On Compact Perturbations of Locally Definitizable Selfadjoint Relations in Krein Spaces

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Abstract. The aim of this paper is to prove two perturbation results for a selfadjoint operator A in a Krein space $\mathcal{H}$ which can roughly be described as follows: (1) If Δ is an open subset of $\overline{\mathbb{R}}$ and all spectral subspaces for A corresponding to compact subsets of Δ have finite rank of negativity, the same is true for a selfadjoint operator B in $\mathcal{H}$ for which the difference of the resolvents of A and B is compact. (2) The property that there exists some neighbourhood Δ_∞ of ∞ such that the restriction of A to a spectral subspace for A corresponding to Δ_∞ is a nonnegative operator in $\mathcal{H}$, is preserved under relative $\mathfrak{S}_p$ perturbations in form sense if the resulting operator is again selfadjoint. The assertion (1) is proved for selfadjoint relations A and B. (1) and (2) generalize some known results.

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1. Introduction

Let A be a definitizable selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e. the resolvent set $\rho(A)$ is nonempty and there exists a polynomial p such that $[p(A)x, x] \geq 0$ holds for all $x \in \mathcal{D}(p(A))$. Then A has a spectral function (see [17]) and with the help of this spectral function the real points of the spectrum $\sigma(A)$ can be classified in points of positive and negative type and critical points. Spectral points of positive and negative type can also be characterized with the help of the resolvent of A (see e.g. [7], [11]) or by approximative eigensequences (see [15], [18]), which allows, in a convenient way, to carry over the sign type classification of spectral points to non-definitizable selfadjoint operators and relations in Krein spaces.

Sign types of spectral points, which are a special feature of the spectral theory in Krein spaces, are closely connected with spectral decomposition properties. For example, if any point of some bounded closed interval $[a, b]$ is either of positive or of negative type with respect to some selfadjoint operator A, then A can be decomposed into a direct orthogonal sum of a definitizable selfadjoint operator A_1 with spectrum in $[a, b]$ and a selfadjoint operator A_2 such that $\sigma(A_2) \cap (a, b) = \emptyset$, that is, A is locally definitizable. In view of these connections between sign types and decomposability, results on stability properties of the sets of spectral points of positive and negative type play an important role in the perturbation theory in Krein spaces.

In [18] it was shown, for a bounded selfadjoint operator A, that if all points of a bounded closed interval Δ are either regular or of positive type with respect to A, then with the exception of no more than a finite number of points the same is true after a symmetric compact perturbation K. Moreover, on the spectral subspaces corresponding to $A + K$ and subintervals of Δ the inner product $[\cdot, \cdot]$ has a finite number of negative squares. A similar result was proved in [8]. In [8] A is not assumed to be bounded, but there are additional assumptions.

The first objective of the present paper is to generalize these results. In Theorem 2.4 we consider unbounded selfadjoint operators and selfadjoint linear relations and drop the additional conditions from [8]. We allow that the unperturbed and the perturbed operator are selfadjoint with respect to different Krein space inner products. It is assumed that the difference of the Gram operators of these inner products fulfils some "local" compactness condition which is usual in local scattering theory. Essentially, the proof of Theorem 2.4 is a variant of the proof of Theorem 5.1 in [18]. Instead of the Lyubich-Matsaev spectral subspace results here we make use of a functional calculus for unitary operators in Krein spaces with finite order growth of the resolvent in a neighbourhood of some arcs of the unit circle (see [7]). For different inner products with compact difference of the corresponding Gram operators, for a bounded unperturbed operator and a compact perturbation Theorem 2.4 is a consequence of the perturbation result [19, Theorem 6.1] on holomorphic operator functions (see also [1]). If the difference of the Gram operators is compact and there is a real point which is regular for the unperturbed and the perturbed relation, Theorem 2.4 can be deduced from [19, Theorem 6.1] with the help of a linear fractional transformation.

The second objective of this paper is to generalize a result from [9]. In Section 3 we consider selfadjoint operators in a Krein space which can be decomposed as a direct orthogonal sum of a bounded selfadjoint and a nonnegative selfadjoint operator. Then the spectrum of positive type as well as the spectrum of negative type may have ∞ as an accumulation point. Such operators and a class of relatively compact perturbations in form sense were studied in [9]. In one of the main results of [9] it is proved that under some conditions the perturbed operator admits a decomposition of the same type. Making use of the perturbation result for unbounded operators of Section 2, here we improve that result from [9] by dropping an assumption on the spectral properties of the perturbed operator. In contrast to [9] we do not exclude the case where the spectral function of the unperturbed operator is unbounded near ∞ and, at the same time, the unperturbed and the perturbed operator are selfadjoint with respect to different Krein space inner products.

2. Sign types of spectral points of two selfadjoint relations with compact resolvent difference

2.1. Notations and definitions

In Section 2 we study linear relations in a separable Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e. linear subspaces of $\mathcal{H}^2$. Linear operators in $\mathcal{H}$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse, the adjoint etc., we refer to [3].

