and its inverse are given by

$$\begin{aligned} \|\iota_{\zeta,\kappa}\|_{L^2(\Sigma;\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} &= \sup_{x'\in\mathbb{R}^{\theta-1}} (1 + |\nabla\zeta(x')|^2)^{-1/4}, \\ \|\iota_{\zeta,\kappa}^{-1}\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\Sigma;\mathbb{C}^N)} &= \sup_{x'\in\mathbb{R}^{\theta-1}} (1 + |\nabla\zeta(x')|^2)^{1/4}. \end{aligned} \tag{4.21}$$

Note that the definition of $H^r(\Sigma;\mathbb{C}^N)$, $r \in [0,2]$, see [25] or [8, Section 2.1], implies that $\iota_{\zeta,\kappa}$ also acts as an isomorphic operator from $H^r(\Sigma;\mathbb{C}^N)$ to $H^r(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ for $r \in [0,2]$. Recall that in this case $\iota_{\zeta,\kappa}$ can also be viewed as a bounded operator from $\mathcal{B}^r(\Sigma)$ to $\mathcal{B}^r(\mathbb{R}^{\theta-1})$ which has the same norm as the operator acting from $H^r(\Sigma;\mathbb{C}^N)$ to $H^r(\mathbb{R}^{\theta-1};\mathbb{C}^N)$.

Next, we introduce for $\varepsilon \in (0,\varepsilon_2)$ the operators

$$\begin{aligned} D_\varepsilon^{\zeta,\kappa}(z) &:= \iota_{\zeta,\kappa} \overline{B}_\varepsilon(z) \iota_{\zeta,\kappa}^{-1} : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \\ D_0^{\zeta,\kappa}(z) &:= \iota_{\zeta,\kappa} B_0(z) \iota_{\zeta,\kappa}^{-1} : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}). \end{aligned} \tag{4.22}$$

The results from Proposition 4.6 and (4.19) imply that $D_\varepsilon^{\zeta,\kappa}(z)$ is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $\varepsilon \in (0,\varepsilon_2)$ and for $r \in (0,1/2)$ the inequality

$$\begin{aligned} \|D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z)\|_{1/2\to 0} &= \|\iota_{\zeta,\kappa}\big(B_0(z) - \overline{B}_\varepsilon(z)\big)\iota_{\zeta,\kappa}^{-1}\|_{1/2\to 0} \\ &\leq C\big(\|B_0(z) - B_\varepsilon(z)\|_{1/2\to 0} + \|B_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{1/2\to 0}\big) \\ &\leq C\big(\|B_0(z) - B_\varepsilon(z)\|_{1/2\to 0} + \|B_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{0\to 0}\big) \\ &\leq C\varepsilon^{1/2-r}, \quad \varepsilon \in (0,\varepsilon_2), \end{aligned} \tag{4.23}$$

holds. In particular, $D_\varepsilon^{\zeta,\kappa}(z) f$ converges for $\varepsilon \to 0$ to $D_0^{\zeta,\kappa}(z) f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ for $f \in \mathcal{B}^{1/2}(\mathbb{R}^{\theta-1})$. Furthermore, $\mathcal{B}^{1/2}(\mathbb{R}^{\theta-1})$ is by [20, Lemma 1.2.19] and [25, the lines above eq. (3.22)] a dense subset of $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Combining these considerations with the uniform boundedness of $D_\varepsilon^{\zeta,\kappa}(z)$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ shows that for all $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$

$$D_\varepsilon^{\zeta,\kappa}(z) f \to D_0^{\zeta,\kappa}(z) f \quad \text{in } \mathcal{B}^0(\mathbb{R}^{\theta-1}), \text{ as } \varepsilon \to 0. \tag{4.24}$$

Using (4.18) and (4.20), and setting

$$x_{\zeta,\kappa} = \kappa(\cdot, \zeta(\cdot)) \quad \text{and} \quad \nu_{\zeta,\kappa} = \nu \circ x_{\zeta,\kappa} = \frac{\kappa(-\nabla\zeta, 1)}{\sqrt{1 + |\nabla\zeta|^2}} \tag{4.25}$$

yields for $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$ and a.e. $(t, x') \in (-1,1) \times \mathbb{R}^{\theta-1}$

$$D_\varepsilon^{\zeta,\kappa}(z)f(t)(x') = \int_{-1}^{1} \int_{\mathbb{R}^{\theta-1}} G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \varepsilon(t-s)\nu_{\zeta,\kappa}(x'))\sqrt{1 + |\nabla\zeta(y')|^2} f(s)(y')\, dy'\, ds. \tag{4.26}$$

Another useful representation is given by

$$D_\varepsilon^{\zeta,\kappa}(z)f(t) = \int_{-1}^{1} d_{\varepsilon(t-s)}^{\zeta,\kappa}(z)f(s)\, ds, \quad f \in \mathcal{B}^0(\mathbb{R}^{\theta-1}), t \in (-1,1), \tag{4.27}$$

where the integral is considered as a Bochner integral and

$$\begin{aligned} &d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) : L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N), \\ &d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)g(x') = \int_{\mathbb{R}^{\theta-1}} G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\sqrt{1 + |\nabla\zeta(y')|^2} g(y')\, dy', \end{aligned} \tag{4.28}$$

for $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2) \setminus \{0\}$. For the interaction strengths $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$, we also define the matrix-valued function

$$Q_{\eta,\tau}^{\zeta,\kappa} := V \circ x_{\zeta,\kappa} = \eta \circ x_{\zeta,\kappa} I_N + \tau \circ x_{\zeta,\kappa}\beta. \tag{4.29}$$

There holds $Q_{\eta,\tau}^{\zeta,\kappa} = \iota_{\zeta,\kappa} V \iota_{\zeta,\kappa}^{-1}$ in the sense of operators in $L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$.

4.4.1 | Hyperplanes and constant interaction strengths

In this section, we assume that ζ is a constant function having the value $y_0 \in \mathbb{R}$, that is, $\Sigma = \Sigma_{\zeta,\kappa} = \Sigma_{y_0,\kappa}$ is an affine $(\theta-1)$-dimensional hyperplane in $\mathbb{R}^\theta$. We also assume in this section that the interaction strengths are constant and given by $\eta, \tau \in \mathbb{R}$. This implies that $Q_{\eta,\tau}^{y_0,\kappa}$ is equal to the constant matrix

$$Q_{\eta,\tau} := \eta I_N + \tau\beta \tag{4.30}$$

in this case. The main goal of this section is to show that for every compact set $S \subset \mathbb{R}^2$ satisfying

$$\max_{(\eta,\tau)\in S} \eta^2 - \tau^2 < \frac{\pi^2}{4}$$

there exists a $\delta_2 = \delta_2(S)$ such that

$$\|(I + D_\varepsilon^{y_0,\kappa}(z)Q_{\eta,\tau})^{-1}\|_{0\to 0}$$

is uniformly bounded with respect to $(\varepsilon, y_0, (\eta,\tau), \kappa) \in (0, \delta_2) \times \mathbb{R} \times S \times \mathrm{SO}(\theta)$; cf. Corollary 4.18. This result will play a major role when we prove the uniform boundedness of $(I + B_\varepsilon(z)Vq)^{-1}$ in Section 4.4.2. To do so we proceed as follows: We start by using the Fourier transform to transform $D_\varepsilon^{y_0,\kappa}(z)$ into a decomposable direct integral operator with frequency dependent fiber operators, see the considerations up to (4.36). Then, up to Lemma 4.16 we find and analyze suitable approximations for the fiber operators for high and low frequencies. Finally, we use these results to prove the main statements (Proposition 4.17 and Corollary 4.18) of this section.

Before we start, let us fix some notations. In the present setting, the normal vector ν is constant and given by κe_θ, where e_θ is the θth euclidean unit vector. Let us also mention that ε_2 from Proposition 4.4 is according to [8, eq. (3.11)] given by

$$\varepsilon_2 = \min\left\{\frac{\varepsilon_1}{2}, \frac{1}{2\|D\widetilde{\nu}\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})}}\right\},$$

where, ε_1 is chosen as below (1.2), see also [8, Proposition 2.4] and $\widetilde{\nu}$ is an extension C_b^1 of ν to $\mathbb{R}^\theta$; cf. [8, the discussion above eq. (3.5)]. Next, we argue that ε_2 can be set independently of κ to ∞. First, ε_1 is chosen such that one can identify Ω_{ε_1} via the map ι in (1.2) with $\Sigma \times (-\varepsilon_1, \varepsilon_1)$ and the eigenvalues of $\varepsilon_1 W(x_\Sigma)$, $x_\Sigma \in \Sigma$, are sufficiently small, so that this identification is bijective. However, in the current case, $\Omega_{\varepsilon_1} = \kappa(\mathbb{R}^{\theta-1} \times (y_0 - \varepsilon_1, y_0 + \varepsilon_1))$ can be identified with

$$\Sigma \times (-\varepsilon_1, \varepsilon_1) = \kappa(\mathbb{R}^{\theta-1} \times \{y_0\}) \times (-\varepsilon_1, \varepsilon_1)$$

for arbitrary $\varepsilon_1 > 0$ and the Weingarten map is zero. Thus, we can set ε_1 independently of y_0 and κ to ∞. Furthermore, as ν is constant, its constant extension is a C_b^1-extension of ν. Hence, $(2\|D\widetilde{\nu}\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})})^{-1} = \infty$ and therefore $\varepsilon_2 = \infty$. Using (4.26) and

$$x_{y_0,\kappa}(x') - x_{y_0,\kappa}(y') + \varepsilon(t-s)\nu_{y_0,\kappa}(x') = \kappa(x'-y', \varepsilon(t-s)),\ x', y' \in \mathbb{R}^{\theta-1}, t, s \in (-1,1),$$

shows that $D_\varepsilon^{y_0,\kappa}(z)$ has for $\varepsilon \in (0,\infty)$ and $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$ the representation

$$D_\varepsilon^{y_0,\kappa}(z) f(t)(x') = \int_{-1}^{1} \int_{\mathbb{R}^{\theta-1}} G_z(\kappa(x'-y', \varepsilon(t-s))) f(s)(y')\, dy'\, ds \tag{4.31}$$

for a.e. $(x',t) \in \mathbb{R}^{\theta-1} \times (-1,1)$, which proves that $D_\varepsilon^{y_0,\kappa}(z)$ is independent of $y_0 \in \mathbb{R}$. Furthermore, (4.24) implies that also $D_0^{y_0,\kappa}(z)$ is independent of y_0. Thus, w.l.o.g. we can set $y_0 = 0$. Before we state the next result, we define for convenience the matrices $\widetilde{\alpha}_j := \alpha \cdot \kappa e_j$, $j \in \{1,\dots,\theta\}$, $\widetilde{\alpha} \cdot \xi := \sum_{j=1}^{\theta} \widetilde{\alpha}_j \xi_j$, $\xi \in \mathbb{R}^\theta$, and $\widetilde{\alpha}' \cdot \xi' = \sum_{j=1}^{\theta-1} \widetilde{\alpha}_j \xi_j'$, $\xi' \in \mathbb{R}^{\theta-1}$. Note that $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_\theta$ satisfy the same anti-commutation relations as $\alpha_1, \dots, \alpha_\theta$; cf. (vi) of Section 1.1.

We start by calculating the Fourier transform of the function $G_z(\kappa(\cdot, \widetilde{\varepsilon}))$ for fixed $\widetilde{\varepsilon} \neq 0$; cf. [31, eqs. (44)–(45)] for similar considerations. Recall that $\mathcal{F}$ is the $(\theta-1)$-dimensional Fourier transform defined in Section 1.1 (xi).

Lemma 4.10. *Let $z \in \rho(H)$, G_z be the integral kernel of $(H-z)^{-1}$ given by (4.4)–(4.5), and $\widetilde{\varepsilon} \neq 0$. Then,*

$$\mathcal{F} G_z(\kappa(\cdot, \widetilde{\varepsilon})) = \left(\frac{\widetilde{\alpha}' \cdot (\cdot) + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\cdot|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(\widetilde{\varepsilon}) \right) \frac{i e^{|\widetilde{\varepsilon}| i \sqrt{z^2 - m^2 - |\cdot|^2}}}{2\sqrt{(2\pi)^{\theta-1}}}.$$

Proof. Let $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_{1,2}$ be defined as in Section 1.1 (xi). First, we consider $\mathcal{F}_1 G_z(\kappa(\cdot))$. Since one has $G_z(\kappa(\cdot)) \in L^1(\mathbb{R}^\theta; \mathbb{C}^{N\times N}) \subset \mathcal{S}'(\mathbb{R}^\theta; \mathbb{C}^{N\times N})$, see (4.4)–(4.5), the expression $\mathcal{F}_1 G_z(\kappa(\cdot))$ is well-defined in $\mathcal{S}'(\mathbb{R}^\theta; \mathbb{C}^{N\times N})$. Moreover, $\mathcal{F}_1 G_z(\kappa(\cdot)) = \mathcal{F}_2^{-1} \mathcal{F}_{1,2} G_z(\kappa(\cdot))$. Thus, we calculate $\mathcal{F}_{1,2} G_z(\kappa(\cdot))$ next. The function G_z satisfies the equation $(-i(\alpha \cdot \nabla) + m\beta - zI_N) G_z = \delta I_N$, with δ denoting the δ distribution supported in $\{0\}$. Hence, the standard rules for the Fourier transform, see [33, Chapter IX], show

$$(\alpha \cdot (\cdot) + m\beta - zI_N) \mathcal{F}_{1,2} G_z = \frac{1}{\sqrt{(2\pi)^\theta}} I_N \quad \text{in } \mathcal{S}'(\mathbb{R}^\theta; \mathbb{C}^{N\times N}).$$

Furthermore, since $G_z \in L^1(\mathbb{R}^\theta; \mathbb{C}^{N\times N})$, we have by the Riemann–Lebesgue lemma $\mathcal{F}_{1,2} G_z \in C_0(\mathbb{R}^\theta; \mathbb{C}^{N\times N})$. Using the properties of α_j, $j \in \{1,\dots,\theta\}$, and β yields $(\alpha \cdot \xi + m\beta - zI_N)^{-1} = \frac{\alpha \cdot \xi + m\beta + zI_N}{|\xi|^2 + m^2 - z^2}$ for $\xi \in \mathbb{R}^\theta$. Thus,

$$(\mathcal{F}_{1,2} G_z)(\kappa\xi) = \frac{\alpha \cdot (\kappa\xi) + m\beta + zI_N}{(|\kappa\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}}, \quad \xi \in \mathbb{R}^\theta.$$

Since $\kappa \in \mathrm{SO}(\theta)$, we obtain

$$\mathcal{F}_{1,2} G_z(\kappa(\cdot))(\xi) = (\mathcal{F}_{1,2} G_z)(\kappa\xi) = \frac{\widetilde{\alpha} \cdot \xi + m\beta + zI_N}{(|\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}}, \quad \xi \in \mathbb{R}^\theta. \tag{4.32}$$

Next, we determine $\mathcal{F}_2^{-1}\mathcal{F}_{1,2}G_z$. We claim that for a.e. $(\xi', x_\theta) \in \mathbb{R}^\theta$ the equation

$$(\mathcal{F}_2^{-1}\mathcal{F}_{1,2}G_z(\kappa(\cdot)))(\xi', x_\theta) = \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(x_\theta)\right)\frac{ie^{|x_\theta| i\sqrt{z^2-m^2-|\xi'|^2}}}{2\sqrt{(2\pi)^{\theta-1}}} \tag{4.33}$$

holds. We verify (4.33) by applying $\mathcal{F}_2$ and comparing the result with (4.32). Since the right-hand side of (4.33) decays exponentially for $|x_\theta| \to \infty$, we can use the integral representation of the Fourier transform. Hence, simple integration gives us for $\xi = (\xi', \xi_\theta) \in \mathbb{R}^\theta$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(x_\theta)\right) \cdot \frac{ie^{i(-x_\theta\xi_\theta + |x_\theta|\sqrt{z^2-m^2-|\xi'|^2})}}{2\sqrt{(2\pi)^{\theta-1}}}\, dx_\theta = \frac{\widetilde{\alpha} \cdot \xi + m\beta + zI_N}{(|\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}},$$

which verifies (4.33). Therefore, $\mathcal{F}_1 G_z(\kappa(\cdot)) = \mathcal{F}_2^{-1}\mathcal{F}_{1,2}G_z(\kappa(\cdot))$ can be represented by the function

$$\mathbb{R}^\theta \ni (\xi', x_\theta) \mapsto \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(x_\theta)\right)\frac{ie^{|x_\theta| i\sqrt{z^2-m^2-|\xi'|^2}}}{2\sqrt{(2\pi)^{\theta-1}}}.$$

