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Schrödinger Operators with Oblique Transmission Conditions in $\mathbb{R}^2$

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Abstract: In this paper we study the spectrum of self-adjoint Schrödinger operators in $L^2(\mathbb{R}^2)$ with a new type of transmission conditions along a smooth closed curve $\Sigma \subseteq \mathbb{R}^2$. Although these *oblique* transmission conditions are formally similar to δ'-conditions on Σ (instead of the normal derivative here the Wirtinger derivative is used) the spectral properties are significantly different: it turns out that for attractive interaction strengths the discrete spectrum is always unbounded from below. Besides this unexpected spectral effect we also identify the essential spectrum, and we prove a Krein-type resolvent formula and a Birman-Schwinger principle. Furthermore, we show that these Schrödinger operators with oblique transmission conditions arise naturally as non-relativistic limits of Dirac operators with electrostatic and Lorentz scalar δ-interactions justifying their usage as models in quantum mechanics.

1. Introduction

In many quantum mechanical applications one considers particles moving in an external potential field which is localized near a set Σ of measure zero. Such strongly localized fields can be modeled by singular potentials that are supported on Σ only; of particular importance in this regard are δ and δ'-interactions. To be more precise, assume that Σ splits $\mathbb{R}^2$ into a bounded domain Ω_+ and an unbounded domain $\Omega_- = \mathbb{R}^2 \setminus \overline{\Omega_+}$, and consider the formal Schrödinger differential expressions

$$\mathcal{H}_{\delta,\alpha} = -\Delta + \alpha\delta_\Sigma \quad \text{and} \quad \mathcal{H}_{\delta',\alpha} = -\Delta + \alpha\delta'_\Sigma, \quad \alpha \in \mathbb{R}. \tag{1.1}$$

These singular perturbations of the free Schrödinger operator $-\Delta$ are characterized by certain transmission conditions along the interface Σ for the functions in the operator domain. For δ-interactions one considers functions $f : \mathbb{R}^2 \to \mathbb{C}$ such that the restrictions $f_\pm = f \restriction \Omega_\pm$ satisfy the transmission conditions

$$f_+ = f_- \quad \text{and} \quad -\frac{\alpha}{2}\big(f_+ + f_-\big) = \big(\partial_\nu f_+ - \partial_\nu f_-\big) \quad \text{on } \Sigma, \tag{1.2}$$

while δ'-interactions are modeled by the transmission conditions

$$f_+ - f_- = -\frac{\alpha}{2}\big(\partial_\nu f_+ + \partial_\nu f_-\big) \quad \text{and} \quad \partial_\nu f_+ = \partial_\nu f_- \quad \text{on } \Sigma; \tag{1.3}$$

here $\partial_\nu f_\pm$ is the normal derivative and $\nu = (\nu_1, \nu_2)$ the unit normal vector field on Σ pointing outwards of Ω_+. The spectra and resonances of the self-adjoint realizations associated with the formal expressions (1.1) in $L^2(\mathbb{R}^2)$ are well understood, see, e.g., [8,9,12,13,15–18,24]. In particular, the essential spectrum is given by $[0, \infty)$ and the discrete spectrum consists of at most finitely many points for every interaction strength $\alpha < 0$, while there is no negative spectrum if $\alpha \geq 0$.

In contrast to the transmission conditions (1.2) and (1.3) we are interested in a new type of transmission conditions of the form

$$(\nu_1 + i\nu_2)\big(f_+ - f_-\big) = -\alpha\big(\partial_{\bar z} f_+ + \partial_{\bar z} f_-\big) \quad \text{and} \quad \partial_{\bar z} f_+ = \partial_{\bar z} f_- \quad \text{on } \Sigma, \tag{1.4}$$

where $\alpha \in \mathbb{R}$ and $\partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2)$ is the Wirtinger derivative. In the sequel such jump conditions will be referred to as *oblique* transmission conditions. Note that the conditions (1.4) can be rewritten as

$$f_+ - f_- = -\frac{\alpha}{2}\big(\partial_\nu f_+ + \partial_\nu f_- + i\partial_t f_+ + i\partial_t f_-\big) \quad \text{and} \quad \partial_{\bar z} f_+ = \partial_{\bar z} f_- \quad \text{on } \Sigma, \tag{1.5}$$

where ∂_t denotes the tangential derivative. Thus, on a formal level there is some analogy to the δ'-transmission conditions in (1.3), but it will turn out that the properties of the corresponding self-adjoint realization in $L^2(\mathbb{R}^2)$ differ significantly from those of Schrödinger operators with δ'-interactions.

To make matters mathematically rigorous, assume that the curve Σ is the boundary of a bounded and simply connected C^∞-domain Ω_+ with open complement $\Omega_- = \mathbb{R}^2 \setminus \overline{\Omega_+}$, denote the L^2-based Sobolev space of first order by H^1, let $\gamma_D^\pm : H^1(\Omega_\pm) \to L^2(\Sigma)$ be the Dirichlet trace operators, and define for $\alpha \in \mathbb{R}$ the Schrödinger operator with oblique transmission conditions by

$$\begin{aligned} T_\alpha f &= (-\Delta f_+) \oplus (-\Delta f_-), \\ \operatorname{dom} T_\alpha &= \Big\{ f \in H^1(\Omega_+) \oplus H^1(\Omega_-) \,\Big|\, \partial_{\bar z} f_+ \oplus \partial_{\bar z} f_- \in H^1(\mathbb{R}^2), \\ &\qquad (\nu_1 + i\nu_2)\big(\gamma_D^+ f_+ - \gamma_D^- f_-\big) = -\alpha\big(\gamma_D^+(\partial_{\bar z} f_+) + \gamma_D^-(\partial_{\bar z} f_-)\big)\Big\}. \end{aligned} \tag{1.6}$$

The next theorem is the main result in this paper. We discuss the spectral properties of the Schrödinger operators T_α and, in particular, we show in item (ii) that for every $\alpha < 0$ the operator T_α is necessarily unbounded from below and the discrete spectrum in $(-\infty, 0)$ is infinite and accumulates to $-\infty$. In items (iii) and (iv) we shall make use of the potential operator $\Psi_\lambda : L^2(\Sigma) \to L^2(\mathbb{R}^2)$ and the single layer boundary integral operator $S(\lambda) : L^2(\Sigma) \to L^2(\Sigma)$ defined in (2.2) and (2.4), respectively.

Theorem 1.1. *For any $\alpha \in \mathbb{R}$ the operator T_α is self-adjoint in $L^2(\mathbb{R}^2)$ and the essential spectrum is given by*

$$\sigma_{\text{ess}}(T_\alpha) = [0, \infty).$$

Furthermore, the following statements hold:

(i) *If $\alpha \geq 0$, then $\sigma_{\text{disc}}(T_\alpha) = \emptyset$ and T_α is a nonnegative operator in $L^2(\mathbb{R}^2)$.*

(ii) *If $\alpha < 0$, then $\sigma_{\mathrm{disc}}(T_\alpha)$ is infinite, unbounded from below, and does not accumulate to 0. Moreover, for every fixed $n \in \mathbb{N}$ the n-th discrete eigenvalue $\lambda_n \in \sigma_{\mathrm{disc}}(T_\alpha)$ (ordered non-increasingly) admits the asymptotic expansion*

$$\lambda_n = -\frac{4}{\alpha^2} + \mathcal{O}(1) \quad \text{for } \alpha \to 0^-,$$

where the dependence on n appears in the $\mathcal{O}(1)$-term.

(iii) *For $\lambda \in \mathbb{C}\setminus[0,\infty)$ the Birman-Schwinger principle is valid:*

$$\lambda \in \sigma_{\mathrm{p}}(T_\alpha) \quad \Longleftrightarrow \quad 1 \in \sigma_{\mathrm{p}}\big(\alpha\lambda S(\lambda)\big).$$

(iv) *For $\lambda \in \rho(T_\alpha) = \mathbb{C}\setminus([0,\infty) \cup \sigma_{\mathrm{p}}(T_\alpha))$ the operator $I - \alpha\lambda S(\lambda)$ is boundedly invertible in $L^2(\Sigma)$ and the resolvent formula*

$$(T_\alpha - \lambda)^{-1} = (-\Delta - \lambda)^{-1} + \alpha\Psi_\lambda\big(I - \alpha\lambda S(\lambda)\big)^{-1}\Psi_{\overline{\lambda}}^*$$

holds, where $-\Delta$ is the free Schrödinger operator defined on $H^2(\mathbb{R}^2)$.

To illustrate the significance of Theorem 1.1 we show that Schrödinger operators with oblique transmission conditions arise naturally as non-relativistic limits of Dirac operators with electrostatic and Lorentz scalar δ-interactions. To motivate this, consider one-dimensional Dirac operators with δ'-interactions of strength $\alpha \in \mathbb{R}$ supported in the point $\Sigma = \{0\}$. These are first order differential operators in $L^2(\mathbb{R})^2$ and the singular interaction is modeled by transmission conditions for functions in the operator domain, which for sufficiently smooth $f = (f_1, f_2) \in L^2(\mathbb{R})^2$ are given by

$$f_1(0+) - f_1(0-) = i\frac{\alpha c}{2}\big(f_2(0+) + f_2(0-)\big) \quad \text{and} \quad f_2(0+) = f_2(0-), \tag{1.7}$$

where $c > 0$ is the speed of light. It is known that the associated self-adjoint Dirac operators converge in the non-relativistic limit to a Schrödinger operator with a δ'-interaction of strength α; cf. [2,19] and also [10,11] for generalizations. It is not difficult to see that (1.7) can be rewritten as the transmission conditions associated with a Dirac operator with a combination of an electrostatic and a Lorentz scalar δ-interaction of strengths $\eta = -\frac{\alpha c^2}{2}$ and $\tau = \frac{\alpha c^2}{2}$, respectively, as they were studied in dimension one recently in [7] and in higher space dimensions in, e.g., [3,5–7].