The resolvent set $\rho(S)$ of a closed linear relation S is the set of all $z \in \mathbb{C}$ such that $(S-z)^{-1} \in \mathcal{L}(\mathcal{H})$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if S is a bounded operator and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. An eigenvalue $\lambda \in \mathbb{C}$ of a closed linear relation S is called *normal* if the root manifold $\mathcal{L}_\lambda$ corresponding to λ is finite-dimensional and there is a projection P with $P\mathcal{H} = \mathcal{L}_\lambda$ which reduces S, i.e. S is the direct sum in $\mathcal{H}^2$ of the subspaces $S \cap (P\mathcal{H})^2$ and $S \cap ((1-P)\mathcal{H})^2$ of $\mathcal{H}^2$, such that $\lambda \in \rho(S \cap ((1-P)\mathcal{H})^2)$. The set of normal eigenvalues of S will be denoted by $\sigma_{p,norm}(S)$. The *essential spectrum* of S is defined by $\sigma_{ess}(S) = \sigma(S) \backslash \sigma_{p,norm}(S)$.

We recall the definitions of the approximate point spectrum and the spectra of positive and negative type of a closed linear relation S (see [11]). For equivalent descriptions of the spectra of positive and negative type we refer to [11, Theorem 3.18].

Definition 2.1. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* S, denoted by $\sigma_{ap}(S)$, if there exists a sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2, \ldots$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$. We define the *extended approximate point spectrum* $\widetilde{\sigma}_{ap}(S)$ *of* S by $\widetilde{\sigma}_{ap}(S) := \sigma_{ap}(S) \cup \{\infty\}$ if $0 \in \sigma_{ap}(S^{-1})$, and $\widetilde{\sigma}_{ap}(S) := \sigma_{ap}(S)$ if $0 \notin \sigma_{ap}(S^{-1})$.

We remark, that the boundary points of $\widetilde{\sigma}(S)$ in $\overline{\mathbb{C}}$ belong to $\widetilde{\sigma}_{ap}(S)$.

Definition 2.2. A point $\lambda \in \sigma_{ap}(S)$ is said to be of *positive type* (*negative type*) *with respect to* S, if for every sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2 \ldots$, with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$ we have

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty}\, [x_n, x_n] < 0).$$

If $\infty \in \widetilde{\sigma}_{ap}(S)$, ∞ is said to be of *positive type* (*negative type*) *with respect to* S if for every sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2 \ldots$, with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|y_n\| = 1$

we have

$$\liminf_{n\to\infty}\,[y_n,y_n]>0 \quad (\text{resp. } \limsup_{n\to\infty}\,[y_n,y_n]<0).$$

The set of all points of positive type (negative type) with respect to S will be denoted by $\sigma_{++}(S)$ (resp. $\sigma_{--}(S)$).

If A is selfadjoint, all points of positive or negative type belong to $\overline{\mathbb{R}}$. Analogously, if U is a unitary operator in the Krein space $(\mathcal{H},[\cdot,\cdot])$, all points of positive or negative type lie on the unit circle $\mathbb{T}$.

If $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ are separable Hilbert spaces $\mathfrak{S}_\infty(\mathcal{K}_1,\mathcal{K}_2)$ denotes the set of all compact operators from $\mathcal{K}_1$ to $\mathcal{K}_2$. If $s_1(A)\geq s_2(A)\geq\dots$ are the s-numbers of $A\in\mathfrak{S}_\infty(\mathcal{K}_1,\mathcal{K}_2)$, i.e. the eigenvalues of $(A^*A)^{\frac{1}{2}}$ where multiplicity is counted, we set

$$\mathfrak{S}_p(\mathcal{K}_1,\mathcal{K}_2):=\Big\{A\in\mathfrak{S}_\infty(\mathcal{K}_1,\mathcal{K}_2):\Big(\sum_j s_j(A)^p\Big)^{\frac{1}{p}}=:\|A\|_{\mathfrak{S}_p}<\infty\Big\},\quad p\in[1,\infty).$$

Let $\mathfrak{S}_p(\mathcal{K}):=\mathfrak{S}_p(\mathcal{K},\mathcal{K})$, $p\in[1,\infty)\cup\{\infty\}$; we will simply write $\mathfrak{S}_p$ when no confusion can arise. By $\mathcal{F}$ we denote the class of operators of finite rank.

2.2. A criterion for compact resolvent difference

Let, in the following, $(\mathcal{H},(\cdot,\cdot))$ be a separable Hilbert space and let G_1 and G_2 be bounded selfadjoint operators in $\mathcal{H}$ with $0\in\rho(G_1)\cap\rho(G_2)$. We define the inner products $[\cdot,\cdot]_1:=(G_1\cdot,\cdot)$ and $[\cdot,\cdot]_2:=(G_2\cdot,\cdot)$ in $\mathcal{H}$. Then $\mathcal{H}_1:=(\mathcal{H},[\cdot,\cdot]_1)$ and $\mathcal{H}_2:=(\mathcal{H},[\cdot,\cdot]_2)$ are Krein spaces. We do not exclude that $\mathcal{H}_1$ or $\mathcal{H}_2$ is a Hilbert space. Let A_1 and A_2 be selfadjoint relations in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that the difference of the resolvents of A_1 and A_2 is compact, i.e. the following condition (I) is fulfilled.