Furthermore, since $G_z(\kappa(\cdot, \widetilde{\varepsilon})) \in L^1(\mathbb{R}^{\theta-1}; \mathbb{C}^{N\times N})$ for $\widetilde{\varepsilon} \neq 0$, this shows that for $\widetilde{\varepsilon} \neq 0$ and $\xi' \in \mathbb{R}^{\theta-1}$, there holds

$$\begin{aligned}
\mathcal{F}G_z(\kappa(\cdot, \widetilde{\varepsilon}))(\xi') &= \frac{1}{\sqrt{(2\pi)^{\theta-1}}} \int_{\mathbb{R}^{\theta-1}} G_z(\kappa(x', \widetilde{\varepsilon}))e^{-i\langle x', \xi'\rangle}\, dx' \\
&= \mathcal{F}_1 G_z(\kappa(\cdot))(\xi', \widetilde{\varepsilon}) \\
&= \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(\widetilde{\varepsilon})\right)\frac{ie^{|\widetilde{\varepsilon}| i\sqrt{z^2-m^2-|\xi'|^2}}}{2\sqrt{(2\pi)^{\theta-1}}}.
\end{aligned}$$

□

Proposition 4.11. *Let $z \in \rho(H)$, $\varepsilon > 0$, and $\mathcal{F}$ be the $(\theta - 1)$-dimensional Fourier transform defined in Section 1.1 (xi). Then, there holds for $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$*

$$\mathcal{F}D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1}f(t)(\xi') = \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s)\right) \cdot \frac{ie^{|\varepsilon(t-s)| i\sqrt{z^2-m^2-|\xi'|^2}}}{2} f(s)(\xi')\, ds \tag{4.34}$$

and

$$\mathcal{F}D_0^{0,\kappa}(z)\mathcal{F}^{-1}f(t)(\xi') = \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s)\right)\frac{if(s)(\xi')}{2}\, ds \tag{4.35}$$

for a.e $(t, \xi') \in (-1, 1) \times \mathbb{R}^{\theta-1}$. In particular, the operators $D_\varepsilon^{0,\kappa}(z)$ and $D_0^{0,\kappa}(z)$ depend continuously on $\kappa \in \mathrm{SO}(\theta)$ with respect to the operator norm.

Proof. We start with the case $\varepsilon > 0$. Equation (4.31) shows

$$D_\varepsilon^{0,\kappa}(z)f(t) = \int_{-1}^{1} G_z(\kappa(\cdot, \varepsilon(t-s))) * f(s)\, ds$$

for a.e. $t \in (-1,1)$ and all $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$. Thus, Lemma 4.10 and [33, Theorem IX.4] prove the statement for $\varepsilon > 0$. It remains to consider the operator $D_0^{0,\kappa}(z)$. We start by defining

$$\widetilde{D}_0^{0,\kappa}(z) : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1})$$

$$\widetilde{D}_0^{0,\kappa}(z)f(t)(\xi') := \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \right) \frac{i f(s)(\xi')}{2}\, ds.$$

Next, let $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$. Using (4.34) and dominated convergence, see [20, Proposition 1.2.5], one obtains that $\mathcal{F} D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1} f$ converges for $\varepsilon \to 0$ to $\widetilde{D}_0^{0,\kappa}(z) f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Thus, the boundedness of $\mathcal{F}$ and $\mathcal{F}^{-1}$ in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ (and therefore also in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$; cf. Section 1.1 (x)) implies that $D_\varepsilon^{0,\kappa}(z)f$ converges to $\mathcal{F}^{-1}\widetilde{D}_0^{0,\kappa}(z)\mathcal{F}f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Moreover, by (4.24) $D_\varepsilon^{0,\kappa}(z)f$ converges to $D_0^{0,\kappa}(z)f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Hence, $D_0^{0,\kappa}(z)f = \mathcal{F}^{-1}\widetilde{D}_0^{0,\kappa}(z)\mathcal{F}f$ which proves (4.35).

It remains to verify the continuous dependence on κ. Taking the definition of $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_\theta$, see the text before Lemma 4.10, and $D_\varepsilon^{0,\kappa}(z)$ and $D_0^{0,\kappa}(z)$ in (4.34) and (4.35), respectively, into account, one sees that all terms that depend on κ can be taken out of the integrals. This implies the claimed continuity. □

The structure of $\mathcal{F} D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1}$ and $\mathcal{F} D_0^{0,\kappa}(z)\mathcal{F}^{-1}$ inspires us to change our viewpoint. Namely, instead of viewing theses operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ we consider them as operators in the isometrically isomorphic space $L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N))$. In [34, Chapter XIII.16] and generally in the context of direct integrals the notation $\int_{\mathbb{R}^{\theta-1}}^{\oplus} L^2((-1,1);\mathbb{C}^N)\, d\xi'$ for this space is also common. Considered as operators acting in this space $\mathcal{F} D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1}$ and $\mathcal{F} D_0^{0,\kappa}(z)\mathcal{F}^{-1}$ are decomposable direct integral operators with fibers which are defined for $\varepsilon \geq 0$, $\xi' \in \mathbb{R}^{\theta-1}$, and $z \in \rho(H)$ by

$$\mathfrak{D}_{\varepsilon,\xi'}(z) : L^2((-1,1);\mathbb{C}^N) \to L^2((-1,1);\mathbb{C}^N),$$

$$\mathfrak{D}_{\varepsilon,\xi'}(z)f(t) := \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \right) \cdot \frac{i e^{|\varepsilon(t-s)| i \sqrt{z^2-m^2-|\xi'|^2}}}{2} f(s)\, ds. \tag{4.36}$$

Remark 4.12. These operators still depend on the rotation matrix $\kappa \in \mathrm{SO}(\theta)$ since $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_\theta$ depend on κ. However, since we use these operators only as auxiliary operators in this section, we omit κ.

Next, we explain the ideas described before (4.36) in a more rigorous way. Using

$$\begin{aligned}\mathcal{B}^0(\mathbb{R}^{\theta-1}) &= L^2((-1,1); L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)) \\ &\simeq L^2((-1,1)\times\mathbb{R}^{\theta-1};\mathbb{C}^N) \simeq L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)),\end{aligned}$$

see [20, Corollary 1.2.23 and Proposition 1.2.24], allows us to define the isometric isomorphism

$$\mathfrak{i} : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)),$$

$$\mathfrak{i} f(\xi')(t) := \mathcal{F} f(t)(\xi') \quad \text{for a.e. } (\xi',t) \in \mathbb{R}^{\theta-1}\times(-1,1).$$

Thus, by Proposition 4.11 and (4.36), we obtain for $\varepsilon \geq 0$ that

$$\mathfrak{i} D_\varepsilon^{0,\kappa}(z)\mathfrak{i}^{-1} : L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)) \to L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N))$$

acts for $f \in L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N))$ and fixed $\xi' \in \mathbb{R}^{\theta-1}$ as $\mathfrak{D}_{\varepsilon,\xi'}(z)f(\xi')$, that is, $\mathfrak{i} D_\varepsilon^{0,\kappa}(z)\mathfrak{i}^{-1}$ is a direct integral operator which can be decomposed in the fiber operators $\mathfrak{D}_{\varepsilon,\xi'}(z)$; cf. [34, Chapter XIII.16].

Now, we use the theory of direct integrals in order to transfer results regarding $\mathfrak{D}_{\varepsilon,\xi'}(z)$ to $D_\varepsilon^{0,\kappa}(z)$ and vice versa. We formulate this in the upcoming lemma which follows from [34, Theorem XIII.83 and Theorem XIII.84]. There, we denote by $\mathcal{L}(L^2((-1,1);\mathbb{C}^N))$ the space of all bounded operators in $L^2((-1,1);\mathbb{C}^N)$.

Lemma 4.13. *Let $\mathfrak{M} \in L^\infty(\mathbb{R}^{\theta-1}; \mathcal{L}(L^2((-1,1);\mathbb{C}^N)))$ and M be defined by*

$$M : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \qquad \mathfrak{i} M \mathfrak{i}^{-1} f(\xi') := \mathfrak{M}(\xi') f(\xi').$$

Then,

$$\|M\|_{0\to 0} = \|\mathfrak{M}\|_{L^\infty(\mathbb{R}^{\theta-1};\mathcal{L}(L^2((-1,1);\mathbb{C}^N)))} = \operatorname{ess\,sup}_{\xi'\in\mathbb{R}^{\theta-1}} \|\mathfrak{M}(\xi')\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}$$

and the operator M is continuously invertible if and only if $\mathfrak{M}$ is continuously invertible a.e. and the inequality $\|\mathfrak{M}^{-1}\|_{L^\infty(\mathbb{R}^{\theta-1};\mathcal{L}(L^2((-1,1);\mathbb{C}^N)))} < \infty$ holds. Furthermore, in this case $\|M^{-1}\|_{0\to 0} = \|\mathfrak{M}^{-1}\|_{L^\infty(\mathbb{R}^{\theta-1};\mathcal{L}(L^2((-1,1);\mathbb{C}^N)))}$.

Next, we study the operator $\mathfrak{D}_{\varepsilon,\xi'}(z)$ in detail. For this purpose, we introduce for $\rho \in (0,\infty)$ and $w' \in \mathbb{R}^{\theta-1}$ with $|w'| = 1$ the auxiliary operator

$$\begin{aligned} \mathfrak{H}_{\rho,w'} &: L^2((-1,1);\mathbb{C}^N) \to L^2((-1,1);\mathbb{C}^N) \\ (\mathfrak{H}_{\rho,w'} f)(t) &:= \int_{-1}^{1} \big(\widetilde{\alpha}' \cdot w' + i\widetilde{\alpha}_\theta \operatorname{sign}(t-s)\big) \frac{e^{-\rho|t-s|}}{2} f(s)\, ds. \end{aligned}$$

It is not difficult to check that $\mathfrak{H}_{\rho,w'}$ is a self-adjoint Hilbert–Schmidt operator.

Lemma 4.14. *Let $\varepsilon > 0$, $\xi' \in \mathbb{R}^{\theta-1} \setminus \{0\}$, and $z \in \rho(H)$. Then, there exists a constant $C_1 > 0$ which only depends on m and z such that*

$$\begin{aligned} \|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{D}_{0,\xi'}(z)\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} &\le C_1\varepsilon(1+|\xi'|), \\ \|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} &\le \frac{C_1}{1+|\xi'|}. \end{aligned}$$

Proof. In this proof $C > 0$ denotes a constant which may change in between lines, but only depends on m and z. We start by estimating the kernel of the integral operator $\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{D}_{0,\xi'}(z)$. It is easy to see that we can estimate this kernel by

$$\begin{aligned} C\left|1 - e^{\varepsilon|t-s| i\sqrt{z^2-m^2-|\xi'|^2}}\right| &\le C\varepsilon|t-s|\sqrt{|z^2-m^2-|\xi'|^2|} \\ &\le C\varepsilon(1+|\xi'|). \end{aligned}$$

With this estimate, one finds a constant C_1 which only depends on m and z such that

$$\|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{D}_{0,\xi'}(z)\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \le C_1\varepsilon(1+|\xi'|).$$

Next, we estimate the kernel of $\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}$ by

$$\begin{aligned} &\frac{1}{2}\left|\left(e^{-\varepsilon|t-s||\xi'|} - e^{\varepsilon|t-s| i\sqrt{m^2-z^2-|\xi'|^2}}\right)\left(\widetilde{\alpha}' \cdot \frac{\xi'}{|\xi'|} + \widetilde{\alpha}_\theta i \operatorname{sign}(t-s)\right)\right| \\ &\quad + \frac{1}{2}\left| e^{\varepsilon|t-s| i\sqrt{z^2-m^2-|\xi'|^2}} \left(\widetilde{\alpha}' \cdot \frac{\xi'}{|\xi'|}\right)\left(1 - \frac{i|\xi'|}{\sqrt{z^2-m^2-|\xi'|^2}}\right)\right| \\ &\quad + \frac{1}{2}\left| e^{\varepsilon|t-s| i\sqrt{z^2-m^2-|\xi'|^2}} \left(\frac{m\beta + zI_N}{\sqrt{z^2-m^2-|\xi'|^2}}\right)\right|. \end{aligned} \tag{4.37}$$

The first term in (4.37) can be estimated for $\varepsilon|t-s| \le 1$ by

$$C\left|e^{-\varepsilon|t-s||\xi'|} - e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\right| \le C\varepsilon|t-s|\left||\xi'| + i\sqrt{z^2-m^2-|\xi'|^2}\right|$$
$$\le C\frac{|z^2-m^2|}{\left||\xi'| - i\sqrt{z^2-m^2-|\xi'|^2}\right|}$$
$$\le \frac{C}{1+|\xi'|},$$

where we used $z \in \rho(H) = \mathbb{C}\setminus((-\infty,-|m|]\cup[|m|,\infty))$ and $\operatorname{Im}\sqrt{w} > 0$ for $w \in \mathbb{C}\setminus[0,\infty)$. For $\varepsilon|t-s| > 1$, we get

$$C\left|e^{-\varepsilon|t-s||\xi'|} - e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\right| \le C(e^{-|\xi'|} + e^{-\operatorname{Im}\sqrt{z^2-m^2-|\xi'|^2}}) \le \frac{C}{1+|\xi'|}.$$

Similarly as we estimated the first term in the case $\varepsilon|t-s| \le 1$, the second term in (4.37) can be estimated by

$$C\left|1 - \frac{i|\xi'|}{\sqrt{z^2-m^2-|\xi'|^2}}\right| = C\frac{|z^2-m^2|}{\left|\sqrt{z^2-m^2-|\xi'|^2} + i|\xi'|\right|\left|\sqrt{z^2-m^2-|\xi'|^2}\right|}$$
$$\le \frac{C}{(1+|\xi'|)^2} \le \frac{C}{1+|\xi'|}.$$

One also sees that the third term in (4.37) is smaller than $\frac{C}{1+|\xi'|}$ for sufficiently large C. Summing up, we have that the kernel of $\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}$ can be bounded by $\frac{C}{1+|\xi'|}$ and therefore if C_1 is chosen sufficiently large, then

$$\|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \le \frac{C_1}{1+|\xi'|}.$$

□

Lemma 4.15. *Let $\rho > 0$, $w' \in \mathbb{R}^{\theta-1}$ with $|w'| = 1$, and q be as in (1.4). Then,* $\sigma(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}) \subset [-2/\pi, 2/\pi]$.