To find a counterpart of the above result in dimension two, consider a Dirac operator with electrostatic and Lorentz scalar δ-shell interactions of strength η and τ, respectively, supported on Σ, which is formally given by

$$\mathcal{A}_{\eta,\tau} = A_0 + (\eta I_2 + \tau\sigma_3)\,\delta_\Sigma; \tag{1.8}$$

here A_0 is the unperturbed Dirac operator, I_2 is the 2×2-identity matrix and $\sigma_3 \in \mathbb{C}^{2\times 2}$ is given in (3.1). The differential expression $\mathcal{A}_{\eta,\tau}$ gives rise to a self-adjoint operator $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2)^2$, see (3.3). If one chooses, as above, $\eta = -\frac{\alpha c^2}{2}$ and $\tau = \frac{\alpha c^2}{2}$ and computes the non-relativistic limit, then instead of a Schrödinger operator with a δ'-interaction one gets the somewhat unexpected limit T_α. Of course, this is compatible with the one-dimensional result described above, as the one-dimensional counterparts of (1.3) and (1.5) coincide, since there are no tangential derivatives in $\mathbb{R}$. However, in higher dimensions Schrödinger operators with oblique transmission conditions should be viewed as the non-relativistic counterparts of Dirac operators with transmission conditions generalizing (1.7). Related results on non-relativistic limits of three-dimensional Dirac operators with singular interactions can be found in [4,5,21]. The precise result about the non-relativistic limit described above is stated in the following theorem and shown in Sect. 3.

Theorem 1.2. *Let $\alpha \in \mathbb{R}$. Then for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ one has*

$$\lim_{c\to\infty} \Big(A_{-\alpha c^2/2,\alpha c^2/2} - (\lambda + c^2/2)\Big)^{-1} = \begin{pmatrix} (T_\alpha - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where the convergence is in the operator norm and the convergence rate is $\mathcal{O}\left(\frac{1}{c}\right)$.

Notations Throughout this paper $\Omega_+ \subseteq \mathbb{R}^2$ is a bounded and simply connected C^∞-domain and $\Omega_- = \mathbb{R}^2\setminus\overline{\Omega_+}$ is the corresponding exterior domain with boundary $\Sigma = \partial\Omega_- = \partial\Omega_+$. The unit normal vector field on Σ pointing outwards of Ω_+ is denoted by ν. Moreover, for $z \in \mathbb{C}\setminus[0,\infty)$ we choose the square root $\sqrt{z}$ such that $\mathrm{Im}\sqrt{z} > 0$ holds. The modified Bessel function of order $j \in \mathbb{N}_0$ is denoted by K_j.

For $s \geq 0$ the spaces $H^s(\mathbb{R}^2)^n$, $H^s(\Omega_\pm)^n$, and $H^s(\Sigma)^n$ are the standard L^2-based Sobolev spaces of $\mathbb{C}^n$-valued functions defined on $\mathbb{R}^2$, $\Omega_\pm$, and Σ, respectively. If $n = 1$ we simply write $H^s(\mathbb{R}^2)$, $H^s(\Omega_\pm)$, and $H^s(\Sigma)$. For negative $s < 0$ we define the spaces $H^s(\mathbb{R}^2)^n$ and $H^s(\Sigma)^n$ as the anti-dual spaces of $H^{-s}(\mathbb{R}^2)^n$ and $H^{-s}(\Sigma)^n$, respectively. We denote the restrictions of functions $f : \mathbb{R}^2 \to \mathbb{C}^n$ onto $\Omega_\pm$ by $f_\pm$; in this sense we write $H^1(\mathbb{R}^2\setminus\Sigma)^n = H^1(\Omega_+)^n \oplus H^1(\Omega_-)^n$ and identify $f \in H^1(\mathbb{R}^2\setminus\Sigma)^n$ with $f_+ \oplus f_-$, where $f_\pm \in H^1(\Omega_\pm)^n$. In the following $\gamma_D^\pm : H^1(\Omega_\pm) \to L^2(\Sigma)$ denote the Dirichlet trace operators and we shall write $\gamma_D : H^1(\mathbb{R}^2) \to L^2(\Sigma)$ for the Dirichlet trace on $H^1(\mathbb{R}^2)$; sometimes these trace operators are also viewed as bounded mappings to $H^{1/2}(\Sigma)$.

For a Hilbert space $\mathcal{H}$ we write $\mathcal{L}(\mathcal{H})$ for the space of all everywhere defined, linear, and bounded operators on $\mathcal{H}$. Furthermore, the domain, kernel, and range of a linear operator T from a Hilbert space $\mathcal{G}$ to $\mathcal{H}$ are denoted by $\mathrm{dom}\, T$, $\ker T$, and $\mathrm{ran}\, T$, respectively. The resolvent set, the spectrum, the essential spectrum, the discrete spectrum, and the point spectrum of a self-adjoint operator T are denoted by $\rho(T)$, $\sigma(T)$, $\sigma_{\mathrm{ess}}(T)$, $\sigma_{\mathrm{disc}}(T)$, and $\sigma_{\mathrm{p}}(T)$. The eigenvalues of compact self-adjoint operators $K \in \mathcal{L}(\mathcal{H})$ are denoted by $\mu_n(K)$ and are ordered by their absolute values.

2. Proof of Theorem 1.1

In this section the main result of this paper will be proved. For this, some families of integral operators are used. Define for $\lambda \in \mathbb{C}\setminus[0,\infty)$ the function L_λ by

$$L_\lambda(x) = \frac{\sqrt{\lambda}}{2\pi} K_1\big(-i\sqrt{\lambda}|x|\big)\frac{x_1 - ix_2}{|x|}, \quad x = (x_1, x_2) \in \mathbb{R}^2\setminus\{0\}, \tag{2.1}$$

and the operator $\Psi_\lambda : L^2(\Sigma) \to L^2(\mathbb{R}^2)$ by

$$\Psi_\lambda\varphi(x) = \int_\Sigma L_\lambda(x - y)\varphi(y)d\sigma(y), \quad \varphi \in L^2(\Sigma),\ x \in \mathbb{R}^2\setminus\Sigma. \tag{2.2}$$

Moreover, for $\lambda \in \mathbb{C}\setminus[0,\infty)$ we make use of the single layer potential $SL(\lambda) : L^2(\Sigma) \to H^1(\mathbb{R}^2)$ and the single layer boundary integral operator $S(\lambda) : L^2(\Sigma) \to L^2(\Sigma)$ associated with $-\Delta - \lambda$ that are defined by

$$SL(\lambda)\varphi(x) = \int_\Sigma \frac{1}{2\pi} K_0\big(-i\sqrt{\lambda}|x-y|\big)\varphi(y)d\sigma(y), \quad \varphi \in L^2(\Sigma), \quad x \in \mathbb{R}^2\setminus\Sigma, \tag{2.3}$$

and

$$S(\lambda)\varphi(x) = \int_\Sigma \frac{1}{2\pi} K_0\big(-i\sqrt{\lambda}|x-y|\big)\varphi(y)d\sigma(y), \quad \varphi \in L^2(\Sigma),\ x \in \Sigma. \tag{2.4}$$

It is known that $SL(\lambda)$ and $S(\lambda)$ are bounded and $\operatorname{ran} S(\lambda) \subseteq H^1(\Sigma)$; cf. [25, Theorem 6.12 and Theorem 7.2]. In particular, $S(\lambda)$ gives rise to a compact operator in $H^s(\Sigma)$ for every $s \in [0,1]$. Furthermore, $S(\lambda)$ is self-adjoint and positive for $\lambda < 0$ (see *Step 1* in the proof of Proposition 2.2). Some properties of Ψ_λ and $S(\lambda)$ that are important in the proof of Theorem 1.1 are summarized in the following two propositions; cf. Appendix A for the proof of Propositions 2.1 and 2.2.

Proposition 2.1. *Let $\lambda \in \mathbb{C}\setminus[0,\infty)$ and let Ψ_λ be given by (2.2). Then*

$$\Psi_\lambda = -2i\partial_z SL(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^2) \tag{2.5}$$

is bounded and the following is true:

(i) *Ψ_λ gives rise to a bijective mapping $\Psi_\lambda : H^{1/2}(\Sigma) \to \mathcal{H}_\lambda$, where*

$$\mathcal{H}_\lambda := \big\{ f \in H^1(\mathbb{R}^2\setminus\Sigma) \mid \partial_{\bar z} f_+ \oplus \partial_{\bar z} f_- \in H^1(\mathbb{R}^2), (-\Delta - \lambda) f_\pm = 0 \text{ on } \Omega_\pm \big\}.$$

(ii) *$\Psi_\lambda^* : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is a compact operator, $\Psi_\lambda^* = -2i\gamma_D \partial_{\bar z}(-\Delta - \bar\lambda)^{-1}$, and* $\operatorname{ran} \Psi_\lambda^* \subseteq H^{1/2}(\Sigma)$.

(iii) *For all $\varphi \in H^{1/2}(\Sigma)$ the jump relations*

$$\begin{aligned} i(\nu_1 + i\nu_2)\big(\gamma_D^+(\Psi_\lambda\varphi)_+ - \gamma_D^-(\Psi_\lambda\varphi)_-\big) &= \varphi, \\ -i\big(\gamma_D^+\partial_{\bar z}(\Psi_\lambda\varphi)_+ + \gamma_D^-\partial_{\bar z}(\Psi_\lambda\varphi)_-\big) &= \lambda S(\lambda)\varphi, \end{aligned}$$

hold.

For $\lambda < 0$ denote by $\mu_n(S(\lambda))$ the discrete eigenvalues of the positive self-adjoint operator $S(\lambda)$ ordered non-increasingly and with multiplicities taken into account.