(**I**): There exists a $\mu\in\mathbb{C}$ such that

$$\mu\in\rho(A_1)\cap\rho(A_2)$$

and

$$(A_1-\mu)^{-1}-(A_2-\mu)^{-1}\in\mathfrak{S}_\infty \tag{2.1}$$

hold.

Condition (I) implies that we have $(A_1-\mu')^{-1}-(A_2-\mu')^{-1}\in\mathfrak{S}_\infty$ for any point $\mu'\in\rho(A_1)\cap\rho(A_2)$. This follows from the relation

$$\begin{aligned}&(A_1-\mu')^{-1}-(A_2-\mu')^{-1}=\\&=\big(1+(\mu'-\mu)(A_2-\mu')^{-1}\big)\big((A_1-\mu)^{-1}-(A_2-\mu)^{-1}\big)\big(1+(\mu'-\mu)(A_1-\mu')^{-1}\big).\end{aligned}$$

The following proposition contains a criterion for (2.1). Observe that G_1A_1 and G_2A_2 are selfadjoint relations in the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$. Therefore, $\mathbb{C}\backslash\mathbb{R}$ belongs to $\rho(G_kA_k)$, $k = 1, 2$.

Proposition 2.3. *Let $p \in [1,\infty) \cup \{\infty\}$ and assume that $G_1 - G_2 \in \mathfrak{S}_p$ and that there is a $\mu \in \mathbb{C}\backslash\mathbb{R}$ with $\mu \in \rho(A_1) \cap \rho(A_2)$. Then*

$$(A_1 - \mu)^{-1} - (A_2 - \mu)^{-1} \in \mathfrak{S}_p \tag{2.2}$$

if and only if

$$(G_1A_1 - \mu)^{-1} - (G_2A_2 - \mu)^{-1} \in \mathfrak{S}_p.$$

Proof. As

$$\begin{aligned}(G_1A_1 - \mu)^{-1} - (G_2A_2 - \mu)^{-1} &= (A_1 - G_1^{-1}\mu)^{-1}G_1^{-1} - (A_2 - G_2^{-1}\mu)^{-1}G_2^{-1} \\ &= (A_1 - G_1^{-1}\mu)^{-1}(G_1^{-1} - G_2^{-1}) + \big((A_1 - G_1^{-1}\mu)^{-1} - (A_2 - G_2^{-1}\mu)^{-1}\big)G_2^{-1}\end{aligned}$$

it is sufficient to prove that

$$(A_1 - G_1^{-1}\mu)^{-1} - (A_2 - G_2^{-1}\mu)^{-1} \in \mathfrak{S}_p$$

is equivalent to (2.2). This equivalence follows from the relation

$$\begin{aligned}&\big((A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\big)\big(1 + (\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1}\big) \\ &- \big(1 + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})\big)\big((A_1 - \lambda G_1^{-1})^{-1} - (A_2 - \lambda G_2^{-1})^{-1}\big) \in \mathfrak{S}_p\end{aligned} \tag{2.3}$$

since $0 \in \rho\big(1 + (\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1}\big)$ and $0 \in \rho\big(1 + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})\big)$. Indeed, we have

$$\big(1 + (\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1}\big)^{-1} = 1 + (\lambda - \lambda G_1^{-1})(A_1 - \lambda)^{-1}$$

and

$$\big(1 + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})\big)^{-1} = 1 + (A_2 - \lambda G_2^{-1})^{-1}(\lambda G_2^{-1} - \lambda).$$

It remains to verify (2.3). Evidently, we have

$$\begin{aligned}&(A_2 - \lambda)^{-1}(\lambda - \lambda G_1^{-1})(A_1 - \lambda G_1^{-1})^{-1} \\ &\qquad - (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})(A_1 - \lambda G_1^{-1})^{-1} =: S \in \mathfrak{S}_p.\end{aligned}$$

Addition of this relation and the relations

$$\begin{aligned}(A_1 - \lambda)^{-1} - (A_1 - \lambda G_1^{-1})^{-1} + (A_1 - \lambda)^{-1}(\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1} &= 0, \\ -(A_2 - \lambda)^{-1} + (A_2 - \lambda G_2^{-1})^{-1} + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})(A_2 - \lambda G_2^{-1})^{-1} &= 0\end{aligned}$$

gives (2.3).

□

2.3. Preservation of Pontryagin local spectral subspaces

Let, in the rest of Section 2, Ω be a domain of the extended complex plane $\overline{\mathbb{C}}$ which is symmetric with respect to the real axis $\mathbb{R}$ such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$. We denote the open upper half-plane by $\mathbb{C}^+$. We assume that A_1 and A_2 satisfy, besides condition (I), the following condition.