Proof. To shorten notation we set $\widetilde{\alpha}_\pm := \widetilde{\alpha}'\cdot w' \pm i\widetilde{\alpha}_\theta$. Then, one has for $f \in L^2((-1,1);\mathbb{C}^N)$

$$\mathfrak{H}_{\rho,w'}f(t) = \frac{1}{2}\int_t^1 e^{-\rho|t-s|}\widetilde{\alpha}_- f(s)\,ds + \frac{1}{2}\int_{-1}^t e^{-\rho|t-s|}\widetilde{\alpha}_+ f(s)\,ds.$$

Using the anti-commutation rules for $\widetilde{\alpha}'\cdot w'$ and $\widetilde{\alpha}_\theta$, and $(\widetilde{\alpha}\cdot w')^2 = \widetilde{\alpha}_\theta^2 = I_N$, see the definition of $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_\theta$ before Lemma 4.10 and (vi) in Section 1.1, shows that $\operatorname{ran}\widetilde{\alpha}_+ \perp \operatorname{ran}\widetilde{\alpha}_-$. Hence, as $q \ge 0$, we have for $f \in L^2((-1,1);\mathbb{C}^N)$

$$\|\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}f\|^2_{L^2((-1,1);\mathbb{C}^N)} = \frac{1}{4}\int_{-1}^1 q(t)\left|\int_t^1 e^{-\rho|t-s|}\widetilde{\alpha}_-\sqrt{q(s)}f(s)\,ds\right|^2 dt$$
$$+ \frac{1}{4}\int_{-1}^1 q(t)\left|\int_{-1}^t e^{-\rho|t-s|}\widetilde{\alpha}_+\sqrt{q(s)}f(s)\,ds\right|^2 dt. \tag{4.38}$$

We start by estimating the first term on the right-hand side. To do so, we assume that Q is the primitive function of q such that $Q(1) = 0$ and therefore by $\int_{-1}^1 q(s)\,ds = 1$, $Q(-1) = -1$. Applying the Cauchy–Schwarz inequality and Fubini's theorem yields

$$\frac{1}{4}\int_{-1}^1 q(t)\left|\int_t^1 e^{-\rho|t-s|}\widetilde{\alpha}_-\sqrt{q(s)}f(s)\,ds\right|^2 dt$$
$$= \frac{1}{4}\int_{-1}^1 q(t)\left|\int_t^1 \frac{\sqrt{\cos(\frac{\pi}{2}Q(s))}}{\sqrt{\cos(\frac{\pi}{2}Q(s))}}e^{-\rho|t-s|}\widetilde{\alpha}_-\sqrt{q(s)}f(s)\,ds\right|^2 dt$$

$$\leq \frac{1}{4}\int_{-1}^{1} q(t)\left(\int_t^1 \cos\left(\tfrac{\pi}{2}Q(s)\right)q(s)\,ds\right)\left(\int_t^1 \frac{1}{\cos(\frac{\pi}{2}Q(s))}|\widetilde{\alpha}_- f(s)|^2\,ds\right)dt$$

$$= \frac{1}{2\pi}\int_{-1}^{1} -\sin\left(\tfrac{\pi}{2}Q(t)\right)q(t)\left(\int_t^1 \frac{1}{\cos\left(\frac{\pi}{2}Q(s)\right)}|\widetilde{\alpha}_- f(s)|^2\,ds\right)dt$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\left(\int_{-1}^{s} -\sin\left(\tfrac{\pi}{2}Q(t)\right)q(t)\,dt\right)\frac{1}{\cos\left(\frac{\pi}{2}Q(s)\right)}|\widetilde{\alpha}_- f(s)|^2\,ds$$

$$= \frac{1}{\pi^2}\int_{-1}^{1}|\widetilde{\alpha}_- f(s)|^2\,ds.$$

The same trick with $Q+1$ instead of Q yields that the second term of the right-hand side of Equation (4.38) can be estimated by $\frac{1}{\pi^2}\int_{-1}^{1}|\widetilde{\alpha}_+ f(s)|^2\,ds$. Using these estimates and $\operatorname{ran}\widetilde{\alpha}_+ \perp \operatorname{ran}\widetilde{\alpha}_-$ gives us

$$\begin{aligned}\|\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}f\|^2_{L^2((-1,1);\mathbb{C}^N)} &\leq \frac{1}{\pi^2}\int_{-1}^{1}|\widetilde{\alpha}_- f(s)|^2 + |\widetilde{\alpha}_+ f(s)|^2\,ds\\ &= \frac{1}{\pi^2}\int_{-1}^{1}|(\widetilde{\alpha}_- + \widetilde{\alpha}_+)f(s)|^2\,ds\\ &= \frac{1}{\pi^2}\int_{-1}^{1}|2(\widetilde{\alpha}'\cdot w')f(s)|^2\,ds\\ &= \frac{4}{\pi^2}\|f\|^2_{L^2((-1,1);\mathbb{C}^N)}.\end{aligned}$$

Since $\mathfrak{H}_{\rho,w'}$ is self-adjoint in $L^2((-1,1);\mathbb{C}^N)$, we obtain $\sigma(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}) \subset [-2/\pi, 2/\pi]$. □

Having studied the spectrum of $\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}$, we employ this knowledge to study the bounded invertibility of the operator $I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q$. Recall that $Q_{\eta,\tau} = \eta I_N + \tau\beta$ for $\eta,\tau\in\mathbb{R}$.

Lemma 4.16. *Let $\rho > 0$, $w' \in \mathbb{R}^{\theta-1}$ with $|w'| = 1$, $\eta,\tau\in\mathbb{R}$, $Q_{\eta,\tau} = \eta I_N + \tau\beta$, $d = \eta^2 - \tau^2$,*

$$c(d) := \begin{cases}\frac{4d}{\pi^2}, & d \geq 0,\\ 0, & d < 0,\end{cases}$$

and q be as in (1.4). If $d < \frac{\pi^2}{4}$, then $I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q$ is continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and the norm of the inverse is bounded by the constant

$$C_2 = C_2(\eta,\tau) := 4\|q\|_{L^\infty((-1,1))}(|\eta| + |\tau|)\frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{(1 - c(d))\pi} + 1. \tag{4.39}$$

Proof. Using $\mathfrak{H}_{\rho,w'}Q_{\eta,\tau} = Q_{\eta,-\tau}\mathfrak{H}_{\rho,w'}$ and $Q_{\eta,-\tau}Q_{\eta,\tau} = dI_N$, which follows from the anti-commutation relations for $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_\theta$, discussed before Lemma 4.10, and β, gives us

$$I - d(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q})^2 = (I - \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})(I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}).$$

If $d < \frac{\pi^2}{4}$, then Lemma 4.15 implies $1 \in \rho(d(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q})^2)$ and therefore the operator $I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}$ is also continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and

$$\begin{aligned}&\|(I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\ &\quad= \|(I - d(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q})^2)^{-1}(I - \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\ &\quad\le \frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{1 - c(d)}.\end{aligned}$$

Moreover, $(I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q)^{-1} = I - \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}(I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})^{-1}\sqrt{q}$ and hence

$$\begin{aligned}&\|(I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\ &\quad= \|\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}(I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})^{-1}\sqrt{q} - I\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\ &\quad\le \|q\|_{L^\infty((-1,1))}(|\eta| + |\tau|)\|\mathfrak{H}_{\rho,w'}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{1 - c(d)} + 1\\ &\quad\le 4\|q\|_{L^\infty((-1,1))}(|\eta| + |\tau|)\frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{(1 - c(d))\pi} + 1,\end{aligned}$$

where we used Lemma 4.15 (for the constant function $q = 1/2$) to estimate the term $\|\mathfrak{H}_{\rho,w'}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}$ by $4/\pi$. □

In the last part of Section 4.4.1, we use our findings to prove norm estimates for the operator $(I + D_\varepsilon^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}$.

Proposition 4.17. *Let $z \in \mathbb{C}\setminus\mathbb{R}$, $\kappa \in \mathrm{SO}(\theta)$, $\eta, \tau \in \mathbb{R}$ such that $d = \eta^2 - \tau^2 < \frac{\pi^2}{4}$, and q be as in (1.4). Moreover, let*

$$C_3 = C_3(\eta,\tau,\kappa) := 2\max\{C_2, \|(I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0}\}, \tag{4.40}$$

with $C_2 = C_2(\eta,\tau)$ as in (4.39), and

$$\delta_1 = \delta_1(\eta,\tau,\kappa) := (C_3C_1(|\eta| + |\tau|)\|q\|_{L^\infty((-1,1))})^{-2}, \tag{4.41}$$

with C_1 from Lemma 4.14. Then,

$$\sup_{\varepsilon\in(0,\delta_1)} \|(I + D_\varepsilon^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0} \le C_3 < \infty.$$

Proof. We claim that $C_3 < \infty$. Since $d < \frac{\pi^2}{4}$, the constant C_2 from (4.39) is finite. Thus, it suffices to show the estimate $\|(I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0} < \infty$ in order to proof the claim $C_3 < \infty$.

Applying Proposition 4.9 (for $V = Q_{\eta,\tau} = \eta I_2 + \tau\beta = \text{const.}$ and $r = 0$) shows that $I + B_0(z)Q_{\eta,\tau}q$ is continuously invertible in $\mathcal{B}^0(\Sigma_{0,\kappa})$. Let us shortly explain why Proposition 4.9 is indeed applicable. The inequality $d < \frac{\pi^2}{4}$ implies that $d \neq (2k+1)^2\pi^2$, $k \in \mathbb{N}_0$, and therefore (1.10) is fulfilled. In the same way as in Proposition 4.9, we set $(\tilde{\eta},\tilde{\tau}) = \mathrm{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta,\tau)$. Then, $\tilde{d} = \tilde{\eta}^2 - \tilde{\tau}^2 = 4\tan\big(\frac{\sqrt{d}}{2}\big) < 4$, that is, $\tilde{d}$ fulfils (1.12). Hence, the assumptions of Proposition 4.9 are satisfied and its application is justified. Together with (4.21) and (4.22), this implies that the operator $I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q = \iota_{0,\kappa}(I + B_0(z)Q_{\eta,\tau}q)\iota_{0,\kappa}^{-1}$ is continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, which proves

$$\|(I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0} < \infty.$$

Hence, $C_3 < \infty$.

According to Lemma 4.14, we have

$$\begin{aligned}&\big\|\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q-\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\&\qquad\le C_1\|q\|_{L^\infty((-1,1))}\frac{|\eta|+|\tau|}{1+|\xi'|}=\frac{\delta_1^{-1/2}}{C_3(1+|\xi'|)}\end{aligned}$$

for $\xi'\in\mathbb{R}^{\theta-1}\setminus\{0\}$ and all $\varepsilon>0$. Hence, if we choose $R:=\delta_1^{-1/2}-1$, then the choices of C_3, δ_1 and R, and Lemma 4.16 yield for $0\neq|\xi'|\ge R$ and $\varepsilon>0$

$$\begin{aligned}\big\|(I+\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)^{-1}(\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q-\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}&\le C_2\frac{\delta_1^{-1/2}}{C_3(1+R)}\\&\le\frac{C_3}{2}\cdot\frac{\delta_1^{-1/2}}{C_3\delta_1^{-1/2}}\\&=\frac{1}{2}.\end{aligned}$$

In particular,

$$\mathfrak{P}_{\varepsilon,\xi'}:=I+(I+\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)^{-1}(\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q-\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)$$

is continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and the norm of its inverse can be bounded by 2 for $0\neq|\xi'|\ge R$ and $\varepsilon>0$. This implies that also

$$I+\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q=(I+\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)\mathfrak{P}_{\varepsilon,\xi'}$$

is continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and by Lemma 4.16 the corresponding norm estimate

$$\begin{aligned}&\|(I+\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\&\qquad=\|\mathfrak{P}_{\varepsilon,\xi'}^{-1}(I+\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\&\qquad\le 2C_2\\&\qquad\le C_3\end{aligned}\tag{4.42}$$

is valid for $0\neq|\xi'|\ge R$ and $\varepsilon>0$.

Having found an estimate for $0\neq|\xi'|\ge R$, we aim to find a similar estimate for $0\neq|\xi'|\le R$. Again, according to Lemma 4.14, we have for $\xi'\in\mathbb{R}^{\theta-1}\setminus\{0\}$ and $\varepsilon>0$

$$\begin{aligned}&\|\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q-\mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\&\qquad\le C_1\|q\|_{L^\infty((-1,1))}(|\eta|+|\tau|)\varepsilon(1+|\xi'|)=\varepsilon\frac{\delta_1^{-1/2}(1+|\xi'|)}{C_3}.\end{aligned}$$

Moreover, Lemma 4.13 (with $\mathfrak{M}(\xi')=I+\mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q$ for $\xi'\in\mathbb{R}^{\theta-1}$) and (4.40) imply

$$\begin{aligned}&\operatorname{ess\,sup}_{\xi'\in\mathbb{R}^{\theta-1}}\|(I+\mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\&\qquad=\|(I+D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0}\le\frac{C_3}{2}.\end{aligned}$$

Hence, as $1 + R = \delta_1^{-1/2}$, we can estimate similarly as in the first part of the proof for $\varepsilon \in (0, \delta_1)$

$$\begin{aligned}
&\operatorname{ess\,sup}_{|\xi'|\le R} \|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&= \operatorname{ess\,sup}_{|\xi'|\le R} \Big\| \big[I + (I + \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big(\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q - \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q\big)\big]^{-1} \\
&\qquad \cdot (I + \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q)^{-1}\Big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\le \frac{1}{1 - \frac{C_3}{2}\cdot\frac{\varepsilon\delta_1^{-1/2}(1+R)}{C_3}} \cdot \frac{C_3}{2} \\
&= \frac{1}{1 - \frac{C_3}{2}\cdot\frac{\varepsilon\delta_1^{-1}}{C_3}} \cdot \frac{C_3}{2} \le C_3.
\end{aligned} \tag{4.43}$$

Combining (4.42) and (4.43), and applying Lemma 4.13 gives us

$$\begin{aligned}
&\|(I + D_\varepsilon^{0,\varkappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0} \\
&\quad = \max\Big\{\operatorname{ess\,sup}_{|\xi'|\ge R}\|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}, \\
&\qquad\qquad \operatorname{ess\,sup}_{|\xi'|< R}\|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\Big\} \\
&\quad \le C_3
\end{aligned}$$

for $\varepsilon \in (0, \delta_1)$. □

Corollary 4.18. *Let $z \in \mathbb{C}\setminus\mathbb{R}$, q be as in (1.4), $S \subset \mathbb{R}^2$ be compact and $\max_{(\eta,\tau)\in S} \eta^2 - \tau^2 < \frac{\pi^2}{4}$. Then, there exists a $\delta_2 = \delta_2(S) > 0$ such that*

$$\sup_{(\varepsilon,y_0,(\eta,\tau),\varkappa)\in(0,\delta_2)\times\mathbb{R}\times S\times \mathrm{SO}(\theta)} \|(I + D_\varepsilon^{y_0,\varkappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0} < \infty.$$

Proof. Since $D_\varepsilon^{y_0,\varkappa}(z) = D_\varepsilon^{0,\varkappa}(z)$, see the text below (4.31), the assertion follows directly from Proposition 4.17 if we can show

$$\sup_{((\eta,\tau),\varkappa)\in S\times\mathrm{SO}(\theta)} C_3(\eta,\tau,\varkappa) < \infty \quad \text{and} \quad \inf_{((\eta,\tau),\varkappa)\in S\times\mathrm{SO}(\theta)} \delta_1(\eta,\tau,\varkappa) > 0,$$

with C_3 and δ_1 as in Proposition 4.17. Note also that as S is bounded the first inequality and (4.41) imply the second inequality in the last displayed formula. Moreover, the assumption $\max_{(\eta,\tau)\in S}\eta^2 - \tau^2 < \frac{\pi^2}{4}$ implies $\max_{(\eta,\tau)\in S} C_2(\eta,\tau) < \infty$, where C_2 was defined in (4.39). Hence, it follows from (4.40) that the inequality $\sup_{((\eta,\tau),\varkappa)\in S\times\mathrm{SO}(\theta)} C_3(\eta,\tau,\varkappa) < \infty$ is fulfilled if

$$\sup_{((\eta,\tau),\varkappa)\in S\times\mathrm{SO}(\theta)} \|(I + D_0^{0,\varkappa}(z)Q_{\eta,\tau}q)^{-1}\|_{0\to 0} < \infty. \tag{4.44}$$

Obviously, the operator $D_0^{0,\varkappa}(z)Q_{\eta,\tau}q$ depends continuously on η, τ with respect to the operator norm in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. According to Proposition 4.11 $D_0^{0,\varkappa}(z)Q_{\eta,\tau}q$ depends also continuously on $\varkappa$ with respect to the operator norm. Moreover, Proposition 4.17 gives us that the operator $I + D_0^{0,\varkappa}(z)Q_{\eta,\tau}q$ is continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ for $((\eta,\tau),\varkappa) \in S\times\mathrm{SO}(\theta)$. Thus, as $S\times\mathrm{SO}(\theta)$ is compact, (4.44) is indeed true. □

4.4.2 | Σ is a rotated C_b^2-graph

After treating the case of affine hyperplanes, we turn to the case where Σ is a rotated C_b^2-graph as in (4.1), that is, there exist $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ and $\kappa \in \mathrm{SO}(\mathbb{R}^\theta)$ such that

$$\Sigma = \Sigma_{\zeta,\kappa} = \{\kappa(x',\zeta(x')) : x' \in \mathbb{R}^{\theta-1}\}.$$

Moreover, recall that $x_{\zeta,\kappa}(x') = \kappa(x',\zeta(x'))$ and $\nu_{\zeta,\kappa}(x') = \nu(x_{\zeta,\kappa}(x'))$, $x' \in \mathbb{R}^{\theta-1}$, see (4.25). According to [8, Proposition A.2 (i), first line of the proof of Proposition 2.4, and eq. (3.11)], there exists a $C_4 = C_4(\Sigma) > 0$ such that for all $x', y' \in \mathbb{R}^{\theta-1}$ and $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2)$

$$C_4^{-1}(|x'-y'| + |\widetilde{\varepsilon}|) \le |x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')| \le C_4(|x'-y'| + |\widetilde{\varepsilon}|), \tag{4.45}$$

with $\varepsilon_2 > 0$ chosen as in Proposition 4.6. We are going to prove the uniform boundedness of $(I + B_\varepsilon(z)Vq)^{-1}$ in $\mathcal{B}^0(\Sigma)$ with respect to $\varepsilon \in (0,\varepsilon_3)$ for a suitable $\varepsilon_3 \in (0,\varepsilon_2)$. It follows from (4.19), (4.21), (4.22), and (4.29) that this is equivalent to proving the uniform boundedness of $(I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1} = \iota_{\zeta,\kappa}(I + \overline{B}_\varepsilon(z)Vq)^{-1}\iota_{\zeta,\kappa}^{-1}$.