Proposition 2.2. *Let $S(\lambda)$ be defined by (2.4) and let $n \in \mathbb{N}$ be fixed. Then the following holds:*

(i) *The function $(-\infty, 0) \ni \lambda \mapsto \lambda\mu_n(S(\lambda))$ is continuous, strictly monotonically increasing and*

$$\lim_{\lambda\to 0^-} \lambda\mu_n(S(\lambda)) = 0 \quad \text{and} \quad \lim_{\lambda\to-\infty} \lambda\mu_n(S(\lambda)) = -\infty.$$

(ii) *For $a < 0$ the unique solution $\lambda_n(a) \in (-\infty, 0)$ of $\lambda\mu_n(S(\lambda)) = a$ (see* (i)*) admits the asymptotic expansion $\lambda_n(a) = -4a^2 + \mathcal{O}(1)$ for $a \to -\infty$, where the dependence on n appears in the $\mathcal{O}(1)$-term.*

Proof of Theorem 1.1. Step 1. We verify that T_α is symmetric in $L^2(\mathbb{R}^2)$. Observe first that for $f \in \operatorname{dom} T_\alpha$ we have $\partial_{\bar z} f_\pm \in H^1(\Omega_\pm)$ and $\Delta f_\pm = 4\partial_z \partial_{\bar z} f_\pm \in L^2(\Omega_\pm)$, and hence T_α is well-defined. Moreover, as $C_0^\infty(\mathbb{R}^2 \setminus \Sigma) \subseteq \operatorname{dom} T_\alpha$ it is also clear that $\operatorname{dom} T_\alpha$ is dense. In order to show that T_α is symmetric, we note that integration by parts in $\Omega_\pm$ yields for $f, g \in \operatorname{dom} T_\alpha$

$$\begin{aligned}(-\Delta f_\pm, g_\pm)_{L^2(\Omega_\pm)} &= (-4\partial_z \partial_{\bar z} f_\pm, g_\pm)_{L^2(\Omega_\pm)} \\ &= 4(\partial_{\bar z} f_\pm, \partial_{\bar z} g_\pm)_{L^2(\Omega_\pm)} \mp 2\big((\nu_1 - i\nu_2)\gamma_D^\pm(\partial_{\bar z} f_\pm), \gamma_D^\pm g_\pm\big)_{L^2(\Sigma)} \\ &= 4(\partial_{\bar z} f_\pm, \partial_{\bar z} g_\pm)_{L^2(\Omega_\pm)} \mp 2\big(\gamma_D^\pm(\partial_{\bar z} f_\pm), (\nu_1 + i\nu_2)\gamma_D^\pm g_\pm\big)_{L^2(\Sigma)}.\end{aligned} \tag{2.6}$$

Now, consider (2.6) for $f = g$ and add the equations for Ω_+ and Ω_-. Then, using $\gamma_D^+(\partial_{\bar z} f_+) = \gamma_D^-(\partial_{\bar z} f_-)$ and the transmission condition for $f \in \operatorname{dom} T_\alpha$, one finds that

$$\begin{aligned}\big(T_\alpha f, f\big)_{L^2(\mathbb{R}^2)} &= 4\big(\|\partial_{\bar z} f_+\|^2_{L^2(\Omega_+)} + \|\partial_{\bar z} f_-\|^2_{L^2(\Omega_-)}\big) \\ &\qquad - \big(\gamma_D^+(\partial_{\bar z} f_+) + \gamma_D^-(\partial_{\bar z} f_-), (\nu_1 + i\nu_2)(\gamma_D^+ f_+ - \gamma_D^- f_-)\big)_{L^2(\Sigma)} \\ &= 4\|\partial_{\bar z} f_+ \oplus \partial_{\bar z} f_-\|^2_{L^2(\mathbb{R}^2)} + \alpha \|\gamma_D^+(\partial_{\bar z} f_+) + \gamma_D^-(\partial_{\bar z} f_-)\|^2_{L^2(\Sigma)} \in \mathbb{R}.\end{aligned} \tag{2.7}$$

Since this holds for all $f \in \operatorname{dom} T_\alpha$, we conclude that T_α is symmetric.

Step 2. Proof of the Birman-Schwinger principle in (iii): To show the first implication, assume that $\lambda \in \mathbb{C}\setminus[0,\infty)$ with $1 \in \sigma_p(\alpha\lambda S(\lambda))$ and choose $\varphi \in \ker\,(I - \alpha\lambda S(\lambda))\setminus\{0\}$. Then it follows from the mapping properties of $S(\lambda)$ that $\varphi = \alpha\lambda S(\lambda)\varphi \in H^{1/2}(\Sigma)$ holds. Therefore, Proposition 2.1 (i) implies that $f := \Psi_\lambda \varphi \in \mathcal{H}_\lambda$ fulfils $f \neq 0$, $f \in H^1(\mathbb{R}^2 \setminus \Sigma)$, $\partial_{\bar z} f_+ \oplus \partial_{\bar z} f_- \in H^1(\mathbb{R}^2)$ and, as $\varphi \in \ker\,(1 - \alpha\lambda S(\lambda))\setminus\{0\}$, Proposition 2.1 (iii) implies

$$i(\nu_1 + i\nu_2)\big(\gamma_D^+ f_+ - \gamma_D^- f_-\big) = \varphi = \alpha\lambda S(\lambda)\varphi = -i\alpha\big(\gamma_D^+(\partial_{\bar z} f_+) + \gamma_D^-(\partial_{\bar z} f_-)\big).$$

Hence, $f \in \operatorname{dom} T_\alpha$. Moreover, as $f \in \mathcal{H}_\lambda$, we conclude $f \in \ker\,(T_\alpha - \lambda)\setminus\{0\}$ and hence $\lambda \in \sigma_p(T_\alpha)$.

To show the second implication, we assume that $\lambda \in \sigma_p(T_\alpha)$ is given and we choose $f \in \ker\,(T_\alpha - \lambda)\setminus\{0\}$. Then, by Proposition 2.1 (i) there exists a unique $\varphi \in H^{1/2}(\Sigma)$ such that $f = \Psi_\lambda \varphi$. Moreover, using $f \in \operatorname{dom} T_\alpha$ and Proposition 2.1 (iii) one finds that

$$0 = i(\nu_1 + i\nu_2)\big(\gamma_D^+ f_+ - \gamma_D^- f_-\big) + i\alpha\big(\gamma_D^+(\partial_{\bar z} f_+) + \gamma_D^-(\partial_{\bar z} f_-)\big) = (I - \alpha\lambda S(\lambda))\varphi.$$

Since $\varphi \neq 0$, we conclude $1 \in \sigma_p(\alpha\lambda S(\lambda))$.

Step 3. Next, we prove that T_α is a self-adjoint operator and the resolvent formula in (iv). Let $\lambda \in \mathbb{C}\setminus([0,\infty) \cup \sigma_p(T_\alpha))$ be fixed. First, we show that $I - \alpha\lambda S(\lambda)$ gives rise to a bijective map in $H^s(\Sigma)$ for every $s \in [0,1]$. Recall that $S(\lambda)$ is compact in $H^s(\Sigma)$. Since $I - \alpha\lambda S(\lambda)$ is injective for our choice of λ by the Birman-Schwinger principle in (iii), Fredholm's alternative shows that $I - \alpha\lambda S(\lambda)$ is indeed bijective.

Recall that T_α is symmetric; cf. *Step 1*. Hence, to show that T_α is self-adjoint, it suffices to verify that $\operatorname{ran}(T_\alpha - \lambda) = L^2(\mathbb{R}^2)$ holds for $\lambda \in \mathbb{C}\setminus([0,\infty) \cup \sigma_p(T_\alpha))$. Fix such a λ, let $f \in L^2(\mathbb{R}^2)$, and define

$$g = (-\Delta - \lambda)^{-1} f + \alpha\Psi_\lambda(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\bar\lambda}^* f, \tag{2.8}$$

which is well-defined by the considerations above. Since $\Psi_{\overline{\lambda}}^* f \in H^{1/2}(\Sigma)$ by Proposition 2.1 (ii) and $(I - \alpha\lambda S(\lambda))^{-1}$ is bijective in $H^{1/2}(\Sigma)$, we conclude with Proposition 2.1 (i) that $\Psi_\lambda(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\overline{\lambda}}^* f \in \mathcal{H}_\lambda \subseteq H^1(\mathbb{R}^2\backslash\Sigma)$. In particular, with $(-\Delta - \lambda)^{-1} f \in H^2(\mathbb{R}^2)$ this implies that $g \in H^1(\mathbb{R}^2\backslash\Sigma)$ and $\partial_{\overline{z}} g_+ \oplus \partial_{\overline{z}} g_- \in H^1(\mathbb{R}^2)$. Moreover, with Proposition 2.1(ii)–(iii) we obtain that

$$\begin{aligned} i(\nu_1 + i\nu_2)&\big(\gamma_D^+ g_+ - \gamma_D^- g_-\big) + i\alpha\big(\gamma_D^+(\partial_{\overline{z}} g_+) + \gamma_D^-(\partial_{\overline{z}} g_-)\big) \\ &= \alpha(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\overline{\lambda}}^* f - \alpha\Psi_{\overline{\lambda}}^* f - \alpha^2\lambda S(\lambda)(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\overline{\lambda}}^* f \\ &= \alpha(I - \alpha\lambda S(\lambda))(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\overline{\lambda}}^* f - \alpha\Psi_{\overline{\lambda}}^* f = 0 \end{aligned}$$

and hence, $g \in \operatorname{dom} T_\alpha$. As $\Psi_\lambda(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\overline{\lambda}}^* f \in \mathcal{H}_\lambda$ by Proposition 2.1 (i), we conclude

$$\begin{aligned} (-\Delta - \lambda)\, g_\pm &= (-\Delta - \lambda)\left((-\Delta - \lambda)^{-1} f\right)_\pm + \alpha\,(-\Delta - \lambda)\left(\Psi_\lambda(I - \alpha\lambda S(\lambda))^{-1}\Psi_{\overline{\lambda}}^* f\right)_\pm \\ &= (-\Delta - \lambda)\left((-\Delta - \lambda)^{-1} f\right)_\pm = f_\pm, \end{aligned}$$

i.e. $(T_\alpha - \lambda)g = f$. Since $f \in L^2(\mathbb{R}^2)$ was arbitrary, we conclude that $\operatorname{ran}(T_\alpha - \lambda) = L^2(\mathbb{R}^2)$ and that T_α is self-adjoint. Moreover, the resolvent formula in item (iv) follows from (2.8).