(**II**): There exists a point $\mu_0 \in \Omega \cap \mathbb{C}^+$ with $\mu_0 \in \rho(A_1) \cap \rho(A_2)$. Moreover, either

$$\Omega \backslash \overline{\mathbb{R}} \subset \rho(A_1) \cup \sigma_{p,norm}(A_1) \tag{2.4}$$

or

$$\Omega \backslash \overline{\mathbb{R}} \subset \rho(A_2) \cup \sigma_{p,norm}(A_2). \tag{2.5}$$

Then (2.1) holds with μ replaced by μ_0. The relations (2.4) and (2.5) can be expressed with the help of similar relations for the bounded operators $(A_1 - \mu_0)^{-1}$ and $(A_2 - \mu_0)^{-1}$, respectively. Then, from (2.1) and well-known perturbation results, it follows that (2.4) and (2.5) are equivalent.

Theorem 2.4. *Let A_1 and A_2 be selfadjoint relations in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that the conditions* (I) *and* (II) *are fulfilled. Assume that for every (in $\overline{\mathbb{R}}$) compact set $\Delta_0 \subset \Omega \cap \overline{\mathbb{R}}$ there exists a finite union Δ_1 of open connected subsets of $\Omega \cap \overline{\mathbb{R}}$ with $\Delta_0 \subset \Delta_1$, $\overline{\Delta_1} \subset \Omega \cap \overline{\mathbb{R}}$ and a selfadjoint projection F_1 in $\mathcal{H}_1$ such that $(F_1\mathcal{H}_1, [\cdot,\cdot]_1)$ is a Pontryagin space with finite rank of negativity and the following holds:*

(i) *If, for some $\lambda \in \rho(A_1)$ and some bounded operator T,*

$$T(A_1 - \lambda)^{-1} = (A_1 - \lambda)^{-1}T,$$

then $F_1T = TF_1$.

(ii) $\widetilde{\sigma}(A_1 \cap (F_1\mathcal{H}_1)^2) \subset \widetilde{\sigma}(A_1) \cap \overline{\Delta_1}$.

(iii) $\widetilde{\sigma}(A_1 \cap ((1 - F_1)\mathcal{H}_1)^2) \subset \widetilde{\sigma}(A_1) \backslash \Delta_1$.

(iv) $(G_1 - G_2)F_1 \in \mathfrak{S}_\infty$.

Then for every in $\overline{\mathbb{R}}$ compact subset $\Delta_0 \subset \Omega \cap \overline{\mathbb{R}}$ there exists a finite union Δ_2 of open connected subsets of $\Omega \cap \overline{\mathbb{R}}$ with $\Delta_0 \subset \Delta_2$, $\overline{\Delta_2} \subset \Omega \cap \overline{\mathbb{R}}$ and a selfadjoint projection F_2 in $\mathcal{H}_2$ such that $(F_2\mathcal{H}_2, [\cdot,\cdot]_2)$ is a Pontryagin space with finite rank of negativity, and (i)-(iv) *holds with F_1, A_1, $\mathcal{H}_1$, Δ_1 replaced by F_2, A_2, $\mathcal{H}_2$, Δ_2.*

From this theorem, with the help of the spectral function for locally definite relations ([7], [18], [11]), we obtain the following corollary.

Corollary 2.5. *Let A_1 and A_2 be selfadjoint relations in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that the conditions* (I) *and* (II) *are fulfilled. Assume that $G_1 - G_2 \in \mathfrak{S}_\infty$ and $\Omega \cap \overline{\mathbb{R}} \subset \rho(A_1) \cup \sigma_{++}(A_1)$. Then the conclusion of Theorem* 2.4 *is true.*

Proof of Theorem 2.4. **1.** We consider the linear fractional transformations ψ and ϕ defined by

$$\psi(\lambda) = -1 + (\mu_0 - \overline{\mu}_0)(\lambda - \overline{\mu}_0)^{-1} \quad \text{and} \quad \varphi(z) = (\overline{\mu}_0 z + \mu_0)(z+1)^{-1},$$

where μ_0 is as in condition (II). ψ maps the open upper half-plane $\mathbb{C}^+$ onto the open unit disc $\mathbb{D}$, and $\psi \circ \varphi$ is the identity mapping. $\psi(\Omega)$ is a domain of $\overline{\mathbb{C}}$ symmetric with respect to the unit circle $\mathbb{T}$, which contains neighbourhoods of 0 and ∞, and it holds $\psi(\Omega) \cap \mathbb{T} \neq \emptyset$. We define the operators

$$U_k := \psi(A_k) = -1 + (\mu_0 - \overline{\mu}_0)(A_k - \overline{\mu}_0)^{-1}, \quad k = 1, 2. \tag{2.6}$$

U_k is a unitary operator in the Krein space $\mathcal{H}_k$. Then (2.1) with $\mu = \overline{\mu}_0$ implies

$$U_1 - U_2 \in \mathfrak{S}_\infty. \tag{2.7}$$

Condition (II) implies

$$\psi(\Omega)\backslash\mathbb{T} \subset \big(\rho(U_1) \cup \sigma_{p,norm}(U_1)\big) \cap \big(\rho(U_2) \cup \sigma_{p,norm}(U_2)\big).$$