We start by analyzing $D_\varepsilon^{\zeta,\kappa}(z)$ locally. To do so we need to introduce further notations. For $x_0' \in \mathbb{R}^{\theta-1}$, we define

$$\zeta_{x_0'}(x') := \zeta(x_0') + \langle \nabla\zeta(x_0'), x' - x_0'\rangle, \qquad x' \in \mathbb{R}^{\theta-1}. \tag{4.46}$$

Moreover, we define the localization parameter $a_\varepsilon := \varepsilon^{1/6}$ for $\varepsilon \in (0,\varepsilon_2)$. Next, we introduce a family of auxiliary operators. For this, we choose a C^∞-function ω with $0 \le \omega \le 1$, $\omega = 1$ on $\mathbb{R}^{\theta-1} \setminus B(0,1)$ and $\omega = 0$ on $B(0,1/2)$. We use this function to cut out the singular part of the integral kernel of $D_\varepsilon^{\zeta,\kappa}(z)$; cf. (4.26). More precisely, in analogy with (4.27) and (4.28), we define for $\varepsilon \in (0,\varepsilon_2)$ and $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2) \setminus \{0\}$ the operators

$$\begin{aligned} &e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z) : L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N), \\ &e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)g(x') := \int_{\mathbb{R}^{\theta-1}} G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\omega\Big(\frac{x'-y'}{a_\varepsilon}\Big)\sqrt{1+|\nabla\zeta(y')|^2}g(y')\,dy', \end{aligned} \tag{4.47}$$

and

$$E_\varepsilon(z) : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \qquad E_\varepsilon(z)f(t) := \int_{-1}^1 e_{\varepsilon(t-s)}^{a_\varepsilon}(z)f(s)\,ds. \tag{4.48}$$

We start by proving preliminary results for $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ and $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$. Afterwards, we transfer these results to $E_\varepsilon(z)$ and $D_\varepsilon^{\zeta,\kappa}(z)$ in Proposition 4.24. For the estimates regarding $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ and $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$, the following lemma turns out to be useful.

Lemma 4.19. *Let $z \in \rho(H)$ and G_z be defined by (4.4)–(4.5). Then, there exist $C_5 = C_5(z,m) > 0$ and $C_6 = C_6(z,m) > 0$ such that for all $x \in \mathbb{R}^\theta \setminus \{0\}$ and $j \in \{1,\dots,\theta\}$*

$$\begin{aligned} |G_z(x)| &\le C_5|x|^{1-\theta}e^{-C_6|x|}, \\ |\partial_j G_z(x)| &\le C_5|x|^{-\theta}e^{-C_6|x|}. \end{aligned} \tag{4.49}$$

In particular, there holds for all $x', y' \in \mathbb{R}^{\theta-1}$, $j \in \{1,\dots,\theta\}$, and $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2)$

$$\begin{aligned} |G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))| &\le C_5C_4^{\theta-1}(|x'-y'| + |\widetilde{\varepsilon}|)^{1-\theta}e^{-\frac{C_6}{C_4}|x'-y'|}, \\ |\partial_j G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))| &\le C_5C_4^{\theta}(|x'-y'| + |\widetilde{\varepsilon}|)^{-\theta}e^{-\frac{C_6}{C_4}|x'-y'|}. \end{aligned} \tag{4.50}$$

Proof. Rough estimations and asymptotic expansions of the modified Bessel functions K_1 and K_0, see [27, §10.25 (ii) and section 10.30 (i)] and [8, eq. (B.8)], lead to

$$|G_z(x)| \leq C(1+|x|^{1-\theta})e^{-\operatorname{Im}\sqrt{z^2-m^2}|x|}$$

and

$$|\partial_j G_z(x)| \leq C(1+|x|^{-\theta})e^{-\operatorname{Im}\sqrt{z^2-m^2}|x|}$$

for all $x \in \mathbb{R}^\theta \setminus \{0\}$ and $j \in \{1, \dots, \theta\}$, where $C = C(m,z) > 0$ is a constant which only depends on m and z. Thus, (4.49) is valid if one chooses $C_6 \in (0, \operatorname{Im}\sqrt{z^2-m^2})$ and

$$C_5 = \sup_{x\in\mathbb{R}^\theta\setminus\{0\}, l\in\{\theta-1,\theta\}} C\frac{1+|x|^{-l}}{|x|^{-l}}e^{-(\operatorname{Im}\sqrt{m^2-z^2}-C_6)|x|} < \infty.$$

Furthermore, combining these estimates with (4.45) implies (4.50). □

Lemma 4.20. *Let $z \in \rho(H)$, $\varepsilon \in (0, \varepsilon_2)$, $a_\varepsilon = \varepsilon^{1/6}$, and $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2) \setminus \{0\}$. Then, the operator $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ acts as a bounded operator from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and*

$$\|e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\frac{1+|\log(\varepsilon)|}{a_\varepsilon},$$

where $C > 0$ does not depend on $\widetilde{\varepsilon}$ and ε. Moreover, for $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ the mapping

$$(-2\varepsilon_2, 2\varepsilon_2) \setminus \{0\} \ni \widetilde{\varepsilon} \mapsto e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)f \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$$

is continuous.

Proof. We aim to prove the assertion by applying Lemma A.1. To do so, it is necessary to find suitable estimates for the integral kernel of $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$, which is for $x', y' \in \mathbb{R}^{\theta-1}$ given by

$$k(x',y') := G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\omega\left(\tfrac{x'-y'}{a_\varepsilon}\right)\sqrt{1+|\nabla\zeta(y')|^2}.$$

We notice as $G_z \in C^\infty(\mathbb{R}^\theta \setminus \{0\};\mathbb{C}^{N\times N})$, $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$, and $\omega \in C_b^\infty(\mathbb{R}^{\theta-1};\mathbb{R})$ and as ω cuts out the singularity of G_z, we have $k \in C_b^1(\mathbb{R}^{\theta-1}\times\mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$. Furthermore, using (4.50), $0 \leq \omega \leq 1$, $\operatorname{supp}\omega \subset \mathbb{R}^{\theta-1}\setminus B(0,1/2)$ and $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ immediately gives us for $x' \neq y' \in \mathbb{R}^{\theta-1}$

$$\begin{aligned}|k(x',y')| &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\left(\tfrac{x'-y'}{a_\varepsilon}\right)(|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta}e^{-c|x'-y'|} \\ &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\left(\tfrac{x'-y'}{a_\varepsilon}\right)|x'-y'|^{1-\theta}e^{-c|x'-y'|},\end{aligned}$$

where $c = \frac{C_6}{C_4}$ with $C_4 > 0$ from (4.45) and C_6 from Lemma 4.19. Next, we estimate the derivatives with respect to x' of k. The derivative with respect to x_l', $l \in \{1, \dots \theta-1\}$, is given for for $x' \neq y' \in \mathbb{R}^{\theta-1}$ by

$$\begin{aligned}\frac{d}{dx_l'}k(x',y') = \sum_{j=1}^{\theta}\Bigg(&(\partial_j G_z)(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')) \\ &\cdot\frac{d}{dx_l'}(x_{\zeta,\kappa}(x')[j] + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')[j])\omega\left(\tfrac{x'-y'}{a_\varepsilon}\right)\sqrt{1+|\nabla\zeta(y')|^2}\Bigg) \\ &+ G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\frac{1}{a_\varepsilon}(\partial_l\omega)\left(\tfrac{x'-y'}{a_\varepsilon}\right)\sqrt{1+|\nabla\zeta(y')|^2},\end{aligned}$$

where $v[j]$ denotes the jth component of a vector v. Applying (4.50), the properties of ω and ζ again, we can estimate for $x' \neq y' \in \mathbb{R}^{\theta-1}$

$$\begin{aligned}\Big|\frac{d}{dx'_l}k(x',y')\Big| &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\big(\tfrac{x'-y'}{a_\varepsilon}\big)\Big((|x'-y'|+|\widetilde{\varepsilon}|)^{-\theta}e^{-c|x'-y'|} \\ &\qquad + \frac{1}{a_\varepsilon}(|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta}e^{-c|x'-y'|}\Big) \\ &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\big(\tfrac{x'-y'}{a_\varepsilon}\big)\Big(|x'-y'|^{-\theta}+\frac{1}{a_\varepsilon}|x'-y'|^{1-\theta}\Big)e^{-c|x'-y'|}.\end{aligned}$$

Thus, if we set $\widetilde{k}(z') := C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\big(\frac{z'}{a_\varepsilon}\big)\Big(|z'|^{-\theta}+\frac{1}{a_\varepsilon}|z'|^{1-\theta}\Big)e^{-c|z'|}$ for $z' \in \mathbb{R}^{\theta-1}\setminus\{0\}$, we get

$$|k(x',y')|,\ \sum_{l=1}^{\theta-1}\Big|\frac{d}{dx'_l}k(x',y')\Big| \leq \widetilde{k}(x'-y'), \quad x' \neq y' \in \mathbb{R}^{\theta-1}. \tag{4.51}$$

Hence, by Lemma A.1 the map $e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)$ acts as a bounded operator from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and

$$\|e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}.$$

Now, the norm estimate in the assertion follows from

$$\begin{aligned}\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})} &= C\int_{\mathbb{R}^{\theta-1}}\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\big(\tfrac{z'}{a_\varepsilon}\big)\Big(|z'|^{-\theta}+\frac{1}{a_\varepsilon}|z'|^{1-\theta}\Big)e^{-c|z'|}dz' \\ &\leq C\int_{a_\varepsilon/2}^{\infty}\Big(r^{-\theta}+\frac{1}{a_\varepsilon}r^{1-\theta}\Big)e^{-cr}r^{\theta-2}\,dr \\ &\leq C\Big(\frac{1}{a_\varepsilon}+\frac{1+|\log(a_\varepsilon)|}{a_\varepsilon}\Big) \leq C\frac{1+|\log(\varepsilon)|}{a_\varepsilon}.\end{aligned}$$

Finally, we prove the continuity. To do so, let $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2)\setminus\{0\}$ and $(\widetilde{\varepsilon}_n)_{n\in\mathbb{N}}$ be such that $\widetilde{\varepsilon}_n \in (-2\varepsilon_2, 2\varepsilon_2)\setminus\{0\}$ for all $n \in \mathbb{N}$ and $\widetilde{\varepsilon}_n \overset{n\to\infty}{\longrightarrow} \widetilde{\varepsilon}$. Using the dominated convergence theorem and (4.51) shows that for $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ $e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}f$ and $\partial_l e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}f$, $l \in \{1,\dots,\theta-1\}$, converge pointwise to $e^{a_\varepsilon}_{\widetilde{\varepsilon}}f$ and $\partial_l e^{a_\varepsilon}_{\widetilde{\varepsilon}}f$, $l \in \{1,\dots,\theta-1\}$, respectively. Furthermore, (4.51) shows that $|e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}f|$ and $|\partial_l e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}f|$, $l \in \{1,\dots,\theta-1\}$, are independently of $n \in \mathbb{N}$ pointwise bounded by the function $|f| * \widetilde{k}$, which is by Young's inequality square integrable as $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and $\widetilde{k} \in L^1(\mathbb{R}^{\theta-1})$. Hence, applying the dominated convergence theorem again shows that $e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}f$ converges to $e^{a_\varepsilon}_{\widetilde{\varepsilon}}f$ in $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. □

Lemma 4.21. *Let $z \in \rho(H)$, $\psi \in C^1_b(\mathbb{R}^{\theta-1})$, $x'_0 \in \mathbb{R}^{\theta-1}$, $\zeta_{x'_0}$ be as in (4.46), and $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2)\setminus\{0\}$. Then,*

$$\|[d^{\zeta_{x'_0},\kappa}_{\widetilde{\varepsilon}}(z),\psi]\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log|\widetilde{\varepsilon}||),$$

where $C > 0$ does not depend on $\widetilde{\varepsilon} \neq 0$ and $x'_0 \in \mathbb{R}^{\theta-1}$. Moreover, for $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$, the mapping

$$(-2\varepsilon_2, 2\varepsilon_2)\setminus\{0\} \ni \widetilde{\varepsilon} \mapsto [d^{\zeta_{x'_0},\kappa}_{\widetilde{\varepsilon}}(z),\psi]f \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$$

is continuous.

Proof. We prove this result in the same vein as the previous lemma, that is, we estimate the integral kernel of $[d_{\tilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]$ and its partial derivatives, and apply Lemma A.1. The integral kernel of $[d_{\tilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]$ is given for $x',y'\in\mathbb{R}^{\theta-1}$ by

$$k(x',y') := G_z(x_{\zeta_{x_0'},\kappa}(x') - x_{\zeta_{x_0'},\kappa}(y') + \tilde{\varepsilon}\nu_{\zeta_{x_0'},\kappa}(x'))\sqrt{1+|\nabla\zeta_{x_0'}(y')|^2}(\psi(y')-\psi(x')).$$

Using

$$\begin{aligned} x_{\zeta_{x_0'},\kappa}(x') &= \kappa(x',\zeta(x_0') + \langle\nabla\zeta(x_0'),x'-x_0'\rangle),\\ \nu_{\zeta_{x_0'},\kappa}(x') &= \frac{\kappa(-\nabla\zeta(x_0'),1)}{\sqrt{1+|\nabla\zeta(x_0')|^2}} = \nu_{\zeta,\kappa}(x_0'),\\ \nabla\zeta_{x_0'}(x') &= \nabla\zeta(x_0') \end{aligned} \tag{4.52}$$

for $x'\in\mathbb{R}^{\theta-1}$ shows that k can be simplified to

$$k(x',y') = G_z(\kappa(x'-y',\langle\nabla\zeta(x_0'),x'-y'\rangle) + \tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0'))\sqrt{1+|\nabla\zeta(x_0')|^2}(\psi(y')-\psi(x')).$$

Moreover, with (4.52), $\kappa\in SO(\theta)$, and the Pythagorean theorem, one gets

$$\begin{aligned} |\kappa(x'-y',\langle\nabla\zeta(x_0'),x'-y'\rangle) + \tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0')|^2 &= |x'-y'|^2 + \langle\nabla\zeta(x_0'),x'-y'\rangle^2 + \tilde{\varepsilon}^2\\ &\le |x'-y'|^2(1+\|\nabla\zeta\|^2_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta-1})}) + \tilde{\varepsilon}^2. \end{aligned}$$

In particular, we can choose $C_4'>0$ which does not depend on x_0' and $\tilde{\varepsilon}$ such that

$$\begin{aligned} (C_4')^{-1}(|x'-y'|+|\tilde{\varepsilon}|) &\le |\kappa(x'-y',\langle\nabla\zeta(x_0'),x'-y'\rangle) + \tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0')|\\ &\le C_4'(|x'-y'|+|\tilde{\varepsilon}|). \end{aligned} \tag{4.53}$$