Step 4. Next, we show $\sigma_{\text{ess}}(T_\alpha) = [0, \infty)$. For this fix some $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Since $\Psi_{\overline{\lambda}}^* : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is compact by Proposition 2.1 (ii), the resolvent formula in (iv) implies that $(T_\alpha - \lambda)^{-1} - (-\Delta - \lambda)^{-1}$ is a compact operator in $L^2(\mathbb{R}^2)$. Consequently, Weyl's Theorem [27, Theorem XIII.14] yields that $\sigma_{\text{ess}}(T_\alpha) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$.

Step 5. Proof of (i): Let $\alpha \geq 0$. Then, (2.7) implies that T_α is non-negative and hence, $\sigma(T_\alpha) \subset [0, \infty)$. Since the latter set coincides with $\sigma_{\text{ess}}(T_\alpha)$, see *Step 4*, we conclude $\sigma_{\text{disc}}(T_\alpha) = \emptyset$.

Step 6. Proof of (ii): Let $\alpha < 0$. Since $\sigma_{\text{ess}}(T_\alpha) = [0, \infty)$, it follows from the Birman-Schwinger principle in (iii) that

$$\sigma_{\text{disc}}(T_\alpha) = \{\lambda_n \mid n \in \mathbb{N}\} = \left\{\lambda < 0 \mid \exists n \in \mathbb{N} \text{ such that } \lambda\mu_n(S(\lambda)) = \alpha^{-1}\right\}$$

holds. Note that by Proposition 2.2 the equation $\lambda\mu_n(S(\lambda)) = \alpha^{-1}$ has a unique solution λ_n for all $n \in \mathbb{N}$. Moreover, for any $n \in \mathbb{N}$ there cannot be infinitely many $k \neq n$ with $\lambda_n = \lambda_k$, since otherwise $\alpha^{-1} < 0$ would be an eigenvalue with infinite multiplicity of the self-adjoint and compact operator $\lambda_n S(\lambda_n)$. Thus $\sigma_{\text{disc}}(T_\alpha)$ is indeed an infinite set. Furthermore, as $S(\lambda)$ is a positive self-adjoint operator in $L^2(\Sigma)$; cf. *Step 1* in the proof of Proposition 2.2, we have by definition $\mu_n(S(\lambda)) \geq \mu_{n+1}(S(\lambda))$ implying $\lambda\mu_n(S(\lambda)) \leq \lambda\mu_{n+1}(S(\lambda))$. Therefore, the monotonicity of the map $\lambda \mapsto \lambda\mu_n(S(\lambda))$ from Proposition 2.2 yields $\lambda_{n+1} \leq \lambda_n$ for all $n \in \mathbb{N}$. This shows that 0 cannot be an accumulation point of the sequence $(\lambda_n)_{n\in\mathbb{N}}$ and as $\sigma_{\text{ess}}(T_\alpha) \cap (-\infty, 0) = \emptyset$ the sequence $(\lambda_n)_{n\in\mathbb{N}}$ has no finite accumulation points, that is, $\sigma_{\text{disc}}(T_\alpha)$ must be unbounded from below.

It remains to prove the asymptotic expansion in item (ii). By the above considerations λ_n is determined as the unique solution of $\lambda\mu_n(S(\lambda)) = \alpha^{-1}$. Clearly, if $\alpha \to 0^-$, then $a := \alpha^{-1} \to -\infty$. Hence, it follows from Proposition 2.2 (ii) with $a = \alpha^{-1}$ that $\lambda_n = -\frac{4}{\alpha^2} + \mathcal{O}(1)$ for $\alpha \to 0^-$ and that the dependence on n appears in the $\mathcal{O}(1)$-term. □

3. Proof of Theorem 1.2

In this section we show that T_α is the non-relativistic limit of a family of Dirac operators with electrostatic and Lorentz scalar δ-shell potentials formally given by (1.8), whose interaction strengths are suitably scaled. First, we recall the rigorous definition of the operator $A_{\eta,\tau}$ associated with (1.8), see [5–7] for details. Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

be the Pauli spin matrices and denote the 2×2 identity matrix by I_2. Furthermore, for $x = (x_1, x_2) \in \mathbb{R}^2$ we will use the abbreviations

$$\sigma \cdot x = \sigma_1 x_1 + \sigma_2 x_2 \quad \text{and} \quad \sigma \cdot \nabla = \sigma_1 \partial_1 + \sigma_2 \partial_2. \tag{3.2}$$

We define Dirac operators with electrostatic and Lorentz scalar δ-shell interactions of strengths $\eta, \tau \in \mathbb{R}$ in $L^2(\mathbb{R}^2)^2$ by

$$\begin{aligned} A_{\eta,\tau} f &= \left(-ic(\sigma\cdot\nabla) + \frac{c^2}{2}\sigma_3\right) f_+ \oplus \left(-ic(\sigma\cdot\nabla) + \frac{c^2}{2}\sigma_3\right) f_-, \\ \operatorname{dom} A_{\eta,\tau} &= \Big\{ f \in H^1(\Omega_+)^2 \oplus H^1(\Omega_-)^2 \;\Big| \\ &\qquad ic(\sigma\cdot\nu)\left(\gamma_D^+ f_+ - \gamma_D^- f_-\right) + \frac{1}{2}(\eta I_2 + \tau\sigma_3)\left(\gamma_D^+ f_+ + \gamma_D^- f_-\right) = 0 \Big\}. \end{aligned} \tag{3.3}$$

It is shown in [6,7] that $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2)^2$, whenever $\eta^2 - \tau^2 \neq 4c^2$, and as in [5] one sees that these operators are the self-adjoint realisations of the formal differential expression (1.8). In the above definition we are using units such that $\hbar = 1$ and consider the mass $m = \frac{1}{2}$, but we keep the speed of light c as a parameter for the discussion of the non-relativistic limit $c \to \infty$.

Throughout this section we make use of the self-adjoint free Dirac operator A_0, which coincides with the operator$A_{0,0}$ given in (3.3) and which is defined on $H^1(\mathbb{R}^2)^2$. For $\lambda \in \rho(A_0) = \mathbb{C}\setminus\big((-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2}, \infty)\big)$ the integral kernel of the resolvent of A_0 is given by $G_\lambda(x-y)$, where $G_\lambda(x)$ is defined for $x \in \mathbb{R}^2\setminus\{0\}$ by

$$\begin{aligned} G_\lambda(x) = \frac{1}{2\pi c}\sqrt{\frac{\lambda^2}{c^2} - \frac{c^2}{4}} K_1\left(-i\sqrt{\frac{\lambda^2}{c^2} - \frac{c^2}{4}}|x|\right)\frac{1}{|x|}(\sigma\cdot x) \\ + \frac{1}{2\pi c} K_0\left(-i\sqrt{\frac{\lambda^2}{c^2} - \frac{c^2}{4}}|x|\right)\left(\frac{\lambda}{c} I_2 + \frac{c}{2}\sigma_3\right); \end{aligned} \tag{3.4}$$

cf. [6, equation (3.2)]. With this function we define the two families of integral operators

$$\begin{aligned} \Phi_\lambda \varphi(x) &= \int_\Sigma G_\lambda(x-y)\varphi(y)\mathrm{d}\sigma(y), \quad \varphi \in L^2(\Sigma)^2,\ x \in \mathbb{R}^2\setminus\Sigma, \\ \mathcal{C}_\lambda \varphi(x) &= \lim_{\varepsilon\to 0^+} \int_{\Sigma\setminus B(x,\varepsilon)} G_\lambda(x-y)\varphi(y)\mathrm{d}\sigma(y), \quad \varphi \in L^2(\Sigma)^2,\ x \in \Sigma, \end{aligned} \tag{3.5}$$

where $B(x,\varepsilon)$ is the ball of radius ε centered at x. Both operators $\Phi_\lambda : L^2(\Sigma)^2 \to L^2(\mathbb{R}^2)^2$ and $\mathcal{C}_\lambda : L^2(\Sigma)^2 \to L^2(\Sigma)^2$ are well-defined and bounded; cf. [6, Proposition 3.3 and equation (3.7)].

In the following lemma, which is a preparation for the proof of Theorem 1.2, we will use the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

products of scalar operators and matrices are understood componentwise, e.g.

$$(-\Delta - \lambda)^{-1} M_1 = \begin{pmatrix} (-\Delta - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix} : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2.$$

Lemma 3.1. *Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then there exists a constant $K > 0$, depending only on λ and Σ, such that the estimates*

$$\left\| \left(A_0 - (\lambda + c^2/2)\right)^{-1} - (-\Delta - \lambda)^{-1} M_1 \right\| \leq \frac{K}{c}, \tag{3.6a}$$

$$\left\| c\Phi_{\lambda + c^2/2} M_3 - \Psi_\lambda M_2 \right\| \leq \frac{K}{c}, \tag{3.6b}$$

$$\left\| c M_3 \Phi^*_{\lambda + c^2/2} - M_2^\top \Psi^*_\lambda \right\| \leq \frac{K}{c}, \tag{3.6c}$$

$$\left\| c^2 M_3 \mathcal{C}_{\lambda + c^2/2} M_3 - \lambda S(\lambda) M_3 \right\| \leq \frac{K}{c}, \tag{3.6d}$$

are valid for all sufficiently large $c > 0$.