Let Δ_0 be a subset of $\Omega \cap \overline{\mathbb{R}}$ which is compact in $\overline{\mathbb{R}}$. We choose Δ_1 and F_1 as in the assumptions of the theorem. Then F_1 commutes with U_1 and we have

$$\begin{aligned} &\sigma(U_1|F_1\mathcal{H}_1) \subset \sigma(U_1) \cap \overline{\psi(\Delta_1)}, \\ &\sigma(U_1|(1-F_1)\mathcal{H}_1) \subset \sigma(U_1)\backslash\psi(\Delta_1). \end{aligned} \tag{2.8}$$

Let

$$F_1\mathcal{H}_1 = \mathcal{K}_+[\dot{+}]\mathcal{K}_- \tag{2.9}$$

be a fundamental decomposition of $F_1\mathcal{H}_1$ and let $F_{1,+}$ and $F_{1,-}$ be the corresponding projections in $F_1\mathcal{H}_1$. Then $\dim \mathcal{K}_- < \infty$. We write the restriction V of U_1 to $F_1\mathcal{H}_1$ as operator matrix,

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

with respect to the fundamental decomposition (2.9). The operators V_{12}, V_{21}, V_{22} are of finite rank, and by the general form of a unitary operator in a Pontryagin space [4, Supplement] there exists a unitary operator V_+ in the Hilbert space $\mathcal{K}_+$ such that $V_{11} - V_+$ is of finite rank. Let ν be a point of $\psi(\Delta_1)$ and define a unitary operator V' in $F_1\mathcal{H}_1$ by

$$V' = \begin{pmatrix} V_+ & 0 \\ 0 & \nu \end{pmatrix}.$$

Then $V - V'$ is of finite rank. We define a Hilbert scalar product $\langle \cdot, \cdot \rangle_1'$ on $F_1\mathcal{H}_1$ by

$$\langle x, y \rangle_1' := [(F_{1,+} - F_{1,-})x, y]_1, \qquad x, y \in F_1\mathcal{H}_1.$$

V' is a unitary operator also in $(F_1\mathcal{H}_1, \langle \cdot, \cdot \rangle_1')$. We set

$$U_1' := V'F_1 + U_1(1 - F_1)$$

and

$$[x, y]_1' := \langle F_1 x, F_1 y \rangle_1' + [(1 - F_1)x, (1 - F_1)y]_1, \qquad x, y \in \mathcal{H}.$$

Then U_1' is a unitary operator in the Krein space $(\mathcal{H}, [\cdot,\cdot]_1') =: \mathcal{H}_1'$ and we have

$$U_1 - U_1' \in \mathcal{F}. \tag{2.10}$$

Let G_1' be the Gram operator of $[\cdot,\cdot]_1'$ with respect to $(\cdot,\cdot)$. Then $G_1 - G_1'$ is of finite rank. It is sufficient to verify, that the difference of the Gram operators of $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_1'$ with respect to some suitably chosen Hilbert scalar product equivalent to $(\cdot,\cdot)$ has this property. This is easy to see.

If A_1' denotes the selfadjoint relation $\varphi(U_1')$ then $\overline{\mu}_0 \in \rho(A_1')$ and by (2.10)

$$(A_1' - \overline{\mu}_0)^{-1} - (A_1 - \overline{\mu}_0)^{-1} \in \mathcal{F}.$$

By the construction of U_1' the set $\Omega\backslash\overline{\mathbb{R}}$ is contained in $\rho(A_1') \cup \sigma_{p,norm}(A_1')$. Let $\widetilde{F}_1$ be the spectral projection corresponding to the unitary operator V' in the Hilbert space $(F_1\mathcal{H}_1, \langle\cdot,\cdot\rangle_1')$ and the set $\psi(\Delta_1)$, and denote by F_1' the projection $\widetilde{F}_1 F_1$ in $\mathcal{H}_1$. Then $\mathcal{H}_1'$, A_1' and F_1' satisfy the conditions fulfilled by $\mathcal{H}_1$, A_1 and F_1 at the beginning of the proof. In particular, we have

$$(G_1' - G_2)F_1' = (G_1' - G_1)F_1' + (G_1 - G_2)F_1F_1' \in \mathfrak{S}_\infty.$$

Moreover, $(F_1'\mathcal{H}_1', [\cdot,\cdot]_1')$ is a Hilbert space. Therefore, in the following we can and will restrict ourselves to the case when $(F_1\mathcal{H}_1, [\cdot,\cdot]_1)$ is a Hilbert space. Note that this implies $\psi(\Delta_1) \subset \sigma_{++}(U_1) \cup \rho(U_1)$.