Then, (4.49), (4.53), the Lipschitz continuity of $\psi\in C_b^1(\mathbb{R}^{\theta-1})$ and $\zeta\in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ yield

$$\begin{aligned} |k(x',y')| &\le C(|x'-y'|+|\tilde{\varepsilon}|)^{1-\theta}e^{-c'|x'-y'|}\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}|x'-y'|\\ &\le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(|x'-y'|+|\tilde{\varepsilon}|)^{2-\theta}e^{-c'|x'-y'|}, \quad x',y'\in\mathbb{R}^{\theta-1}, \end{aligned}$$

where $c' = \frac{C_6}{C_4'}$ with C_6 from Lemma 4.19 and $C>0$ is independent of x_0' and $\tilde{\varepsilon}$. The derivative with respect to x_l', $x'\in\mathbb{R}^{\theta-1}$, of k is given by

$$\begin{aligned} \frac{d}{dx_l'}k(x',y') = \Bigg(&\sum_{j=1}^{\theta}(\partial_j G_z)(\kappa(x'-y',\langle\nabla\zeta(x_0'),x'-y'\rangle)+\tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0'))(\kappa(e_l',\partial_l\zeta(x_0')))[j](\psi(y')-\psi(x'))\\ &- G_z(\kappa(x'-y',\langle\nabla\zeta(x_0'),x'-y'\rangle)+\tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0'))(\partial_l\psi)(x')\Bigg)\sqrt{1+|\nabla\zeta(x_0')|^2}, \end{aligned}$$

where e_l' denotes the lth Euclidean unit vector in $\mathbb{R}^{\theta-1}$ and $(\kappa(e_l',\partial_l\zeta(x_0')))[j]$ denotes the jth entry of the vector $\kappa(e_l',\partial_l\zeta(x_0'))$. Using (4.49), (4.53), the Lipschitz continuity of $\psi\in C_b^1(\mathbb{R}^{\theta-1})$, and $\zeta\in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ again gives us

$$\begin{aligned} \left|\frac{d}{dx_l'}k(x',y')\right| &\le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\Big((|x'-y'|+|\tilde{\varepsilon}|)^{-\theta}e^{-c'|x'-y'|}|x'-y'| + (|x'-y'|+|\tilde{\varepsilon}|)^{1-\theta}e^{-c'|x'-y'|}\Big)\\ &\le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(|x'-y'|+|\tilde{\varepsilon}|)^{1-\theta}e^{-c'|x'-y'|} \end{aligned}$$

for all $x', y' \in \mathbb{R}^{\theta-1}$, where $C > 0$ can again be chosen independently of x'_0 and $\widetilde{\varepsilon}$. Setting for $z' \in \mathbb{R}^{\theta-1}$

$$\widetilde{k}(z') := C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}((|z'| + |\widetilde{\varepsilon}|)^{2-\theta} + (|z'| + |\widetilde{\varepsilon}|)^{1-\theta})e^{-c'|z'|}$$

leads to

$$|k(x', y')|, \sum_{l=1}^{\theta-1}\left|\frac{d}{dx'_l}k(x', y')\right| \leq \widetilde{k}(x' - y'), \quad x', y' \in \mathbb{R}^{\theta-1}.$$

Now, Lemma A.1 shows

$$\begin{aligned}\|[d_{\widetilde{\varepsilon}}^{\zeta_{x'_0},\kappa}(z), \psi]\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} &\leq C\int_{\mathbb{R}^{\theta-1}} \widetilde{k}(z')\,dz' \\ &\leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_0^\infty ((r + |\widetilde{\varepsilon}|)^{2-\theta} + (r + |\widetilde{\varepsilon}|)^{1-\theta})e^{-c'r}r^{\theta-2}\,dr \\ &\leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_0^\infty (1 + (r + |\widetilde{\varepsilon}|)^{-1})e^{-c'r}\,dr \\ &\leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1 + |\log|\widetilde{\varepsilon}||).\end{aligned}$$

The assertion regarding the continuity can be proven in a similar way as in Lemma 4.20. □

Lemma 4.22. *Let $z \in \rho(H)$, $x'_0 \in \mathbb{R}^{\theta-1}$, $\zeta \in C^2_b(\mathbb{R}^{\theta-1}; \mathbb{R})$ be as described at the beginning of Section 4.4.2, $\zeta_{x'_0}$ be as in (4.46), $\varepsilon \in (0, \varepsilon_2)$, $a_\varepsilon = \varepsilon^{1/6}$, and $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2) \setminus \{0\}$. Then, there exists a $\delta_3 = \delta_3(\zeta) \in (0, \varepsilon_2)$ such that for all $\varepsilon \in (0, \delta_3)$ the inequality*

$$\left\|\chi_{B(x'_0,3a_\varepsilon)}\left(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) - d_{\widetilde{\varepsilon}}^{\zeta_{x'_0},\kappa}(z)\right)\chi_{B(x'_0,3a_\varepsilon)}\right\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq Ca_\varepsilon(1 + |\log|\widetilde{\varepsilon}||)$$

holds, where $C > 0$ does not depend on ε, $\widetilde{\varepsilon}$, and x'_0.

Proof. We prove this statement by estimating the integral kernel of the operator

$$\chi_{B(x'_0,a_\varepsilon)}\left(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) - d_{\widetilde{\varepsilon}}^{\zeta_{x'_0},\kappa}(z)\right)\chi_{B(x'_0,a_\varepsilon)}$$

and applying Lemma A.1. The mentioned integral kernel is given for $x', y' \in \mathbb{R}^{\theta-1}$ by

$$\begin{aligned}k(x', y') := \chi_{B(x'_0,3a_\varepsilon)}(x')\Big(&G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\sqrt{1 + |\nabla\zeta(y')|^2} \\ &-G_z(\kappa(x' - y', \langle\nabla\zeta(x'_0), x' - y'\rangle) + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'_0))\sqrt{1 + |\nabla\zeta(x'_0)|^2}\Big)\chi_{B(x'_0,3a_\varepsilon)}(y').\end{aligned}$$

If $x' \notin B(x'_0, 3a_\varepsilon)$ or $y' \notin B(x'_0, 3a_\varepsilon)$, then $k(x', y') = 0$. Thus, we assume from now on $x', y' \in B(x'_0, 3a_\varepsilon)$. Using the mean value theorem for matrix-valued functions, cf. [8, Lemma A.1], (4.49), and $\zeta \in C^2_b(\mathbb{R}^{\theta-1}; \mathbb{R})$, we find

$$\begin{aligned}|k(x', y')| &\leq \sqrt{\theta}\sup_{v\in[0,1], j\in\{1,\dots\theta\}}|\partial_j G_z(w_v)\sqrt{1 + |\nabla\zeta(y')|^2}||w_1 - w_0| \\ &\quad + \left|G_z(w_0)\left(\sqrt{1 + |\nabla\zeta(y')|^2} - \sqrt{1 + |\nabla\zeta(x'_0)|^2}\right)\right| \\ &\leq C\left(\sup_{v\in[0,1]}|w_v|^{-\theta}|w_1 - w_0| + |w_0|^{1-\theta}|x'_0 - y'|\right)\end{aligned} \tag{4.54}$$

with

$$w_\upsilon = \upsilon(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \tilde{\varepsilon}\nu_{\zeta,\kappa}(x')) + (1-\upsilon)(\kappa(x'-y', \langle \nabla\zeta(x_0'), x'-y'\rangle) + \tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0'))$$

for $\upsilon \in [0,1]$. Note that the mean value theorem is applicable since $w_\upsilon \neq 0$ for $\upsilon \in [0,1]$, see (4.56) below. Next, we want to estimate

$$\begin{aligned}|w_1 - w_0| &= |x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \tilde{\varepsilon}\nu_{\zeta,\kappa}(x') - \kappa(x'-y', \langle \nabla\zeta(x_0'), x'-y'\rangle) - \tilde{\varepsilon}\nu_{\zeta,\kappa}(x_0')| \\ &= |\kappa(x'-y', \zeta(x') - \zeta(y')) - \kappa(x'-y', \langle \nabla\zeta(x_0'), x'-y'\rangle) + \tilde{\varepsilon}(\nu_{\zeta,\kappa}(x') - \nu_{\zeta,\kappa}(x_0'))| \\ &= |\kappa(0, \zeta(x') - \zeta(y') - \langle \nabla\zeta(x_0'), x'-y'\rangle) + \tilde{\varepsilon}(\nu_{\zeta,\kappa}(x') - \nu_{\zeta,\kappa}(x_0'))|.\end{aligned}$$

As $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ and $\kappa \in \mathrm{SO}(\theta)$, there exists a $K = K(\zeta) > 0$ such that

$$|\nu_{\zeta,\kappa}(x') - \nu_{\zeta,\kappa}(x_0')| = \left| \frac{\kappa(-\nabla\zeta(x'),1)}{\sqrt{1+|\nabla\zeta(x')|^2}} - \frac{\kappa(-\nabla\zeta(x_0'),1)}{\sqrt{1+|\nabla\zeta(x_0')|^2}} \right| \leq K|x'-x_0'| \leq 3Ka_\varepsilon$$

and

$$\begin{aligned}|\kappa(0, \zeta(x') - \zeta(y') - \langle \nabla\zeta(x_0'), x'-y'\rangle)| &= |\zeta(x') - \zeta(y') - \langle \nabla\zeta(x_0'), x'-y'\rangle| \\ &= \left| \int_0^1 \langle \nabla\zeta(y' + t(x'-y')) - \nabla\zeta(x_0'), x'-y'\rangle \, dt \right| \\ &\leq K \left| \int_0^1 |t(x'-x_0') + (1-t)(y'-x_0')||x'-y'| \, dt \right| \\ &\leq 3Ka_\varepsilon |x'-y'|,\end{aligned}$$

where we used $x', y' \in B(x_0', 3a_\varepsilon)$. Hence, if $\delta_3 = \delta_3(\zeta) \in (0, \varepsilon_2)$ is chosen sufficiently small, then for all $a_\varepsilon \in (0, \delta_3^{1/6})$ the inequality

$$|w_1 - w_0| \leq K3a_\varepsilon(|x'-y'| + |\tilde{\varepsilon}|) \leq \frac{1}{2C_4}(|x'-y'| + |\tilde{\varepsilon}|) \tag{4.55}$$

holds with $C_4 > 0$ from (4.45). Therefore, we can use (4.45) to estimate $|w_\upsilon|$, $\upsilon \in [0,1]$, from below by

$$\begin{aligned}|w_\upsilon| &= |\upsilon w_1 + (1-\upsilon)w_0| = |w_1 + (1-\upsilon)(w_0 - w_1)| \geq |w_1| - |w_1 - w_0| \\ &\geq \frac{1}{C_4}(|x'-y'| + |\tilde{\varepsilon}|) - \frac{1}{2C_4}(|x'-y'| + |\tilde{\varepsilon}|) = \frac{1}{2C_4}(|x'-y'| + |\tilde{\varepsilon}|).\end{aligned} \tag{4.56}$$

Thus, plugging (4.55) and (4.56) into (4.54) yields for $a_\varepsilon = \varepsilon^{1/6}$ with $\varepsilon \in (0, \delta_3)$

$$|k(x',y')| \leq \begin{cases} 0, & \text{if } x' \notin B(x_0', 3a_\varepsilon) \text{ or } y' \notin B(x_0', 3a_\varepsilon), \\ Ca_\varepsilon(|x'-y'| + |\tilde{\varepsilon}|)^{1-\theta}, & \text{else.} \end{cases}$$

Applying Lemma A.1 yields

$$\begin{aligned}&\left\| \chi_{B(x_0',3a_\varepsilon)} \left(d_{\tilde{\varepsilon}}^{\zeta,\kappa}(z) - d_{\tilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z) \right) \chi_{B(x_0',3a_\varepsilon)} \right\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \\ &\qquad \leq Ca_\varepsilon \int_{B(0,6a_\varepsilon)} (|z'| + |\tilde{\varepsilon}|)^{1-\theta} \, dz'\end{aligned}$$

$$\begin{aligned}
&\leq C a_\varepsilon \int_0^{6a_\varepsilon} (r+|\widetilde{\varepsilon}|)^{1-\theta} r^{\theta-2}\, dr \\
&\leq C a_\varepsilon \int_0^{6a_\varepsilon} (r+|\widetilde{\varepsilon}|)^{-1}\, dr \\
&\leq C a_\varepsilon (1+|\log|\widetilde{\varepsilon}||). \qquad \square
\end{aligned}$$

Corollary 4.23. *Let $z \in \rho(H)$, $x_0' \in \mathbb{R}^{\theta-1}$, $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ be as described at the beginning of Section 4.4.2, $\zeta_{x_0'}$ be as in (4.46), $\varepsilon \in (0,\delta_3)$ with δ_3 chosen as in Lemma 4.22, $a_\varepsilon = \varepsilon^{1/6}$, $\widetilde{\varepsilon} \in (-2\varepsilon_2, 2\varepsilon_2)\setminus\{0\}$, and $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (4.29). Then,*

$$\left\| \chi_{B(x_0',3a_\varepsilon)} \Big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(x_0') \Big) \chi_{B(x_0',3a_\varepsilon)} \right\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C a_\varepsilon (1+|\log|\widetilde{\varepsilon}||),$$

where C does not depend on ε, $\widetilde{\varepsilon}$, and x_0'.

Proof. Lemma 4.22 and $Q_{\eta,\tau}^{\zeta,\kappa} \in C_b^1(\mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$ yield

$$\begin{aligned}
&\left\| \chi_{B(x_0',3a_\varepsilon)} \Big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(x_0') \Big) \chi_{B(x_0',3a_\varepsilon)} \right\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \\
&\leq \left\| \chi_{B(x_0',3a_\varepsilon)} d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) \Big(Q_{\eta,\tau}^{\zeta,\kappa} - Q_{\eta,\tau}^{\zeta,\kappa}(x_0') \Big) \chi_{B(x_0',3a_\varepsilon)} \right\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \\
&\quad + \left\| \chi_{B(x_0',3a_\varepsilon)} \Big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z) \Big) \chi_{B(x_0',3a_\varepsilon)} Q_{\eta,\tau}^{\zeta,\kappa}(x_0') \right\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \\
&\leq C \Big(\| d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) \|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} a_\varepsilon + a_\varepsilon (1+|\log|\widetilde{\varepsilon}||) \Big),
\end{aligned}$$

where C does not depend on ε, $\widetilde{\varepsilon}$, and x_0'. Moreover, (4.28), (4.50), and $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ imply for the integral kernel k of $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$

$$|k(x',y')| \leq C(|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta} e^{-c|x'-y'|}, \quad x',y' \in \mathbb{R}^{\theta-1},$$

where $c = \frac{C_6}{C_4} > 0$ with C_4 from (4.45) and C_6 from (4.50), and therefore Lemma A.1 implies

$$\begin{aligned}
\| d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) \|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} &\leq C \int_{\mathbb{R}^{\theta-1}} (|z'|+|\widetilde{\varepsilon}|)^{1-\theta} e^{-c|z'|}\, dz' \\
&\leq C \int_0^\infty (r+|\widetilde{\varepsilon}|)^{1-\theta} e^{-cr} r^{\theta-2}\, dr \\
&\leq C(1+|\log|\widetilde{\varepsilon}||),
\end{aligned}$$

which completes the proof. $\square$

Now, we transfer in Proposition 4.24 the results regarding $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ and $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$ to $E_\varepsilon(z)$ and $D_\varepsilon^{\zeta,\kappa}(z)$. The first, second, and third estimates in Proposition 4.24 are consequences of Lemma 4.20, Lemma 4.21, and Corollary 4.23, respectively. Recall that $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ is described in the beginning of Section 4.4.2, $\zeta_{x_0'}$ is defined by (4.46), and δ_3 is chosen as in Lemma 4.22.