Proof. We use a similar strategy as in the proof of [4, Proposition 5.2]. In the following let $\lambda \in \mathbb{C}\backslash\mathbb{R}$ be fixed. Then $\lambda + \frac{c^2}{2} \in \mathbb{C}\backslash\mathbb{R}$ and hence all operators in (3.6a)–(3.6d) are well-defined. One verifies by direct calculation that for sufficiently large $c > 0$ and all $t \in [0,1]$

$$0 < \frac{1}{2}\left|\sqrt{\lambda}\right| \leq \left|\sqrt{\lambda + t\frac{\lambda^2}{c^2}}\right| \leq \frac{3}{2}\left|\sqrt{\lambda}\right| \quad \text{and} \quad \frac{1}{2}\mathrm{Im}\sqrt{\lambda} \leq \mathrm{Im}\sqrt{\lambda + t\frac{\lambda^2}{c^2}} \tag{3.7}$$

hold. With the well-known asymptotic expansions of the modified Bessel functions and $K_1'(z) = -K_0(z) - \frac{1}{z}K_1(z)$, (see [1]) one shows that there exist constants $\widehat{K}, \kappa, R > 0$, depending only on λ, such that

$$\left| K_j\left(-i\sqrt{\lambda + t\frac{\lambda^2}{c^2}}|x|\right)\right| \leq \widehat{K}\begin{cases} |x|^{-1}, & \text{for } |x| < R, \\ e^{-\kappa|x|}, & \text{for } |x| \geq R, \end{cases} \tag{3.8}$$

and

$$\left| K_1'\left(-i\sqrt{\lambda + t\frac{\lambda^2}{c^2}}|x|\right)\right| \leq \widehat{K}\begin{cases} |x|^{-2}, & \text{for } |x| < R, \\ e^{-\kappa|x|}, & \text{for } |x| \geq R, \end{cases} \tag{3.9}$$

hold for all $x \in \mathbb{R}^2\backslash\{0\}$, $j \in \{0,1\}$, $t \in [0,1]$, and sufficiently large $c > 0$.

Next, with $G_{\lambda+c^2/2}$ defined by (3.4) we find

$$G_{\lambda+c^2/2}(x) = \frac{1}{2\pi c}\sqrt{\lambda + \frac{\lambda^2}{c^2}} K_1 \left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x| \right) \frac{1}{|x|}(\sigma \cdot x) + \frac{1}{2\pi c} K_0 \left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x| \right) \left(\frac{\lambda}{c} I_2 + c M_1 \right). \tag{3.10}$$

Let

$$U_\lambda(x) = \frac{1}{2\pi} K_0\big(-i\sqrt{\lambda}|x|\big), \quad x \in \mathbb{R}^2 \setminus \{0\},$$

be the integral kernel of the resolvent of the free Laplace operator; cf. [28, Chapter 7.5]. Then

$$G_{\lambda+c^2/2}(x) - U_\lambda(x) M_1 = t_1(x) + t_2(x) + t_3(x)$$

holds, where the matrix-valued functions t_1, t_2, and t_3 are given by

$$t_1(x) = \frac{1}{2\pi c}\sqrt{\lambda + \frac{\lambda^2}{c^2}} K_1 \left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x| \right) \frac{\sigma \cdot x}{|x|},$$

$$t_2(x) = \frac{1}{2\pi} \left(K_0 \left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x| \right) - K_0\big(-i\sqrt{\lambda}|x|\big) \right) M_1,$$

$$t_3(x) = \frac{\lambda}{2\pi c^2} K_0 \left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x| \right) I_2.$$

With (3.7) and (3.8) applied with $t = 1$ one finds that there exist constants $k_1, \kappa, R > 0$, depending only on λ, such that for $j \in \{1, 3\}$ and sufficiently large $c > 0$ one has

$$\big|t_j(x)\big| \leq \frac{k_1}{c} \begin{cases} |x|^{-1}, & \text{for } |x| < R, \\ e^{-\kappa|x|}, & \text{for } |x| \geq R. \end{cases}$$

To estimate t_2, we use $K_0' = -K_1$ and obtain with the fundamental theorem of calculus, (3.7), and (3.8)

$$\begin{aligned} &\left| K_0 \left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x| \right) - K_0 \left(-i\sqrt{\lambda}|x| \right) \right| \leq \int_0^1 \left| \frac{d}{dt} K_0 \left(-i\sqrt{\lambda + t\frac{\lambda^2}{c^2}}|x| \right) \right| dt \\ &= \int_0^1 \frac{|\lambda|^2 |x|}{\left|\sqrt{\lambda + t\frac{\lambda^2}{c^2}}\right|} \frac{1}{2c^2} \left| K_1 \left(-i\sqrt{\lambda + t\frac{\lambda^2}{c^2}}|x| \right) \right| dt \\ &\leq \frac{k_2}{c^2} \begin{cases} 1, & \text{for } |x| < R, \\ e^{-\frac{\kappa}{2}|x|}, & \text{for } |x| \geq R, \end{cases} \end{aligned} \tag{3.11}$$

with a constant k_2 which depends only on λ. Thus, if we define $k_3 = 2k_1 + \frac{k_2 R}{2\pi}$, then

$$\left|G_{\lambda+c^2/2}(x) - U_\lambda(x)M_1\right| \le \frac{k_3}{c}\begin{cases} |x|^{-1}, & \text{for } |x| < R, \\ e^{-\frac{\kappa}{2}|x|}, & \text{for } |x| \ge R. \end{cases}$$

This estimation for the integral kernel yields with the Schur test; cf. [4, Proposition A.3] for a similar argument,

$$\left\|\left(A_0 - (\lambda + c^2/2)\right)^{-1} - (-\Delta - \lambda)^{-1} M_1\right\| \le \frac{K}{c}$$

for all sufficiently large $c > 0$, which is the first claimed estimate (3.6a).

Next, we prove (3.6b). Recall that the integral kernel L_λ of Ψ_λ is given by (2.1). Using that $\sigma_1 M_3 = M_2$, $\sigma_2 M_3 = -iM_2$, and $M_1 M_3 = 0$, we obtain with (3.10) the decomposition

$$cG_{\lambda+c^2/2}(x)M_3 - L_\lambda(x)M_2 = \tau_1(x) + \tau_2(x) + \tau_3(x)$$

with

$$\tau_1(x) = \frac{1}{2\pi}\left(\sqrt{\lambda + \frac{\lambda^2}{c^2}} - \sqrt{\lambda}\right) K_1\left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x|\right)\frac{x_1 - ix_2}{|x|}M_2,$$

$$\tau_2(x) = \frac{\sqrt{\lambda}}{2\pi}\left(K_1\left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x|\right) - K_1\left(-i\sqrt{\lambda}|x|\right)\right)\frac{x_1 - ix_2}{|x|}M_2,$$

$$\tau_3(x) = \frac{\lambda}{2\pi c}K_0\left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x|\right)M_3.$$

Similar as above it can be shown that there exists a $k_4 > 0$, depending only on λ, such that for all $j \in \{1, 2, 3\}$

$$\left|\tau_j(x)\right| \le \frac{k_4}{c}\begin{cases} |x|^{-1}, & \text{for } |x| < R, \\ e^{-\frac{\kappa}{2}|x|}, & \text{for } |x| \ge R; \end{cases}$$

to see the estimate for τ_2 one has to use (3.9). With the help of the Schur test the estimate (3.6b) follows (see also [4, Proposition A.4] for a similar argument); the constant k_4 depends in this case on λ and Σ. The estimate in (3.6c) follows by taking adjoints.

It remains to prove (3.6d). Taking $M_3(\sigma \cdot x)M_3 = 0$, which holds for any $x \in \mathbb{R}^2$, and (3.11) into account we obtain that

$$\begin{aligned}&\left|c^2 M_3 G_{\lambda+c^2/2}(x)M_3 - \lambda U_\lambda(x)M_3\right| \\ &\quad = \frac{|\lambda|}{2\pi}\left|K_0\left(-i\sqrt{\lambda + \frac{\lambda^2}{c^2}}|x|\right) - K_0\left(-i\sqrt{\lambda}|x|\right)\right| \le \frac{k_5}{c^2}\end{aligned}$$

holds for all $x \in \mathbb{R}^2 \setminus \{0\}$. Using the dominated convergence theorem, one sees that

$$\left(c^2 M_3 \mathcal{C}_{\lambda+c^2/2} M_3 f\right)(x) = \int_\Sigma c^2 M_3 G_{\lambda+c^2/2}(x-y)M_3 f(y)d\sigma(y)$$

holds for all $f \in L^2(\Sigma)^2$ and $x \in \Sigma$, i.e. the integral does not have to be understood as principal value. Thus we obtain with the Schur test [23, III. Example 2.4] that

$$\left\| c^2 M_3 \mathcal{C}_{\lambda+mc^2} M_3 - \lambda S(\lambda) M_3 \right\| \leq \frac{K}{c^2}.$$

In this case, the constant K depends on λ and Σ. This yields (3.6d) and finishes the proof of this lemma. □

Now we are prepared to prove Theorem 1.2 and show that $A_{-\alpha c^2/2,\alpha c^2/2}$ converges in the non-relativistic limit to T_α defined in (1.6).