2. In this part of the proof we will show that any point $\lambda \in \psi(\Delta_1)$ either belongs to $\sigma_{++}(U_2) \cup \rho(U_2)$ or is an eigenvalue with (at least) one nonpositive eigenvector with respect to $[\cdot,\cdot]_2$. We proceed as in the proof of Theorem 5.1 from [18]; in addition, we need the following fact.

Claim. For $\lambda \in \psi(\Delta_1)$ and a sequence (x_n), $n = 1, 2, \ldots$, in $\mathcal{H}$, $\|x_n\| = 1$, with $\lim_{n\to\infty} \|(U_1 - \lambda)x_n\| = 0$ which converges weakly to zero we have

$$\liminf_{n\to\infty}\, [x_n, x_n]_1 = \liminf_{n\to\infty}\, [x_n, x_n]_2.$$

Indeed, the inner products $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ are related by

$$[\cdot,\cdot]_2 = [(1 + G_1^{-1}(G_2 - G_1))\cdot, \cdot]_1.$$

By assumption $(G_2 - G_1)F_1 \in \mathfrak{S}_\infty$ and $\lim_{n\to\infty} \|(1 - F_1)x_n\| = 0$, which follows from $\lambda \in \psi(\Delta_1)$ and (iii), we find

$$\begin{aligned}\liminf_{n\to\infty}\, [x_n, x_n]_2 &= \liminf_{n\to\infty}\, [(1 + G_1^{-1}(G_2 - G_1))F_1x_n, F_1x_n]_1 \\ &= \liminf_{n\to\infty}\, [F_1x_n, F_1x_n]_1 = \liminf_{n\to\infty}\, [x_n, x_n]_1,\end{aligned}$$

and the claim is proved.

Let $\lambda \in \psi(\Delta_1)$. It remains to prove, that in the case $\lambda \in \sigma(U_2)\backslash\sigma_{++}(U_2)$ there exists an eigenvector of U_2 corresponding to λ which is nonpositive with respect to $[\cdot,\cdot]_2$. Since λ is a boundary point of $\sigma(U_2)$ and does not belong to $\sigma_{++}(U_2)$ there exists a sequence (x_n), $n = 1, 2, \dots$, in $\mathcal{H}$, $\|x_n\| = 1$, such that

$$\lim_{n\to\infty} \|(U_2 - \lambda)x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} [x_n, x_n]_2 \le 0 \tag{2.11}$$

holds. It is no restriction to assume that the sequence (x_n) converges weakly. Let $x_0 := w - \lim_{n\to\infty} x_n$, then $x_0 \neq 0$, as otherwise (2.7) and the first relation of (2.11) would imply

$$\lim_{n\to\infty} \|(U_1 - \lambda)x_n\| = 0,$$

and, since $\lambda \in \sigma_{++}(U_1) \cup \rho(U_1)$, the claim above would imply

$$\liminf_{n\to\infty}\, [x_n, x_n]_2 = \liminf_{n\to\infty}\, [x_n, x_n]_1 > 0,$$

which contradicts $\lambda \notin \sigma_{++}(U_2) \cup \rho(U_2)$. From (2.11) we have $(U_2 - \lambda)x_0 = 0$. We show that x_0 is nonpositive in $\mathcal{H}_2$. This is evident, if for $y_n := x_n - x_0$, $n = 1, 2, \dots$, $\lim_{n\to\infty} \|y_n\| = 0$ holds. Assume that $\inf \|y_n\| > 0$. By $w - \lim_{n\to\infty} y_n = 0$, (2.7) and (2.11) we have

$$\lim_{n\to\infty} \|(U_1 - \lambda)y_n\| = \lim_{n\to\infty} \|(U_2 - \lambda)y_n\| = 0,$$

hence $\liminf_{n\to\infty}[y_n, y_n]_1 > 0$. Then making use of the claim proved above we find

$$0 < \liminf_{n\to\infty}\, [y_n, y_n]_2 = \liminf_{n\to\infty}\, [x_n, x_n]_2 - [x_0, x_0]_2,$$

and the second relation of (2.11) yields $[x_0, x_0]_2 < 0$.

3. In this part of the proof we show that the set of the points which do not belong to $\sigma_{++}(U_2) \cup \rho(U_2)$ is discrete in $\psi(\Delta_1)$. Moreover we show, that for a suitable $\delta \in (0, 1)$

$$\Lambda := \left\{\mu \in \mathbb{C} \mid \mu = re^{i\vartheta}, e^{i\vartheta} \in \psi(\Delta_1), r \in (\delta, 1) \cup (1, \delta^{-1})\right\}$$

is contained in $\rho(U_1) \cap \rho(U_2)$. Obviously it is sufficient to prove the following: For every $\lambda \in \psi(\Delta_1)$ there exists a neighbourhood $\mathcal{U}(\lambda)$ of λ in $\mathbb{C}$ such that

$$\mathcal{U}(\lambda)\backslash\{\lambda\} \subset \sigma_{++}(U_2) \cup \rho(U_2).$$

For the convenience of the reader we repeat the proof of this fact from [18].