Proposition 4.24. *Let $z \in \rho(H)$, $x_0' \in \mathbb{R}^{\theta-1}$, $\varepsilon \in (0,\delta_3)$, $a_\varepsilon = \varepsilon^{1/6}$, $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (4.29), and $\psi \in C_b^1(\mathbb{R}^{\theta-1})$. Then, the operators $E_\varepsilon(z)$ and $[D_\varepsilon^{\zeta,\kappa}(z),\psi]$ act as bounded operators from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$ and*

$$\|E_\varepsilon(z)\|_{0\to 1} \le C\frac{1+|\log(\varepsilon)|}{a_\varepsilon},$$

$$\|[D_\varepsilon^{\zeta_{x_0'},\varkappa}(z),\psi]\|_{0\to 1} \le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|),$$

$$\left\|\chi_{B(x_0',3a_\varepsilon)}\Big(D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa} - D_\varepsilon^{\zeta_{x_0'},\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}(x_0')\Big)\chi_{B(x_0',3a_\varepsilon)}\right\|_{0\to 0} \le Ca_\varepsilon(1+|\log(\varepsilon)|),$$

where $C > 0$ does not depend on x_0' and ε.

Proof. We start by showing that $[D_\varepsilon^{\zeta_{x_0'},\varkappa}(z),\psi]$ is well defined as an operator from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$. Let $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$ and set $g = [D_\varepsilon^{\zeta_{x_0'},\varkappa}(z),\psi]f$. It follows from (4.27) that g has for $t \in (-1,1)$ the representation

$$g(t) = \int_{-1}^{1} [d_{\varepsilon(t-s)}^{\zeta_{x_0'},\varkappa}(z),\psi]f(s)\,ds. \tag{4.57}$$

Since $[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\varkappa}(z),\psi]$ has the continuity property from Lemma 4.21 and $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$, [20, Proposition 1.1.28] implies that the function

$$(-1,1)\times(-1,1) \ni (t,s) \mapsto [d_{\varepsilon(t-s)}^{\zeta_{x_0'},\varkappa}(z),\psi]f(s) \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$$

is measurable. According to Lemma 4.21, we have

$$\begin{aligned}&\int_{-1}^{1}\left(\int_{-1}^{1}\|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\varkappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\right)^2 dt\\ &\le C\|\psi\|^2_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_{-1}^{1}\left(\int_{-1}^{1}(1+|\log|\varepsilon(t-s)||)\|f(s)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\right)^2 dt.\end{aligned} \tag{4.58}$$

This expression can be estimated using the Cauchy-Schwarz inequality and Fubini's theorem by

$$\begin{aligned}&C\|\psi\|^2_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_{-1}^{1}\bigg(\int_{-1}^{1}(1+|\log|\varepsilon(t-s)||)\,ds\\ &\qquad\qquad\cdot\int_{-1}^{1}(1+|\log|\varepsilon(t-s)||)\|f(s)\|^2_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\bigg)\,dt\\ &\le C\|\psi\|^2_{W^1_\infty(\mathbb{R}^{\theta-1})}\left(\int_{-2}^{2}(1+|\log|\varepsilon s||)\,ds\right)^2\int_{-1}^{1}\|f(s)\|^2_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\\ &\le C\Big(\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|)\|f\|_0\Big)^2.\end{aligned} \tag{4.59}$$

In particular, applying the Cauchy–Schwarz inequality again gives us

$$\begin{aligned}&\int_{-1}^{1}\int_{-1}^{1}\|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\varkappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\,dt\\ &\le\sqrt{2}\sqrt{\int_{-1}^{1}\left(\int_{-1}^{1}\|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\varkappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\right)^2 dt} < \infty.\end{aligned}$$

Thus, Fubini's theorem for Bochner integrals, see [20, Proposition 1.2.7], shows that $g(t)$ in (4.57) is well-defined and measurable as a function from $(-1,1)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. Moreover, (4.57), (4.58), and (4.59) also give us the norm estimate as

$$\begin{aligned}\|[D_\varepsilon^{\zeta_{x_0'},\kappa}(z),\psi]f\|_1^2 &= \|g\|_1^2\\ &= \int_{-1}^1 \|g(t)\|^2_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,dt\\ &\leq \int_{-1}^1\left(\int_{-1}^1 \|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\right)^2 dt\\ &\leq C\Big(\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|)\|f\|_0\Big)^2.\end{aligned}$$

The proof of the assertion regarding $E_\varepsilon(z)$ follows along the same lines if one applies Lemma 4.20 instead of Lemma 4.21. Moreover, since

$$\chi_{B(x_0',3a_\varepsilon)}\left(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\right)\chi_{B(x_0',3a_\varepsilon)}$$

is only considered as an operator in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, one only has to prove the norm estimate in this topology. The norm can be estimated in the same way as we estimated the norm of $E_\varepsilon(z)$ by applying Corollary 4.23 instead of Lemma 4.21. □

As the last part of our local analysis we state a result concerning the inverse of $I + D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')q$ for $x_0' \in \mathbb{R}^{\theta-1}$. This is an important result as these operators are going to play an essential role when constructing the inverse of the mapping $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$. Recall that $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ and Σ are as described in the beginning of Section 4.4.2, $\zeta_{x_0'}$ is given by (4.46), and q is as in (1.4).

Proposition 4.25. *Let $z \in \mathbb{C}\setminus\mathbb{R}$, $\eta,\tau \in C_b^1(\Sigma;\mathbb{R})$, $d = \eta^2 - \tau^2$ such that*

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4},$$

and $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (4.29). Then, there exists a $\delta_4 > 0$ such that the operators $(I + D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')q)^{-1}$ are uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $\varepsilon \in (0,\delta_4)$ and $x_0' \in \mathbb{R}^{\theta-1}$.

Proof. Note that

$$Q_{\eta,\tau}^{\zeta,\kappa}(x_0') = \eta(x_{\zeta,\kappa}(x_0'))I_N + \tau(x_{\zeta,\kappa}(x_0'))\beta = Q_{\eta(x_{\zeta,\kappa}(x_0')),\tau(x_{\zeta,\kappa}(x_0'))}$$

for $x_0' \in \mathbb{R}^{\theta-1}$; cf. (4.29) and (4.30). Moreover, for every $x_0' \in \mathbb{R}^{\theta-1}$ the set $\Sigma_{\zeta_{x_0'},\kappa}$ is an affine hyperplane in $\mathbb{R}^{\theta-1}$ and therefore there exists a $y_0(x_0') \in \mathbb{R}$ and a $\widetilde{\kappa}(x_0') \in \mathrm{SO}(\theta)$ such that

$$\Sigma_{\zeta_{x_0'},\kappa} = \{\kappa(x',\zeta_{x_0'}(x')) : x' \in \mathbb{R}^{\theta-1}\} = \widetilde{\kappa}(x_0')\big(\mathbb{R}^{\theta-1}\times\{y_0(x_0')\}\big) = \Sigma_{y_0(x_0'),\widetilde{\kappa}(x_0')}.$$

Now, let $\overline{B}_\varepsilon^{\Sigma_{\zeta_{x_0'},\kappa}}(z) = \overline{B}_\varepsilon^{\Sigma_{y_0(x_0'),\widetilde{\kappa}(x_0')}}(z)$ be defined as $\overline{B}_\varepsilon(z)$ in (4.18) with Σ substituted by $\Sigma_{\zeta_{x_0'},\kappa} = \Sigma_{y_0(x_0'),\widetilde{\kappa}(x_0')}$. According to Proposition 4.17 the operator $I + D_\varepsilon^{y_0(x_0'),\widetilde{\kappa}(x_0')}(z)Q_{\eta(x_{\zeta,\kappa}(x_0')),\tau(x_{\zeta,\kappa}(x_0'))}q$ is continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Moreover, we get from (4.22)

$$
\begin{aligned}
I + D_\varepsilon^{\zeta_{x_0'},\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}(x_0')q &= \iota_{\zeta_{x_0'},\varkappa}\big(I + \overline{B}_\varepsilon^{\Sigma_{\zeta_{x_0'},\varkappa}}(z)Q_{\eta(x_{\zeta,\varkappa}(x_0')),\tau(x_{\zeta,\varkappa}(x_0'))}q\big)\iota_{\zeta_{x_0'},\varkappa}^{-1} \\
&= \iota_{\zeta_{x_0'},\varkappa}\big(I + \overline{B}_\varepsilon^{\Sigma_{y_0(x_0'),\widetilde{\varkappa}(x_0')}}(z)Q_{\eta(x_{\zeta,\varkappa}(x_0')),\tau(x_{\zeta,\varkappa}(x_0'))}q\big)\iota_{\zeta_{x_0'},\varkappa}^{-1} \\
&= \iota_{\zeta_{x_0'},\varkappa}\iota_{y_0(x_0'),\widetilde{\varkappa}(x_0')}^{-1}\big(I + D_\varepsilon^{y_0(x_0'),\widetilde{\varkappa}(x_0')}(z)Q_{\eta(x_{\zeta,\varkappa}(x_0')),\tau(x_{\zeta,\varkappa}(x_0'))}q\big)\iota_{y_0(x_0'),\widetilde{\varkappa}(x_0')}\iota_{\zeta_{x_0'},\varkappa}^{-1}
\end{aligned}
$$

and from (4.21)

$$
\|\iota_{\zeta_{x_0'},\varkappa}\|_{L^2(\Sigma_{\zeta_{x_0'},\varkappa};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} = \|\iota_{\zeta_{x_0'},\varkappa}^{-1}\|^{-1}_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\Sigma_{\zeta_{x_0'},\varkappa};\mathbb{C}^N)} = (1+|\nabla\zeta(x_0')|^2)^{-1/4},
$$

$$
\|\iota_{y_0(x_0'),\widetilde{\varkappa}(x_0')}\|_{L^2(\Sigma_{y_0(x_0'),\widetilde{\varkappa}(x_0')};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} = \|\iota_{y_0(x_0'),\widetilde{\varkappa}(x_0')}^{-1}\|^{-1}_{L^2(\Sigma_{y_0(x_0'),\widetilde{\varkappa}(x_0')};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} = 1.
$$

These considerations show that $I + D_\varepsilon^{\zeta_{x_0'},\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}(x_0')q$ is also continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ and

$$
\big\|(I + D_\varepsilon^{\zeta_{x_0'},\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}(x_0')q)^{-1}\big\|_{0\to 0} \leq \big\|(I + D_\varepsilon^{y_0(x_0'),\widetilde{\varkappa}(x_0')}(z)Q_{\eta(x_{\zeta,\varkappa}(x_0')),\tau(x_{\zeta,\varkappa}(x_0'))}q)^{-1}\big\|_{0\to 0}
$$

Now, the result follows from applying Corollary 4.18 (for $S = \overline{\operatorname{ran}(\eta,\tau)}$) if one chooses $\delta_4 = \delta_2(\overline{\operatorname{ran}(\eta,\tau)}) > 0$, where δ_2 was introduced in Corollary 4.18. □

Inspired by the local principle in [31, Proposition 5], see also [29, 30], we are going to construct partitions of unity which allow us to globalize the established local results. We start by choosing a partition of unity $(\phi_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ for $\mathbb{R}^{\theta-1}$ with uniformly bounded derivatives which satisfies $\operatorname{supp}\phi_{n'} \subset B(n',1)$ for $n'\in\mathbb{Z}^{\theta-1}$. Moreover, let $(\vartheta_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a sequence of functions with uniformly bounded derivatives which fulfills $0\leq\vartheta_{n'}\leq 1$, $\vartheta_{n'} = 1$ on $B(n',2)$ and $\operatorname{supp}\vartheta_{n'}\subset B(n',3)$ for $n'\in\mathbb{Z}^{\theta-1}$. According to Proposition A.2 such sequences exist. By defining for $a\in(0,\varepsilon_2^{1/6})$ and $n'\in\mathbb{Z}^{\theta-1}$ the functions $\phi_{n'}^a(\cdot) = \phi_{n'}(\cdot/a)$ and $\vartheta_{n'}^a(\cdot) = \vartheta_{n'}(\cdot/a)$ we obtain similar sequences with scaled supports; in particular $(\phi_{n'}^a)_{n'\in\mathbb{Z}^{\theta-1}}$ is a partition of unity for $\mathbb{R}^{\theta-1}$. Furthermore, there exists a $C>0$ which does not depend on a such that

$$
\sup_{n'\in\mathbb{Z}^{\theta-1}} \max\{\|\phi_{n'}^a\|_{W_\infty^1(\mathbb{R}^{\theta-1})}, \|\vartheta_{n'}^a\|_{W_\infty^1(\mathbb{R}^{\theta-1})}\} < \frac{C}{a}. \tag{4.60}
$$

Before we can construct the inverse of $I + D_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\varkappa}q$ we have to deal with series of operators. Let $(A_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a family of bounded operators mapping from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{G}$. If the sequence of partial sums $S_n = \sum_{n'\in\mathbb{Z}^{\theta-1},|n'|\leq n} A_{n'}$ converges in the strong sense to an operator, then we define

$$
\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} A_{n'} := \operatorname*{s-lim}_{n\to\infty} S_n.
$$

According to the Banach–Steinhaus theorem, $\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} A_{n'}$ is again a bounded operator mapping from $\mathcal{H}$ to $\mathcal{G}$. Moreover, the definition of the series implies that if $\mathcal{H}'$ and $\mathcal{G}'$ are Hilbert spaces, and $U : \mathcal{H}'\to\mathcal{H}$ and $V : \mathcal{G}\to\mathcal{G}'$ are bounded operators, then

$$
\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} VA_{n'}U : \mathcal{H}'\to\mathcal{G}'
$$

is a well-defined bounded operator and

$$V\Bigg(\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} A_{n'}\Bigg)U = \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} VA_{n'}U$$

holds. If $(\mathcal{A}_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ is a uniformly bounded sequence of operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, then according to Proposition A.4 the series $\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a \mathcal{A}_{n'} \vartheta_{n'}^a$ converges in the strong sense in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ and we have

$$\Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a \mathcal{A}_{n'} \vartheta_{n'}^a \Big\|_{0\to 0} \le 11^{\theta-1} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}\|_{0\to 0}. \tag{4.61}$$

Moreover, if $(\mathcal{A}_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ is also uniformly bounded as a sequence of operators from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$, then $\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a \mathcal{A}_{n'} \vartheta_{n'}^a$ acts also as a bounded operator from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$ and

$$\Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a \mathcal{A}_{n'} \vartheta_{n'}^a \Big\|_{0\to 1} \le \frac{C}{a} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}\|_{0\to 1} \tag{4.62}$$

holds, with $C > 0$ independent of a, see also Proposition A.4. Before we use these essential observations in the proof of Proposition 4.27, we state a helpful preliminary lemma.