Proof of Theorem 1.2. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ be fixed. Then it follows from [7, Lemma 5.4, Proposition 5.5, Theorem 5.6, and Lemma 5.9] (see also [6, Theorem 4.6]) that the operator $I_2 - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2} : L^2(\Sigma)^2 \to L^2(\Sigma)^2$ is boundedly invertible and the resolvent of $A_{-\alpha c^2/2,\alpha c^2/2} - c^2/2$ is given by

$$\begin{aligned}\left(A_{-\alpha c^2/2,\alpha c^2/2} - (\lambda + c^2/2)\right)^{-1} &= \left(A_0 - (\lambda + c^2/2)\right)^{-1} \\ &\quad + \Phi_{\lambda+c^2/2} \left(I - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2}\right)^{-1} \alpha c^2 M_3 \Phi^*_{\overline{\lambda}+c^2/2}.\end{aligned} \tag{3.12}$$

Because of $M_3 = M_3^2$ it follows from [26, Proposition 2.1.8] that

$$\sigma\left(M_3 \mathcal{C}_{\lambda+c^2/2}\right) \cup \{0\} = \sigma\left(M_3 \mathcal{C}_{\lambda+c^2/2} M_3\right) \cup \{0\}.$$

In particular, this yields that the operator $I - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2} M_3$ is boundedly invertible in $L^2(\Sigma)^2$ for all $c > 0$ and a direct calculation shows

$$(I - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2})^{-1} M_3 = M_3 (I - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2} M_3)^{-1}. \tag{3.13}$$

Recall that for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ also $I - \alpha\lambda S(\lambda)$ is boundedly invertible in $L^2(\Sigma)$; cf. Theorem 1.1 (iv). Hence, we obtain from Lemma 3.1 and [23, IV. Theorem 1.16] that

$$\left\| (I - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2} M_3)^{-1} - (I - \alpha\lambda S(\lambda) M_3)^{-1} \right\| \leq \frac{K}{c} \tag{3.14}$$

holds for all sufficiently large $c > 0$ with a constant $K > 0$ which depends only on λ, α, and Σ.

To conclude, note that (3.12) and (3.13) yield

$$\begin{aligned}\left(A_{-\alpha c^2/2,\alpha c^2/2} - (\lambda + c^2/2)\right)^{-1} &= \left(A_0 - (\lambda + c^2/2)\right)^{-1} \\ &\quad + c\Phi_{\lambda+c^2/2} M_3 (I - \alpha c^2 M_3 \mathcal{C}_{\lambda+c^2/2} M_3)^{-1} \alpha c M_3 \Phi^*_{\overline{\lambda}+c^2/2},\end{aligned}$$

while Theorem 1.1 (iv) and $M_2 M_3 M_2^\top = M_1$ show

$$\begin{aligned}(T_\alpha - \lambda)^{-1} M_1 &= (-\Delta - \lambda)^{-1} M_1 + \Psi_\lambda (I - \alpha\lambda S(\lambda))^{-1} \alpha \Psi^*_{\overline{\lambda}} M_1 \\ &= (-\Delta - \lambda)^{-1} M_1 + \Psi_\lambda M_2 (I - \alpha\lambda S(\lambda) M_3)^{-1} \alpha M_2^\top \Psi^*_{\overline{\lambda}}.\end{aligned}$$

Using Lemma 3.1 and (3.14) the last two displayed formulae finally lead to the claimed convergence result and it also follows that the order of convergence is $\mathcal{O}(\frac{1}{c})$. □

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A. Proof of Propositions 2.1 and 2.2

Recall that for $\lambda \in \mathbb{C}\setminus[0,\infty)$ the operators Ψ_λ, $SL(\lambda)$, and $S(\lambda)$ are defined by (2.2), (2.3), and (2.4), respectively. First, we collect some properties of the single layer potential $SL(\lambda)$ that are needed in the following. It is well-known that $SL(\lambda) : H^{1/2}(\Sigma) \to H^2(\mathbb{R}^2\setminus\Sigma)$ gives rise to a bounded operator, that $(-\Delta-\lambda)SL(\lambda)\varphi = 0$ in $\mathbb{R}^2\setminus\Sigma$, and that for $\varphi \in H^{1/2}(\Sigma)$ the jump relations

$$\gamma_D^+(SL(\lambda)\varphi)_+ = \gamma_D^-(SL(\lambda)\varphi)_- \quad \text{and} \quad \partial_\nu(SL(\lambda)\varphi)_+ - \partial_\nu(SL(\lambda)\varphi)_- = \varphi \tag{A.1}$$

hold; cf. [25] or [22, Section 3.3]. Furthermore, for the single layer boundary integral operator $S(\lambda)$ from (2.4) we have $S(\lambda) = \gamma_D SL(\lambda)$ and for all $\varphi \in L^2(\Sigma)$ the representations

$$SL(\lambda)\varphi = (-\Delta-\lambda)^{-1}\gamma_D'\varphi \quad \text{and} \quad S(\lambda)\varphi = \gamma_D(-\Delta-\lambda)^{-1}\gamma_D'\varphi \tag{A.2}$$

hold (see [22,25]); here $\gamma_D : H^1(\mathbb{R}^2) \to L^2(\Sigma)$ and $\gamma_D' : L^2(\Sigma) \to H^{-1}(\mathbb{R}^2)$ is the anti-dual operator.

Proof of Proposition 2.1. First, we prove item (ii). For $\lambda \in \mathbb{C}\setminus[0,\infty)$ define the operator

$$\widehat{\Psi}_\lambda := -2i\gamma_D\partial_{\bar z}(-\Delta-\bar\lambda)^{-1}. \tag{A.3}$$

Since $(-\Delta-\bar\lambda)^{-1} : L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ and $\gamma_D : H^1(\mathbb{R}^2) \to H^{1/2}(\Sigma)$ are bounded, we get that $\widehat{\Psi}_\lambda : L^2(\mathbb{R}^2) \to H^{1/2}(\Sigma)$ is well-defined and bounded. Furthermore, as $H^{1/2}(\Sigma)$ is compactly embedded in $L^2(\Sigma)$ by Rellich's embedding theorem, the operator $\widehat{\Psi}_\lambda : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is compact. Note that $\widehat{\Psi}_\lambda$ is an integral operator with integral kernel

$$\begin{aligned} k(x,y) &= -2i\partial_{\bar z}\frac{1}{2\pi}K_0\Big(-i\sqrt{\bar\lambda}|x-y|\Big) \\ &= \frac{\sqrt{\bar\lambda}}{2\pi}K_1\Big(-i\sqrt{\bar\lambda}|x-y|\Big)\frac{x_1-y_1+i(x_2-y_2)}{|x-y|} \\ &= \overline{L_\lambda(y-x)}, \end{aligned}$$

where we used $K_0' = -K_1$ in the second step and $\overline{\sqrt{\lambda}} = -\sqrt{\overline{\lambda}}$ in the last step (recall that $\mathrm{Im}\sqrt{\omega} > 0$ for $\omega \in \mathbb{C}\setminus[0,\infty)$). Hence, we conclude that

$$\Psi_\lambda = \widehat{\Psi}_{\overline{\lambda}}^* : L^2(\Sigma) \to L^2(\mathbb{R}^2)$$

is bounded and that all claims in item (ii) are true.

Next, we show (2.5). Let $\varphi \in L^2(\Sigma)$ and $f \in H^1(\mathbb{R}^2)$. Since $\Delta = 4\partial_{\overline{z}}\partial_z = 4\partial_z\partial_{\overline{z}}$, we see that $\partial_{\overline{z}}(-\Delta - \overline{\lambda})^{-1} f = (-\Delta - \overline{\lambda})^{-1}\partial_{\overline{z}} f$. Hence, item (ii) and (A.2) imply

$$\begin{aligned} (\Psi_\lambda\varphi, f)_{L^2(\Sigma)} &= \big(\varphi, -2i\gamma_D\partial_{\overline{z}}(-\Delta - \overline{\lambda})^{-1} f\big)_{L^2(\mathbb{R}^2)} \\ &= \big(\varphi, -2i\gamma_D(-\Delta - \overline{\lambda})^{-1}\partial_{\overline{z}} f\big)_{L^2(\mathbb{R}^2)} \\ &= \big(-2i\partial_z SL(\lambda)\varphi, f\big)_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Since $H^1(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, we conclude that (2.5) is true. In particular, this and the properties of the single layer potential mentioned at the beginning of this appendix imply that

$$\Psi_\lambda : H^{1/2}(\Sigma) \to H^1(\mathbb{R}^2\setminus\Sigma) \tag{A.4}$$

is bounded and for $\varphi \in H^{1/2}(\Sigma)$ we have

$$i\partial_{\overline{z}}\,(\Psi_\lambda\varphi)_\pm = 2\partial_{\overline{z}}\partial_z\,(SL(\lambda)\varphi)_\pm = \frac{1}{2}\Delta\,(SL(\lambda)\varphi)_\pm = -\frac{\lambda}{2}\,(SL(\lambda)\varphi)_\pm\,. \tag{A.5}$$

Since $SL(\lambda)\varphi \in H^1(\mathbb{R}^2)$ it follows that $\partial_{\overline{z}}\,(\Psi_\lambda\varphi)_+ \oplus \partial_{\overline{z}}\,(\Psi_\lambda\varphi)_- \in H^1(\mathbb{R}^2)$ holds for any $\varphi \in H^{1/2}(\Sigma)$.