Assume the contrary. Then there exists a sequence $(\lambda_n) \subset \psi(\Delta_1) \cup (\psi(\Omega)\backslash\mathbb{T})$, $n = 1, 2 \dots$, with $\overline{\lambda}_n \neq \lambda_m^{-1}$ for $n \neq m$ such that $\lim_{n\to\infty} \lambda_n = \lambda$ and λ_n does not belong to $\sigma_{++}(U_2) \cup \rho(U_2)$. If, for some n, $\lambda_n \in \psi(\Delta_1)$, it follows from part 2 of the proof that λ_n is an eigenvalue of U_2 with at least one nonpositive eigenvector ϕ_n in $\mathcal{H}_2$. If $\lambda_n \in \psi(\Omega)\backslash\mathbb{T}$, then λ_n is a normal eigenvalue of U_2 with a $[\cdot,\cdot]_2$-neutral eigenvector ϕ_n. As $\overline{\lambda}_n \neq \lambda_m^{-1}$ if $n \neq m$, we have $[\phi_n, \phi_m]_2 = 0$. Then

$$\mathfrak{L} := \text{clsp}\ \{\phi_n \mid n = 1, 2, \dots\}$$

is a nonpositive invariant subspace of U_2.

We consider the operator $W := (U_2 - \lambda)|\mathfrak{L}$. As all $\lambda_n - \lambda$ are eigenvalues of W, W cannot have closed range and finite-dimensional kernel, since this would imply the existence of a neighbourhood of 0 which consists of eigenvalues of W (see [5]), a contradiction to the fact that λ is no inner point of $\sigma(U_2)$. We remark that W has closed range and finite-dimensional kernel if and only if there exists a subspace $\mathfrak{M} \subset \mathfrak{L}$ with $\operatorname{codim}_{\mathfrak{L}} \mathfrak{M} < \infty$, such that $W|\mathfrak{M}$ is an isomorphism of $\mathfrak{M}$ onto $\mathcal{R}(W|\mathfrak{M})$.

Suppose that $\mathcal{R}(W)$ is not closed or $\dim\ker W = \infty$. Then, for $\epsilon > 0$ and an arbitrary subspace $\mathfrak{M} \subset \mathfrak{L}$ with finite codimension in $\mathfrak{L}$ there exists an $f \in \mathfrak{M}$ such that $\|f\| = 1$ and $\|Wf\| < \epsilon$. Thus we can choose a $(\cdot,\cdot)$-orthonormal sequence $(f_n) \subset \mathfrak{L}$, $n = 1, 2 \ldots$, such that $\lim_{n\to\infty} \|(U_2 - \lambda)f_n\| = 0$. Then, by (2.7), $\lim_{n\to\infty} \|(U_1 - \lambda)f_n\| = 0$ and, since $\psi(\Delta_1) \subset \sigma_{++}(U_1) \cup \rho(U_1)$, we have $\liminf_{n\to\infty}[f_n, f_n]_1 > 0$. Then the claim in part 2 of the proof yields $\liminf_{n\to\infty}[f_n, f_n]_2 > 0$, a contradiction to $f_n \in \mathfrak{L}$.

4. Next we verify that U_1 and U_2 admit the functional calculus introduced in [7]. Furthermore, we find an open set Δ_2 and define a selfadjoint projection F_2 in $\mathcal{H}_2$ with the help of this functional calculus such that the conditions (i)-(iii) of the theorem with F_1, A_1, $\mathcal{H}_1$, Δ_1 replaced by F_2, A_2, $\mathcal{H}_2$, Δ_2 are fulfilled.

Arcs on the unit circle are denoted similarly to real intervals. For example, $(\tilde{a}, \tilde{b})$ denotes the open arc run over by a point moving from $\tilde{a}$ to $\tilde{b}$ in counterclockwise direction.

We choose a finite number of arcs (a_j, b_j), $j = 1, \ldots, n$, of $\mathbb{T}$ such that their closures $[a_j, b_j]$ are pairwise disjoint and for

$$\gamma := \bigcup_{j=1}^{n} (a_j, b_j)$$

the following holds:

(a) $\psi(\Delta_0) \subset \gamma$, $\overline{\gamma} \subset \psi(\Delta_1)$.
(b) The points a_j, b_j, $j = 1, \ldots, n$, belong to $\sigma_{++}(U_2) \cup \rho(U_2)$.
(c) Every component of $\psi(\Delta_1)$ contains exactly one of the arcs (a_j, b_j).