Lemma 4.26. *Let $z \in \rho(H)$, $\varepsilon \in (0, \varepsilon_2)$, $a_\varepsilon = \varepsilon^{1/6}$, $D_\varepsilon^{\zeta,\kappa}(z)$ be as in (4.22), and $E_\varepsilon(z)$ be as in (4.48). Then, for any fixed $n' \in \mathbb{Z}^{\theta-1}$*

$$(1 - \vartheta_{n'}^{a_\varepsilon})E_\varepsilon(z)\phi_{n'}^{a_\varepsilon} = (1 - \vartheta_{n'}^{a_\varepsilon})D_\varepsilon^{\zeta,\kappa}(z)\phi_{n'}^{a_\varepsilon}.$$

Proof. We prove this by showing that the difference of the integral kernels of $(1 - \vartheta_{n'}^{a_\varepsilon})E_\varepsilon(z)\phi_{n'}^{a_\varepsilon}$ and $(1 - \vartheta_{n'}^{a_\varepsilon})D_\varepsilon^{\zeta,\kappa}(z)\phi_{n'}^{a_\varepsilon}$ is zero. By (4.26), (4.47), and (4.48) this difference is given by

$$(1 - \vartheta_{n'}^{a_\varepsilon}(x'))\big(\omega\big(\tfrac{x'-y'}{a_\varepsilon}\big) - 1\big)\phi_{n'}^{a_\varepsilon}(y')G_z(x_{\zeta,\kappa}(x') - x_{\zeta,\kappa}(y') + \varepsilon(t-s)\nu_{\zeta,\kappa}(x'))\sqrt{1 + |\nabla\zeta(y')|^2} \tag{4.63}$$

for all $x', y' \in \mathbb{R}^{\theta-1}$ and $t, s \in (-1, 1)$. If $y' \notin B(a_\varepsilon n', a_\varepsilon)$, then $\frac{y'}{a_\varepsilon} \notin B(n', 1) \supset \operatorname{supp} \phi_{n'}$ and therefore

$$\phi_{n'}^{a_\varepsilon}(y') = \phi_{n'}\big(\tfrac{y'}{a_\varepsilon}\big) = 0.$$

Furthermore, if $x' \in B(a_\varepsilon n', 2a_\varepsilon)$, then $\frac{x'}{a_\varepsilon} \in B(n', 2)$ and hence as $\vartheta_{n'} = 1$ on $B(n', 2)$, we have

$$1 - \vartheta_{n'}^{a_\varepsilon}(x') = 1 - \vartheta_{n'}\big(\tfrac{x'}{a_\varepsilon}\big) = 0.$$

These two observations show that if $x' \in B(a_\varepsilon n', 2a_\varepsilon)$ or $y' \notin B(a_\varepsilon n', a_\varepsilon)$, then (4.63) vanishes. Thus, it remains to consider the case $x' \notin B(a_\varepsilon n', 2a_\varepsilon)$ and $y' \in B(a_\varepsilon n', a_\varepsilon)$. However, this implies $|x' - y'| > a_\varepsilon$. Then, we use $\omega = 1$ on $\mathbb{R}^{\theta-1} \setminus B(0, 1)$, see the text above (4.48), to obtain

$$\omega\big(\tfrac{x'-y'}{a_\varepsilon}\big) - 1 = 0.$$

This shows that (4.63) vanishes for all $x', y' \in \mathbb{R}^{\theta-1}$ and $t, s \in (-1, 1)$. □

Proposition 4.27. *Let $z \in \mathbb{C} \setminus \mathbb{R}$, $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ and Σ be as described in the beginning of Section 4.4.2, $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$, $d = \eta^2 - \tau^2$ satisfy*

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4},$$

$Q_{\eta,\tau}^{\zeta,\varkappa}$ be as in (4.29) and q be as in (1.4). Then, there exists a $\delta_5 \in (0, \varepsilon_2)$, with $\varepsilon_2 > 0$ from Proposition 4.6, such that the operator $I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q$ has a bounded right inverse which is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $\varepsilon \in (0, \delta_5)$.

Proof. The proof is split into four steps. In *Step 1* we define R_ε which will turn out to be a first approximation for the right inverse of $I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q$. Moreover, in this step we also show that R_ε is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to ε. Then, in *Step 2* we calculate $(I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q)R_\varepsilon$. Afterwards, we find in *Step 3* that $(I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q)R_\varepsilon$ equals $I + K_\varepsilon + L_\varepsilon$, where K_ε and L_ε fulfill the inequalities

$$\|K_\varepsilon\|_{0\to1} \le C\frac{1 + |\log(\varepsilon)|}{a_\varepsilon^2} \quad \text{and} \quad \|L_\varepsilon\|_{0\to0} \le C a_\varepsilon(1 + |\log(\varepsilon)|). \tag{4.64}$$

Based on these observations we define in *Step 4* an operator $\widetilde{R}_\varepsilon$ which is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to ε and fulfills $(I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q)\widetilde{R}_\varepsilon = I + \widetilde{L}_\varepsilon$, where $\|\widetilde{L}_\varepsilon\|_{0\to0}$ can be estimated by $C(1 + |\log(\varepsilon)|)\varepsilon^{1/6-r}$ for an $r \in (0, 1/6)$. In particular, this shows that for sufficiently small $\varepsilon > 0$ the right inverse of $I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q$ is given by the operator $\widetilde{R}_\varepsilon(I + \widetilde{L}_\varepsilon)^{-1}$ and uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to ε.

Step 1. We define for $\varepsilon \in (0, \min\{\varepsilon_2, \delta_4\})$, where ε_2 and δ_4 are chosen as in Proposition 4.6 and Proposition 4.25, respectively,

$$R_\varepsilon : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \quad R_\varepsilon := \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon}.$$

Here, $a_\varepsilon = \varepsilon^{1/6}$ and $R_{n',\varepsilon} := (I + D_\varepsilon^{\zeta_{a_\varepsilon n'},\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}(a_\varepsilon n')q)^{-1}$ with $\zeta_{a_\varepsilon n'} = \zeta_{x_0'}$ as in (4.46) for $x_0' = a_\varepsilon n'$. The equality $\vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon} = \phi_{n'}^{a_\varepsilon}$ implies

$$R_\varepsilon = \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}\phi_n^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon}$$

and therefore Proposition 4.25 and (4.61) show that R_ε is well-defined and uniformly bounded by

$$\|R_\varepsilon\|_{0\to0} \le 11^{\theta-1} \sup_{n' \in \mathbb{Z}^{\theta-1}} \|(I + D_\varepsilon^{\zeta_{a_\varepsilon n'},\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}(a_\varepsilon n')q)^{-1}\|_{0\to0} \le C, \tag{4.65}$$

where $C > 0$ does not depend on ε.

Step 2. Applying $I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q$ to R_ε yields

$$\begin{aligned}(I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q)R_\varepsilon &= \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} (I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q)\phi_{n'}^{a_\varepsilon} R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} \\ &= \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(I + D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q)\phi_{n'}^{a_\varepsilon} R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} \\ &\quad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} (1 - \vartheta_{n'}^{a_\varepsilon})D_\varepsilon^{\zeta,\varkappa}(z)Q_{\eta,\tau}^{\zeta,\varkappa}q\phi_{n'}^{a_\varepsilon} R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon}.\end{aligned}$$

Moreover, using Lemma 4.26 gives us

$$
\begin{aligned}
(I+D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)R_\varepsilon &= \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(I+D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)\phi_{n'}^{a_\varepsilon}R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} \\
&\quad + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} (1-\vartheta_{n'}^{a_\varepsilon})E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}q\phi_{n'}^{a_\varepsilon}R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} \\
&= \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(I+D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q - E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}q)\phi_{n'}^{a_\varepsilon}R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} \\
&\quad + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon.
\end{aligned}
$$

Writing $D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q\phi_{n'}^{a_\varepsilon}$ as

$$
\begin{aligned}
&D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q\phi_{n'}^{a_\varepsilon} + \left(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')\right)q\phi_{n'}^{a_\varepsilon} \\
&= \phi_{n'}^{a_\varepsilon}D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q + [D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z),\phi_{n'}^{a_\varepsilon}]Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q \\
&+ \left(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')\right)q\phi_{n'}^{a_\varepsilon}
\end{aligned}
$$

and introducing the operators

$$
L_{n',\varepsilon} := \left(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')\right)q\phi_{n'}^{a_\varepsilon}
$$

and

$$
K_{n',\varepsilon} := [D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z),\phi_{n'}^{a_\varepsilon}]Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q - E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}q\phi_{n'}^{a_\varepsilon}
$$

yields

$$
\begin{aligned}
(I+D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)R_\varepsilon &= \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon}(I+D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q)R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} \\
&\quad + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(K_{n',\varepsilon}+L_{n',\varepsilon})R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon \\
&= I + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(K_{n',\varepsilon}+L_{n',\varepsilon})R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon,
\end{aligned}
\tag{4.66}
$$

where

$$
(I+D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q)R_{n',\varepsilon} = I \quad \text{and} \quad \sum_{n'\in\mathbb{Z}^{\theta-1}} \vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon}\vartheta_{n'}^{a_\varepsilon} = \sum_{n'\in\mathbb{Z}^{\theta-1}} \phi_{n'}^{a_\varepsilon} = 1
$$

were used.

Step 3. We start this step by setting

$$
K_\varepsilon := \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}K_{n',\varepsilon}R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon
$$

and

$$L_\varepsilon := \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} L_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon}.$$

Then, (4.66) shows

$$(I + D_\varepsilon^{\zeta,\varkappa}(z) Q_{\eta,\tau}^{\zeta,\varkappa} q) R_\varepsilon = I + K_\varepsilon + L_\varepsilon. \tag{4.67}$$

Since R_ε and $D_\varepsilon^{\zeta,\varkappa}(z)$ are uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, see *Step 1* and the text above (4.23), respectively, this implies that also $K_\varepsilon + L_\varepsilon$ is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Moreover, $Q_{\eta,\tau}^{\zeta,\varkappa} \in C_b^1(\mathbb{R}^{\theta-1}; \mathbb{C}^{N\times N})$, Proposition 4.24, and (4.60) imply

$$\begin{aligned}
\|K_{n',\varepsilon}\|_{0\to1} &\leq \|[D_\varepsilon^{\zeta_{a_\varepsilon n'},\varkappa}(z), \phi_{n'}^{a_\varepsilon}] Q_{\eta,\tau}^{\zeta,\varkappa}(a_\varepsilon n') q\|_{0\to1} + \|E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\varkappa} q \phi_{n'}^{a_\varepsilon}\|_{0\to1} \\
&\leq C\big(\|[D_\varepsilon^{\zeta_{a_\varepsilon n'},\varkappa}(z), \phi_{n'}^{a_\varepsilon}]\|_{0\to1} + \|E_\varepsilon(z)\|_{0\to1}\big) \\
&\leq C\Big(\|\phi_{n'}^{a_\varepsilon}\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1 + |\log(\varepsilon)|) + \frac{1 + |\log(\varepsilon)|}{a_\varepsilon}\Big) \\
&\leq C\frac{1 + |\log(\varepsilon)|}{a_\varepsilon}, \qquad n' \in \mathbb{Z}^{\theta-1}.
\end{aligned}$$

In turn, with (4.62), (4.65), $Q_{\eta,\tau}^{\zeta,\varkappa} \in C_b^1(\mathbb{R}^{\theta-1}; \mathbb{C}^{N\times N})$, Proposition 4.24, and Proposition 4.25 we get

$$\begin{aligned}
\|K_\varepsilon\|_{0\to1} &\leq \Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} K_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \Big\|_{0\to1} + \|E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\varkappa} q R_\varepsilon\|_{0\to1} \\
&\leq \Big(\frac{C}{a_\varepsilon} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|K_{n',\varepsilon} R_{n',\varepsilon}\|_{0\to1} + \|E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\varkappa} q R_\varepsilon\|_{0\to1}\Big) \\
&\leq \Big(\frac{C}{a_\varepsilon} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|K_{n',\varepsilon}\|_{0\to1} + \|E_\varepsilon(z)\|_{0\to1}\Big) \\
&\leq C\Big(\frac{1}{a_\varepsilon} \cdot \frac{1 + |\log(\varepsilon)|}{a_\varepsilon} + \frac{1 + |\log(\varepsilon)|}{a_\varepsilon}\Big) \\
&\leq C\frac{1 + |\log(\varepsilon)|}{a_\varepsilon^2}.
\end{aligned}$$

Similar we estimate L_ε with (4.61), Proposition 4.25, and Proposition 4.24 by

$$\begin{aligned}
\|L_\varepsilon\|_{0\to0} &= \Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} L_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \Big\|_{0\to0} \\
&= \Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} \chi_{B(a_\varepsilon n', 3a_\varepsilon)} L_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \Big\|_{0\to0} \\
&\leq C \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\chi_{B(a_\varepsilon n', 3a_\varepsilon)} L_{n',\varepsilon}\|_{0\to0} \\
&= C \sup_{n'\in\mathbb{Z}^{\theta-1}} \Big\|\chi_{B(a_\varepsilon n', 3a_\varepsilon)} \Big(D_\varepsilon^{\zeta,\varkappa}(z) Q_{\eta,\tau}^{\zeta,\varkappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\varkappa}(z) Q_{\eta,\tau}^{\zeta,\varkappa}(a_\varepsilon n')\Big) q \phi_{n'}^{a_\varepsilon}\Big\|_{0\to0} \\
&= C \sup_{n'\in\mathbb{Z}^{\theta-1}} \Big\|\chi_{B(a_\varepsilon n', 3a_\varepsilon)} \Big(D_\varepsilon^{\zeta,\varkappa}(z) Q_{\eta,\tau}^{\zeta,\varkappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\varkappa}(z) Q_{\eta,\tau}^{\zeta,\varkappa}(a_\varepsilon n')\Big) q \chi_{B(a_\varepsilon n', 3a_\varepsilon)} \phi_{n'}^{a_\varepsilon}\Big\|_{0\to0} \\
&\leq C a_\varepsilon (1 + |\log(\varepsilon)|).
\end{aligned}$$

This shows that (4.64) is valid and hence completes *Step 3*.

Step 4. We note that Proposition 4.9, (4.20), (4.22), and (4.29) imply that $I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$ is continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ and $\mathcal{B}^{1/2}(\mathbb{R}^{\theta-1})$. Furthermore, since R_ε and $K_\varepsilon + L_\varepsilon$ are uniformly bounded operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, the operator

$$\widetilde{R}_\varepsilon := R_\varepsilon - (I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}(K_\varepsilon + L_\varepsilon)$$

is also uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Moreover, applying $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$ to $\widetilde{R}_\varepsilon$ and (4.67) give us

$$\begin{aligned}
&(I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)\widetilde{R}_\varepsilon \\
&\quad = I + K_\varepsilon + L_\varepsilon - (I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}(K_\varepsilon + L_\varepsilon) \\
&\quad = I + (D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}(K_\varepsilon + L_\varepsilon) \\
&\quad = I + \widetilde{L}_\varepsilon
\end{aligned} \tag{4.68}$$

with

$$\widetilde{L}_\varepsilon := (D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}(K_\varepsilon + L_\varepsilon).$$

Thus, by using the estimates for L_ε and K_ε from *Step 3* and (4.23) for a fixed $r \in (0, 1/6)$ we obtain

$$\begin{aligned}
\|\widetilde{L}_\varepsilon\|_{0\to 0} &\le \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}K_\varepsilon\|_{0\to 0} \\
&\quad + \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}L_\varepsilon\|_{0\to 0} \\
&\le \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{1/2\to 0}\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\|_{1/2\to 1/2}\|K_\varepsilon\|_{0\to 1/2} \\
&\quad + \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{0\to 0}\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\|_{0\to 0}\|L_\varepsilon\|_{0\to 0} \\
&\le \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{1/2\to 0}\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\|_{1/2\to 1/2}\|K_\varepsilon\|_{0\to 1} \\
&\quad + \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{0\to 0}\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\|_{0\to 0}\|L_\varepsilon\|_{0\to 0} \\
&\le C\left(\frac{\varepsilon^{1/2-r}(1 + |\log(\varepsilon)|)}{a_\varepsilon^2} + a_\varepsilon(1 + |\log(\varepsilon)|)\right) \\
&= C(1 + |\log(\varepsilon)|)(\varepsilon^{1/6-r} + \varepsilon^{1/6}) \\
&\le C(1 + |\log(\varepsilon)|)\varepsilon^{1/6-r}.
\end{aligned}$$

This shows that if we choose $\delta_5 > 0$ sufficiently small, then $\|\widetilde{L}_\varepsilon\|_{0\to 0} < \frac{1}{2}$ for all $\varepsilon \in (0, \delta_5)$ and $(I + \widetilde{L}_\varepsilon)^{-1}$ is uniformly bounded with respect to $\varepsilon \in (0, \delta_5)$. Thus, as $\widetilde{R}_\varepsilon$ is also uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, $\widetilde{R}_\varepsilon(I + \widetilde{L}_\varepsilon)^{-1}$ is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $\varepsilon \in (0, \delta_5)$ and by (4.68) it is also the right inverse of $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$. □

Proposition 4.28. *Let Σ be a rotated graph as described in the beginning of Section 4.4.2, $z \in \mathbb{C} \setminus \mathbb{R}$, q be as in (1.4), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$, and $d = \eta^2 - \tau^2$ such that*

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4}.$$

Then, there exists $\varepsilon_3 \in (0, \varepsilon_2)$, with $\varepsilon_2 > 0$ from Proposition 4.6, such that $I + B_\varepsilon(z)Vq$ has a bounded inverse which is uniformly bounded in $\mathcal{B}^0(\Sigma)$ with respect to $\varepsilon \in (0, \varepsilon_3)$.