Now, we show (iii). Let $\varphi \in H^{1/2}(\Sigma)$. With (A.5) we see that

$$-i\left(\gamma_D^+\partial_{\overline{z}}(\Psi_\lambda\varphi)_+ + \gamma_D^-\partial_{\overline{z}}(\Psi_\lambda\varphi)_-\right) = \lambda\gamma_D SL(\lambda)\varphi = \lambda S(\lambda)\varphi$$

holds. Moreover, we obtain with $SL(\lambda)\varphi \in H^2(\mathbb{R}^2\setminus\Sigma)$

$$i(\nu_1 + i\nu_2)\gamma_D^\pm\,(-2i\partial_z SL(\lambda)\varphi)_\pm = \partial_\nu\,(SL(\lambda)\varphi)_\pm - i\partial_t\,(SL(\lambda)\varphi)_\pm\,,$$

where ∂_t is the tangential derivative on Σ. As $SL(\lambda)\varphi \in H^1(\mathbb{R}^2)$, one has the relation $\partial_t\,(SL(\lambda)\varphi)_+ = \partial_t\,(SL(\lambda)\varphi)_-$ and consequently with (A.1)

$$i(\nu_1 + i\nu_2)\left(\gamma_D^+(\Psi_\lambda\varphi)_+ - \gamma_D^-(\Psi_\lambda\varphi)_-\right) = \partial_\nu\,(SL(\lambda)\varphi)_+ - \partial_\nu\,(SL(\lambda)\varphi)_- = \varphi.$$

This finishes the proof of (iii).

It remains to prove item (i). By applying the Wirtinger derivative ∂_z to (A.5) one gets with (2.5) that

$$-\Delta\,(\Psi_\lambda\varphi)_\pm = -4\partial_z\partial_{\overline{z}}\,(\Psi_\lambda\varphi)_\pm = -2i\lambda\partial_z\,(SL(\lambda)\varphi)_\pm = \lambda\,(\Psi_\lambda\varphi)_\pm$$

holds for all $\varphi \in L^2(\Sigma)$ in the distributional sense. This, (A.4), (A.5), and the properties of $SL(\lambda)$ described at the beginning of this appendix show that $\Psi_\lambda\varphi \in \mathcal{H}_\lambda$ for all $\varphi \in H^{1/2}(\Sigma)$ and therefore the mapping $\Psi_\lambda : H^{1/2}(\Sigma) \to \mathcal{H}_\lambda$ is well-defined. Moreover, it follows from (iii) that this mapping is injective. To prove that $\Psi_\lambda : H^{1/2}(\Sigma) \to \mathcal{H}_\lambda$ is surjective, let $f \in \mathcal{H}_\lambda$ be fixed. Define $\varphi = i(\nu_1 + i\nu_2)(\gamma_D^+ f_+ - \gamma_D^- f_-) \in H^{1/2}(\Sigma)$ and $g = \Psi_\lambda\varphi \in \mathcal{H}_\lambda$. By (iii) we have that

$$\gamma_D^+(f - g)_+ - \gamma_D^-(f - g)_- = \gamma_D^+ f_+ - \gamma_D^- f_- + i(\nu_1 - i\nu_2)\varphi = 0.$$

This shows $f - g \in H^1(\mathbb{R}^2)$. Moreover, due to $f, g \in \mathcal{H}_\lambda$ we have that $\partial_{\bar{z}}(f - g) \in H^1(\mathbb{R}^2)$, which implies $f - g \in H^2(\mathbb{R}^2)$. Combining this with $f, g \in \mathcal{H}_\lambda$ we find that $f - g \in \ker(-\Delta - \lambda) = \{0\}$, i.e. $f = g = \Psi_\lambda \varphi$. Thus $\Psi_\lambda : H^{1/2}(\Sigma) \to \mathcal{H}_\lambda$ is also surjective and all claims in assertion (i) are shown. □

Proof of Proposition 2.2. The proof of item (i) is divided into 3 separate steps. In *Step 1* we show that the map $(-\infty, 0) \ni \lambda \mapsto \mu_n(S(\lambda)) \in (0, \infty)$ is continuous and strictly monotonically increasing, and in *Step 2* we verify that the same is true for the map $(-\infty, 0) \ni \lambda \mapsto \lambda\mu_n(S(\lambda)) \in (-\infty, 0)$. Using these results, we complete the proof of assertion (i) in *Step 3*.

Step 1. Let $n \in \mathbb{N}$. We show that the map $(-\infty, 0) \ni \lambda \mapsto \mu_n(S(\lambda)) \in (0, \infty)$ is continuous and strictly monotonically increasing. To verify that $\mu_n(S(\lambda)) > 0$ for $\lambda \in (-\infty, 0)$, it suffices to prove that $S(\lambda)$ is a positive self-adjoint operator. From the definition of $S(\lambda)$ in (2.4) it follows that $S(\lambda)$ is self-adjoint. Next, let $\varphi \in L^2(\Sigma)$ with $\varphi \neq 0$ and set $f := SL(\lambda)\varphi$. Using the properties of $SL(\lambda)$ described at the beginning of this appendix one finds that $f \neq 0$ and

$$\begin{aligned}\big(S(\lambda)\varphi, \varphi\big)_{L^2(\Sigma)} &= \big(\gamma_D f, \partial_\nu f_+ - \partial_\nu f_-\big)_{L^2(\Sigma)} \\ &= \big(f_+, \Delta f_+\big)_{L^2(\Omega_+)} + \|\nabla f_+\|^2_{L^2(\Omega_+)} + \big(f_-, \Delta f_-\big)_{L^2(\Omega_-)} + \|\nabla f_-\|^2_{L^2(\Omega_-)} \\ &\geq \big(f_+, \Delta f_+\big)_{L^2(\Omega_+)} + \big(f_-, \Delta f_-\big)_{L^2(\Omega_-)} = -\lambda\|f\|^2_{L^2(\mathbb{R}^2)} > 0.\end{aligned}$$

Therefore, $\mu_n(S(\lambda)) > 0$ must be true.

Next, we show that $(-\infty, 0) \ni \lambda \mapsto \mu_n(S(\lambda)) \in (0, \infty)$ is monotonically increasing and continuous. With (A.2) one sees that $S(\cdot) : \mathbb{C}\setminus[0, \infty) \to \mathcal{L}(L^2(\Sigma))$ is holomorphic and that $\frac{d}{d\lambda}S(\lambda) = \gamma_D(-\Delta - \lambda)^{-2}\gamma_D'$ holds. In particular, for any $\varphi \in L^2(\Sigma)$ the function $(-\infty, 0) \ni \lambda \mapsto (S(\lambda)\varphi, \varphi)_{L^2(\Sigma)}$ is continuously differentiable and

$$\begin{aligned}\frac{d}{d\lambda}\big(S(\lambda)\varphi, \varphi\big)_{L^2(\Sigma)} &= \big((-\Delta - \lambda)^{-1}\gamma_D'\varphi, (-\Delta - \lambda)^{-1}\gamma_D'\varphi\big)_{L^2(\Sigma)} \\ &= \|SL(\lambda)\varphi\|^2_{L^2(\mathbb{R}^2)} \geq 0\end{aligned}$$

is true. Thus, the min–max principle implies that the map $(-\infty, 0) \ni \lambda \mapsto \mu_n(S(\lambda))$ is monotonically increasing for every $n \in \mathbb{N}$. Furthermore, due to the holomorphy of $S(\cdot) : \mathbb{C}\setminus[0, \infty) \to \mathcal{L}(L^2(\Sigma))$ and the estimate

$$|\mu_n(S(\eta)) - \mu_n(S(\lambda))| \leq \|S(\eta) - S(\lambda)\|, \quad \eta, \lambda < 0,$$

(see [29, Satz 3.17]), we find that $(-\infty, 0) \ni \lambda \mapsto \mu_n(S(\lambda))$ is continuous for $n \in \mathbb{N}$.

It remains to show that the latter map is strictly monotonically increasing. Define for $\alpha \in \mathbb{R}\setminus\{0\}$ the operator-valued function $\mathcal{B}_1 : \mathbb{C}\setminus[0, \infty) \to \mathcal{L}(L^2(\Sigma))$ by $\mathcal{B}_1(\lambda) = I - \alpha S(\lambda)$. By the properties of $S(\lambda)$ it is easy to see that $\mathcal{B}_1$ is holomorphic and $\mathcal{B}_1(\lambda)$ is a Fredholm operator with index 0 for any fixed λ, since $S(\lambda)$ is compact in $L^2(\Sigma)$. Moreover, by [18, Theorem 1.2] there exists a constant $K > 0$ such that

$$\|S(\lambda)\| \leq \frac{K}{\sqrt{2 + |\lambda|}} \ln\sqrt{2 + \frac{1}{|\lambda|}}, \quad \lambda \in \mathbb{C}\setminus[0, \infty). \tag{A.6}$$

Hence, there exists $\lambda_0 < 0$ such that $\|S(\lambda)\| < |\alpha|^{-1}$ is valid for all $\lambda < \lambda_0$. This implies that $\mathcal{B}_1(\lambda)$ is boundedly invertible for every $\lambda < \lambda_0$. Therefore, by [20, Chapter XI., Corollary 8.4] the set

$$\mathcal{M}_{\alpha,1} = \big\{\lambda \in \mathbb{C}\setminus[0, \infty) \,\big|\, \mathcal{B}_1(\lambda) = I - \alpha S(\lambda) \text{ is not invertible}\big\}$$

is at most countable and does not have an accumulation point in $\mathbb{C}\setminus[0,\infty)$. Now assume that $\lambda_1 < \lambda_2 < 0$ satisfy $\mu_n(S(\lambda_1)) = \mu_n(S(\lambda_2)) =: \alpha^{-1}$ for some $n \in \mathbb{N}$. Then it follows from the monotonicity of $\lambda \mapsto \mu_n(S(\lambda))$ that $[\lambda_1, \lambda_2] \subseteq \mathcal{M}_{\alpha,1}$, which is a contradiction to the fact that $\mathcal{M}_{\alpha,1}$ is at most countable. Therefore, the mapping $(-\infty, 0) \ni \lambda \mapsto \mu_n(S(\lambda))$ is continuous and strictly monotonically increasing for $n \in \mathbb{N}$.

Step 2. To show the continuity and strict monotonicity of the map $(-\infty, 0) \ni \lambda \mapsto \lambda\mu_n(S(\lambda))$ for all $n \in \mathbb{N}$, we note first that the continuity follows from the continuity of the map $\lambda \mapsto \mu_n(S(\lambda))$ shown in *Step 1*. In order to prove the monotonicity, we use again $\frac{d}{d\lambda}S(\lambda) = \gamma_D(-\Delta-\lambda)^{-2}\gamma_D'$ and compute for a fixed $\varphi \in L^2(\Sigma)$ and $\lambda \in (-\infty, 0)$ with the help of (2.5) and (A.2)

$$\begin{aligned}\frac{d}{d\lambda}\big(\lambda S(\lambda)\varphi, \varphi\big)_{L^2(\Sigma)} &= \big(S(\lambda)\varphi + \lambda\gamma_D(-\Delta-\lambda)^{-2}\gamma_D'\varphi, \varphi\big)_{L^2(\Sigma)} \\ &= \big(-4\gamma_D(-\Delta-\lambda)^{-1}\partial_{\bar z}\partial_z(-\Delta-\lambda)^{-1}\gamma_D'\varphi, \varphi\big)_{L^2(\Sigma)} \\ &= \|\Psi_\lambda\varphi\|^2_{L^2(\mathbb{R}^2)} \geq 0.\end{aligned}$$

Thus, the min-max principle yields the monotonicity of the mapping $(-\infty, 0) \ni \lambda \mapsto \lambda\mu_n(S(\lambda))$. To see the strict monotonicity, we use a similar strategy as in *Step 1* and define for $\alpha \in \mathbb{R}\setminus\{0\}$ the holomorphic mapping $\mathcal{B}_2 : \mathbb{C}\setminus[0,\infty) \to \mathcal{L}(L^2(\Sigma))$ by $\mathcal{B}_2(\lambda) = I - \alpha\lambda S(\lambda)$. Again, $\mathcal{B}_2(\lambda)$ is a Fredholm operator with index zero for any fixed λ and it follows from (A.6) that there exists $\lambda_3 < 0$ such that $\|\lambda S(\lambda)\| < |\alpha|^{-1}$ holds for all $\lambda \in (\lambda_3, 0)$. In particular, $\mathcal{B}_2(\lambda)$ is boundedly invertible for all $\lambda \in (\lambda_3, 0)$. It follows from [20, Chapter XI., Corollary 8.4] that the set

$$\mathcal{M}_{\alpha,2} = \big\{\lambda \in \mathbb{C}\setminus[0,\infty) \,\big|\, \mathcal{B}_2(\lambda) = I - \alpha\lambda S(\lambda) \text{ is not invertible}\big\}$$

is at most countable and does not have an accumulation point in $\mathbb{C}\setminus[0,\infty)$. Now the same argument as in *Step 1* shows that $(-\infty, 0) \ni \lambda \mapsto \lambda\mu_n(S(\lambda))$ is strictly monotonously increasing.

Step 3. To study the limiting behaviour of $\lambda\mu_n(S(\lambda))$ for $\lambda \to 0$, note that (A.6) implies $\|\lambda S(\lambda)\| \to 0$ for $\lambda \to 0^-$ and hence,

$$\lim_{\lambda\to 0^-} \lambda\mu_n(S(\lambda)) = 0, \quad n \in \mathbb{N}. \tag{A.7}$$

Next, we consider the limit of $\lambda\mu_n(S(\lambda))$ for $\lambda \to -\infty$. For this purpose, results on Schrödinger operators with δ-interactions will be used. Define for $\alpha < 0$ the sesquilinear form

$$\mathfrak{h}_{\delta,\alpha}[f, g] = \big(\nabla f, \nabla g\big)_{L^2(\mathbb{R}^2)} + \alpha\big(\gamma_D f, \gamma_D g\big)_{L^2(\Sigma)}, \quad f, g \in \operatorname{dom}\mathfrak{h}_{\delta,\alpha} = H^1(\mathbb{R}^2).$$

By [9,14] the form $\mathfrak{h}_{\delta,\alpha}$ is semi-bounded and closed, and one can show for the self-adjoint operator $H_{\delta,\alpha}$, which is associated with $\mathfrak{h}_{\delta,\alpha}$ by the first representation theorem, that $\sigma_{\text{ess}}(H_{\delta,\alpha}) = [0,\infty)$, that its discrete spectrum $\sigma_{\text{disc}}(H_{\delta,\alpha})$ is finite, and for $\lambda \in (-\infty, 0)$ one has that

$$\lambda \in \sigma_{\text{p}}(H_{\delta,\alpha}) \quad \Longleftrightarrow \quad -1 \in \sigma_{\text{p}}(\alpha S(\lambda)); \tag{A.8}$$

see for instance [9, Lemma 2.3 and Theorem 4.2] and [8, Theorems 3.5 and 3.14]. Recall that the eigenvalues $\mu_n(S(\lambda))$ are ordered non-increasingly with multiplicities taken into account. If we order the discrete eigenvalues of $H_{\delta,\alpha}$ in an increasing way then the strict monotonicity of $\lambda \mapsto \mu_n(S(\lambda))$ implies that the k-th discrete eigenvalue $E_k(\alpha)$ (if it exists) satisfies the equation $-1 = \alpha\mu_k(S(E_k(\alpha)))$.

Let $n \in \mathbb{N}$. Then by [14, Theorem 1] the operator $H_{\delta,\alpha}$ has at least n negative discrete eigenvalues (counted with multiplicities) if $-\alpha > 0$ is sufficiently large, and the n-th discrete eigenvalue $E_n(\alpha)$ of $H_{\delta,\alpha}$ admits the asymptotic expansion

$$E_n(\alpha) = -\frac{\alpha^2}{4} + \mu_n(H) + \mathcal{O}(\alpha^{-1} \ln|\alpha|), \qquad \alpha \to -\infty. \tag{A.9}$$

Here H is a fixed semibounded differential operator on Σ that is independent of α and has purely discrete spectrum $\mu_1(H) \leq \mu_2(H) \leq \dots$. Thus for $\alpha \to -\infty$ we obtain with (A.8) that

$$\frac{\alpha}{4} + \frac{|\mu_n(H)|+1}{\alpha} \leq E_n(\alpha)\mu_n(S(E_n(\alpha))) = -\frac{E_n(\alpha)}{\alpha} \leq \frac{\alpha}{4} - \frac{|\mu_n(H)|+1}{\alpha}. \tag{A.10}$$

This shows

$$\lim_{\lambda\to-\infty} \lambda\mu_n(S(\lambda)) = -\infty \tag{A.11}$$

and finishes the proof of item (i).

To show item (ii), we note first that by (A.7), (A.11), and the strict monotonicity and continuity of the mapping $\lambda \mapsto \lambda\mu_n(S(\lambda))$ it is clear that for any $a < 0$ there is a unique solution $\lambda_n(a)$ of $\lambda\mu_n(S(\lambda)) = a$. Let $\mu_n(H)$ be as in (A.9), define the numbers $k_\pm = \pm(|\mu_n(H)| + 1)$ and let

$$\alpha_\pm = -2|a|\left(\sqrt{1+\frac{k_\pm}{a^2}}+1\right) = -4|a| - \frac{k_\pm}{|a|} + f_\pm(a) \tag{A.12}$$

with some functions $f_\pm(a) = \mathcal{O}(a^{-3})$ for large $|a| > 0$, where the latter representation holds due to a Taylor series expansion. Then one has

$$a = \frac{\alpha_\pm}{4} - \frac{k_\pm}{\alpha_\pm} \tag{A.13}$$

and it follows with (A.10) that

$$E_n(\alpha_+)\mu_n(S(E_n(\alpha_+))) \leq \frac{\alpha_+}{4} - \frac{|\mu_n(H)|+1}{\alpha_+} = a = \lambda_n(a)\mu_n(S(\lambda_n(a))$$

and

$$\lambda_n(a)\mu_n(S(\lambda_n(a)) = a = \frac{\alpha_-}{4} + \frac{|\mu_n(H)|+1}{\alpha_-} \leq E_n(\alpha_-)\mu_n(S(E_n(\alpha_-))).$$

Since $\lambda \mapsto \lambda\mu_n(S(\lambda))$ is monotone we find

$$E_n(\alpha_+) \leq \lambda_n(a) \leq E_n(\alpha_-). \tag{A.14}$$

From (A.12) we obtain

$$\frac{1}{4}\alpha_\pm^2 = 4a^2 + 2k_\pm + g_\pm(a)$$

with functions $g_\pm(a) = \mathcal{O}(a^{-2})$ for large $|a| > 0$ and hence (A.9) implies

$$\left|E_n(\alpha_\pm) + 4a^2 + 2k_\pm + g_\pm(a) - \mu_n(H)\right| \leq C_1\left|\alpha_\pm^{-1} \ln|\alpha_\pm|\right| \tag{A.15}$$

for some constant $C_1 > 0$. Note that there exist positive constants $C_2, C_3 > 0$ such that $C_2|a| \leq |\alpha_\pm| \leq C_3|a|$ holds for large $|a| > 0$. With this we conclude from (A.15) that

$$\left|E_n(\alpha_\pm) + 4a^2 + 2k_\pm - \mu_n(H)\right| \leq C_4\left|a^{-1} \ln |a|\right| \tag{A.16}$$

holds for some constant $C_4 > 0$ and for large $|a| > 0$. Taking (A.14) and (A.16) into account, one concludes finally that

$$|\lambda_n(a) + 4a^2| \leq 3|\mu_n(H)| + 2 + \mathcal{O}(a^{-1} \ln |a|) = \mathcal{O}(1) \quad \text{for } a \to -\infty.$$

□

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