Proof. We directly get from Proposition 4.27, (4.21), (4.22), and (4.29) that $I + \overline{B}_\varepsilon(z)Vq$ has a right inverse, which is uniformly bounded with respect to $\varepsilon \in (0, \delta_5)$ with δ_5 from the previous proposition. Using (4.19) shows that then $I + B_\varepsilon(z)Vq$ has a right inverse which is uniformly bounded for $\varepsilon \in (0, \varepsilon_3)$ if $\varepsilon_3 > 0$ is chosen small enough. We denote this right inverse $\mathcal{R}_\varepsilon(z)$. Then, the right inverse of $I + VqB_\varepsilon(\overline{z})$ is given by $I - Vq\mathcal{R}_\varepsilon(\overline{z})B_\varepsilon(\overline{z})$. Hence, $(I + VqB_\varepsilon(\overline{z}))^* = I + (B_\varepsilon(\overline{z}))^*Vq$ has the uniformly bounded left-inverse $\mathcal{L}_\varepsilon(z) := I - (Vq\mathcal{R}_\varepsilon(\overline{z})B_\varepsilon(\overline{z}))^*$. Moreover, by [8, Remark 3.11] the estimate

$$\left\| B_\varepsilon(z) - (B_\varepsilon(\overline{z}))^* \right\|_{0\to 0} \leq C\varepsilon, \quad \varepsilon \in (0, \varepsilon_2),$$

holds. Thus,

$$\mathcal{L}_\varepsilon(z)(I + B_\varepsilon(z)Vq) = I + \mathcal{L}_\varepsilon(z)(B_\varepsilon(z) - (B_\varepsilon(\overline{z}))^*)Vq \tag{4.69}$$

converges in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ for $\varepsilon \to 0$ to the identity operator I. In particular, if $\varepsilon_3 > 0$ is chosen small enough, then the right-hand side of (4.69) is invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ for all $\varepsilon \in (0, \varepsilon_3)$, showing that $I + B_\varepsilon(z)Vq$ is invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, that is,

$$\mathcal{R}_\varepsilon(z) = (I + B_\varepsilon(z)Vq)^{-1}.$$

This concludes the proof since we already know that $\mathcal{R}_\varepsilon(z)$ is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. □

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CONFLICT OF INTEREST STATEMENT

The authors have no competing interests to declare that are relevant to the content of this paper. No data are associated with this paper.

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APPENDIX: ADDITIONAL RESULTS FOR SECTION 4.4.2

In this section, we provide results which are used in Section 4.4.2. We begin by stating a convenient version of the Schur test.

Lemma A.1. *Let k be a measurable function in $\mathbb{R}^{\theta-1} \times \mathbb{R}^{\theta-1}$ with values in $\mathbb{C}^{N\times N}$ and $\widetilde{k} \in L^1(\mathbb{R}^{\theta-1})$ such that*

$$|k(x', y')| \le \widetilde{k}(x' - y') \quad \text{for a.e. } x', y' \in \mathbb{R}^{\theta-1}.$$

Then, the operator $K : L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ acting as

$$Kf(x') = \int_{\mathbb{R}^{\theta-1}} k(x', y') f(y') dy'$$

is well-defined and bounded, and $\|K\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq \|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}$. *Moreover, if additionally to the above assumptions* $k \in C^1(\mathbb{R}^{\theta-1}\times\mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$ *and*

$$\sum_{l=1}^{\theta-1}\left|\frac{d}{dx'_l}k(x',y')\right| \leq \widetilde{k}(x'-y') \quad \text{for a.e. } x',y' \in \mathbb{R}^{\theta-1},$$

then K *also acts as bounded operator from* $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ *to* $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ *and* $\|K\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}$.

Proof. The first assertion is an immediate consequence of the Schur test, see for instance [21, Chapter III, Example 2.4]. Next, let us prove the second assertion. We start by choosing $g \in \mathcal{D}(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. Since $k \in C^1(\mathbb{R}^{\theta-1}\times\mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$ and g is compactly supported, dominated convergence shows that Kg is differentiable and

$$\partial_l(Kg)(x') = \int_{\mathbb{R}^{\theta-1}}\frac{d}{dx'_l}k(x',y')g(y')\,dy'.$$

Hence, applying the Schur test shows

$$\|Kg\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}\|g\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}.$$

The rest follows from the fact that $\mathcal{D}(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ is dense in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$, the completeness of $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$, and the continuity of K in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. □

In the upcoming proposition, we construct a partition of unity for $\mathbb{R}^{\theta-1}$ such that the functions have uniformly bounded derivatives.

Proposition A.2. *There exists a partition of unity* $(\phi_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ *for* $\mathbb{R}^{\theta-1}$ *subordinate to* $(B(n',1))_{n'\in\mathbb{Z}^{\theta-1}}$ *such that* $\max_{n'\in\mathbb{Z}^{\theta-1}}\|\phi_{n'}\|_{W^1_\infty(\mathbb{R}^{\theta-1})} < \infty$. *Moreover, there exists a sequence* $(\vartheta_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ *with* $\operatorname{supp}\vartheta_{n'} \subset B(n',3)$, $0 \leq \vartheta_{n'} \leq 1$, $\vartheta_{n'} = 1$ *on* $B(n',2)$ *for all* $n' \in \mathbb{Z}^{\theta-1}$ *and* $\max_{n'\in\mathbb{Z}^{\theta-1}}\|\vartheta_{n'}\|_{W^1_\infty(\mathbb{R}^{\theta-1})} < \infty$.

Proof. Note that since $\theta \in \{2,3\}$, the family $(B(n',3/4))_{n'\in\mathbb{Z}^{\theta-1}}$ is also an open cover of $\mathbb{R}^{\theta-1}$. We start by choosing a function $\phi \in C^\infty(\mathbb{R}^{\theta-1})$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B(0,3/4)$ and $\operatorname{supp}\phi \subset B(0,1)$. Furthermore, we define the function $\widetilde{\phi}_{n'} := \phi(\cdot - n')$ for $n' \in \mathbb{Z}^{\theta-1}$. Then, $0 \leq \widetilde{\phi}_{n'} \leq 1$, $\widetilde{\phi}_{n'} = 1$ on $B(n',3/4)$ and $\operatorname{supp}\widetilde{\phi}_{n'} \subset B(n',1)$. Next, we fix a bijection $\mathcal{Z} : \mathbb{N} \to \mathbb{Z}^{\theta-1}$ and set $\phi_{\mathcal{Z}(1)} := \widetilde{\phi}_{\mathcal{Z}(1)}$ and

$$\phi_{\mathcal{Z}(j)} = (1-\widetilde{\phi}_{\mathcal{Z}(1)})\cdot\dots\cdot(1-\widetilde{\phi}_{\mathcal{Z}(j-1)})\widetilde{\phi}_{\mathcal{Z}(j)}, \quad j \in \mathbb{N}\setminus\{1\}.$$

Then, $\operatorname{supp}\phi_{n'} \subset \operatorname{supp}\widetilde{\phi}_{n'}$, $0 \leq \phi_{n'} \leq 1$ for $n' \in \mathbb{Z}^{\theta-1}$ and one gets via induction for $j \in \mathbb{N}$

$$\sum_{k=1}^{j}\phi_{\mathcal{Z}(k)} = 1 - \prod_{k=1}^{j}(1-\widetilde{\phi}_{\mathcal{Z}(k)}).$$

This implies $\sum_{n'\in\mathbb{Z}^{\theta-1}}\phi_{n'}(x') = \sum_{j=1}^\infty\phi_{\mathcal{Z}(j)}(x') = 1$ for $x' \in \mathbb{R}^{\theta-1}$. Furthermore, let $j \in \mathbb{N}$, $l \in \{1,\dots,\theta-1\}$, and $x' \in \mathbb{R}^{\theta-1}$. We estimate

$$\begin{aligned}
|\partial_l\phi_{\mathcal{Z}(j)}(x')| &= \left|\partial_l(\widetilde{\phi}_{\mathcal{Z}(j)}(x'))\prod_{k=1}^{j-1}(1-\widetilde{\phi}_{\mathcal{Z}(k)}(x')) - \sum_{k=1}^{j-1}\widetilde{\phi}_{\mathcal{Z}(j)}(x')(\partial_l\widetilde{\phi}_{\mathcal{Z}(k)}(x'))\prod_{r=1,r\neq k}^{j-1}(1-\widetilde{\phi}_{\mathcal{Z}(r)}(x'))\right| \\
&\leq \sum_{k=1}^{j}|\partial_l\widetilde{\phi}_{\mathcal{Z}(k)}(x')| = \sum_{k=1,x'\in B(\mathcal{Z}(k),1)}^{j}|\partial_l\widetilde{\phi}_{\mathcal{Z}(k)}(x')| \\
&\leq 2^{\theta-1}\|\partial_l\phi\|_{L^\infty(\mathbb{R}^{\theta-1})},
\end{aligned}$$

where we used that $x' \in \mathbb{R}^{\theta-1}$ can be in at most $2^{\theta-1}$ balls of the form $B(n',1)$ with $n' \in \mathbb{Z}^{\theta-1}$. This shows that the derivatives of the $\phi_{n'}$s are uniformly bounded by $2^{\theta-1}\|\phi\|_{W^1_\infty(\mathbb{R}^{\theta-1};\mathbb{R})}$. Next we construct the sequence $(\vartheta_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$. To do so, we choose $\vartheta \in C^\infty(\mathbb{R}^{\theta-1})$ such that $0 \le \vartheta \le 1$, $\theta = 1$ on $B(0,2)$ and $\operatorname{supp}\vartheta \subset B(0,3)$. Then, we define $\vartheta_{n'} := \vartheta(\cdot - n')$. The constructed sequence has the claimed properties. □

Our next goal is to use the functions $\vartheta_{n'}$, $n' \in \mathbb{Z}$, from Proposition A.2 to construct operators based on a uniformly bounded sequence of operators. We start by providing a useful variant of the Cotlar–Stein lemma.

Lemma A.3. *Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces, let $(\mathcal{A}_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a family of uniformly bounded operators acting from $\mathcal{H}$ to $\mathcal{G}$. Moreover, assume that there exists a number $N \in \mathbb{N}$ such that for every $n' \in \mathbb{Z}^{\theta-1}$ exist at most N indices $m' \in \mathbb{Z}^{\theta-1}$ such $\mathcal{A}^*_{n'}\mathcal{A}_{m'}$ and $\mathcal{A}_{n'}\mathcal{A}^*_{m'}$ are nonzero operators. Then, the sum $\sum_{n'\in\mathbb{Z}^{\theta-1},|n'|<n}\mathcal{A}_{n'}$ converges for $n\to\infty$ in the strong sense to a bounded operator $\mathcal{A}$ (which is also denoted by $\sum^{\text{st.}}_{n'\in\mathbb{Z}^{\theta-1}}\mathcal{A}_{n'}$). Moreover, its norm can be estimated by*

$$\|\mathcal{A}\|_{\mathcal{H}\to\mathcal{G}} \le N \sup_{n'\in\mathbb{Z}^{\theta-1}}\|\mathcal{A}_{n'}\|_{\mathcal{H}\to\mathcal{G}}.$$

Proof. Our assumptions guarantee

$$\sup_{n'\in\mathbb{Z}^{\theta-1}}\sum_{m'\in\mathbb{Z}^{\theta-1}}\|\mathcal{A}_{n'}\mathcal{A}^*_{m'}\|^{1/2}_{\mathcal{G}\to\mathcal{G}} \le N\sup_{n'\in\mathbb{Z}^{\theta-1}}\|\mathcal{A}_{n'}\|_{\mathcal{H}\to\mathcal{G}}$$

and

$$\sup_{n'\in\mathbb{Z}^{\theta-1}}\sum_{m'\in\mathbb{Z}^{\theta-1}}\|\mathcal{A}^*_{n'}\mathcal{A}_{m'}\|^{1/2}_{\mathcal{H}\to\mathcal{H}} \le N\sup_{n'\in\mathbb{Z}^{\theta-1}}\|\mathcal{A}_{n'}\|_{\mathcal{H}\to\mathcal{G}}.$$

Hence, the assertions follow from the Cotlar–Stein lemma, see [16, Lemma 18.6.5]. □

Proposition A.4. *Let $a \in (0,b)$ for a $b > 0$, $(\vartheta_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be the sequence from Proposition A.2, $\vartheta^a_{n'} := \vartheta_{n'}\left(\frac{\cdot}{a}\right)$ for $n' \in \mathbb{Z}^{\theta-1}$, and $(A_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a sequence of uniformly bounded operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Then,*

$$A = \sum^{\text{st.}}_{n'\in\mathbb{Z}^{\theta-1}}\vartheta^a_{n'}A_{n'}\vartheta^a_{n'}$$

is a well-defined operator in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with $\|A\|_{0\to0} \le 11^{\theta-1}\sup_{n'\in\mathbb{Z}^{\theta-1}}\|A_{n'}\|_{0\to0}$. Moreover, if $(A_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ is also a family of uniformly bounded operators acting from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$, then A acts also as a bounded operator from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$ and $\|A\|_{0\to1} \le \frac{C}{a}\sup_{n'\in\mathbb{Z}^{\theta-1}}\|A_{n'}\|_{0\to1}$, where $C > 0$ does not depend on $a \in (0,b)$.

Proof. Let us start by proving the assertion where we consider A and $A_{n'}$, $n' \in \mathbb{Z}^{\theta-1}$, as operators acting from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. We set $\mathcal{A}_{n'} := \vartheta^a_{n'}A_{n'}\vartheta^a_{n'}$. Since a fixed ball $B(an',3a)$ overlaps with at most $11^{\theta-1}$ balls of the type $B(am',3a)$, $m' \in \mathbb{Z}^{\theta-1}$, there exist for every $n' \in \mathbb{Z}^{\theta-1}$ at most $N = 11^{\theta-1}$ indices $m' \in \mathbb{Z}^{\theta-1}$ such that $\mathcal{A}_{n'}\mathcal{A}^*_{m'} \ne 0$ and $\mathcal{A}^*_{n'}\mathcal{A}_{m'} \ne 0$. Moreover, we have for all $n' \in \mathbb{Z}^{\theta-1}$

$$\|\mathcal{A}_{n'}\|_{0\to0} \le \|\vartheta^a_{n'}\|_{0\to0}\|A_{n'}\|_{0\to0}\|\vartheta^a_{n'}\|_{0\to0} \le \|A_{n'}\|_{0\to0}.$$

Thus, by Lemma A.3 we conclude $\|A\|_{0\to0} \le 11^{\theta-1}\sup_{n'\in\mathbb{Z}^{\theta-1}}\|A_{n'}\|_{0\to0}$.

Next, we assume that $A_{n'}$, $n' \in \mathbb{Z}^{\theta-1}$, act as uniformly bounded operators from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$. Using again the fact that a fixed ball $B(an',3a)$ overlaps with at most $11^{\theta-1}$ balls of the type $B(am',3a)$, $m' \in \mathbb{Z}^{\theta-1}$, shows that there exist for every $n' \in \mathbb{Z}^{\theta-1}$ at most $N = 11^{\theta-1}$ indices $m' \in \mathbb{Z}^{\theta-1}$ such that $\mathcal{A}_{n'}\mathcal{A}^{0*1}_{m'} \ne 0$ and $\mathcal{A}^{0*1}_{n'}\mathcal{A}_{m'} \ne 0$, where the expressions ${\mathcal{A}_{n'}}^{0*1}$ and ${\mathcal{A}_{m'}}^{0*1}$ denote the adjoint operators of $\mathcal{A}_{n'}$ and $\mathcal{A}_{m'}$, respectively, considered as operators mapping from

$\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$. Furthermore, for all $n' \in \mathbb{Z}^{\theta-1}$ the inequality

$$\begin{aligned}
\|\mathcal{A}_{n'}\|_{0\to 1} &\le \|\vartheta_{n'}^a\|_{1\to 1}\|A_{n'}\|_{0\to 1}\|\vartheta_{n'}^a\|_{0\to 0}\\
&\le \|\vartheta_{n'}^a\|_{1\to 1}\|A_{n'}\|_{0\to 1}\\
&\le C\|\vartheta_{n'}^a\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\|A_{n'}\|_{0\to 1}\\
&\le \frac{C}{a}\|\vartheta_{n'}\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\|A_{n'}\|_{0\to 1}\\
&\le \frac{C}{a}\|A_{n'}\|_{0\to 1}
\end{aligned}$$

is valid. Thus, Lemma A.3 yields $\|A\|_{0\to 1} \le \frac{C}{a}\sup_{n'\in\mathbb{Z}^{\theta-1}}\|A_{n'}\|_{0\to 1}$. □