Further, we choose arcs $(a_j^{(1)}, a_j^{(2)}) \ni a_j$, $(b_j^{(1)}, b_j^{(2)}) \ni b_j$, $j = 1, \ldots, n$, such that their closures are pairwise disjoint and contained in $\psi(\Delta_1)$. In addition, we assume for the union

$$\gamma_0 := \bigcup_{j=1}^{n} (a_j^{(1)}, a_j^{(2)}) \cup (b_j^{(1)}, b_j^{(2)})$$

that $\overline{\gamma}_0 \cap \psi(\Delta_0) = \emptyset$ and

$$\overline{\gamma}_0 \subset (\sigma_{++}(U_1) \cup \rho(U_1)) \cap (\sigma_{++}(U_2) \cup \rho(U_2)) \,. \tag{2.12}$$

We connect every arc (a_j, b_j) by a smooth simple curve

$$\mathcal{C}_j \subset \rho(U_1) \cap \rho(U_2) \cap \psi(\Omega) \cap \mathbb{D}$$

with the point 0 such that $\mathcal{C}_j \cap \mathcal{C}_k = \{0\}$ for $j \neq k$. Then, making use of the fact that no point of $\psi(\Delta_1)$ is an accumulation point of $\sigma(U_2)\backslash\mathbb{T}$, which was proved in part 3 of the proof, we find an open neighbourhood O in $\overline{\mathbb{C}}$ of

$$\psi(\Delta_1) \cup \bigcup_{j=1}^{n} (\mathcal{C}_j \cup \widehat{\mathcal{C}}_j), \qquad \widehat{\mathcal{C}}_j := \{\overline{z}^{-1} : z \in \mathcal{C}_j\},$$

with the following properties:

(α) O is a $\mathbb{T}$-symmetric domain of $\overline{\mathbb{C}}$, $O \cap \mathbb{D}$ is simply connected.
(β) $O \cap \mathbb{T} = \psi(\Delta_1)$, $O \subset \psi(\Omega)$.
(γ) $O\backslash\mathbb{T} \subset \rho(U_1) \cap \rho(U_2)$.

Let $K := (\overline{\mathbb{C}}\backslash O) \cup (\mathbb{T}\backslash\gamma_0)$. Then we have $\mathbb{T}\backslash K = \gamma_0$.

By (2.12) and on account of [11, Theorem 3.18] there exists an $r_0 \in (0,1)$ such that

$$\sup\left\{\|(U_k - re^{i\varphi})^{-1}\|\,|1-|r|| : e^{i\varphi} \in \gamma_0, r \in (r_0,1) \cup (1, r_0^{-1})\right\} < \infty, \quad k = 1,2,$$

holds. Therefore, by [7, Proposition 1.2] the Riesz-Dunford functional calculi for U_1 and U_2 can be extended by continuity to A_K^2. Here A_K^2 is the space of all functions f defined on $\mathbb{T} \cup K$ such that $f|\mathbb{T} \in C^2(\mathbb{T})$ and f is locally holomorphic on K, with an inductive limit topology introduced in [7].

Let (ϵ_m), $m = 1,2,\ldots$, be a decreasing null sequence of positive numbers. Assume that

$$[a_j^{(2)}, b_j^{(1)}] \subset (a_j e^{i\epsilon_1}, b_j e^{-i\epsilon_1}), \quad j = 1,\ldots,n.$$

We set

$$\gamma_{(m)} := \bigcup_{j=1}^{n} (a_j e^{i\epsilon_m}, b_j e^{-i\epsilon_m}), \quad m = 1,2,\ldots.$$

Then $\gamma = \bigcup_{j=1}^{\infty} \gamma_{(m)}$. Let (χ_m), $m = 1,2,\ldots$, be a sequence of functions belonging to A_K^2 with the following properties:

(1) $\chi_m(z) = 0$ if $z \in (\mathbb{T} \cup K)\backslash\gamma_{(m)}$; $\chi_{m+1}(z) = 1$ if $z \in \gamma_{(m)}$, $m = 1,2,\ldots$.
(2) $0 \leq \chi_m(z) \leq 1$, $z \in \mathbb{T}$, $m = 1,2,\ldots$.

Since the functions χ_m are real on $\mathbb{T}$, the operators $\chi_m(U_2)$ are selfadjoint in $\mathcal{H}_2$. If $m > l$, $\chi_m(z) - \chi_l(z)$ is nonnegative for all $z \in \mathbb{T} \cup K$ and equal to zero outside γ_0. By (2.12) and in view of [7, Proposition 2.1] and [11, Theorem 3.18] the A_K^2 functional calculus restricted to functions with support in γ_0 is positive. Therefore,

$$[(\chi_m(U_2) - \chi_l(U_2))x, x]_2 \geq 0, \quad x \in \mathcal{H},\ m > l.$$

It is easy to see that the selfadjoint operators $\chi_m(U_2)$, $m = 1,2,\ldots$, in $\mathcal{H}_2$ are uniformly bounded. Hence the strong limit

$$s - \lim_{m\to\infty} \chi_m(U_2) =: F_2$$

exists, and F_2 is selfadjoint. Repeating the above reasoning with the functions χ_m replaced by their squares χ_m^2, we find that the strong limit $s - \lim_{m\to\infty} \chi_m^2(U_2)$ exists. It is equal to F_2^2. Since for every $m = 2, 3, \ldots,$ we have

$$\chi_m(z) \geq (\chi_m(z))^2 \geq \chi_{m-1}(z), \quad z \in \mathbb{T},$$

it follows that $F_2 = F_2^2$, that is, F_2 is a selfadjoint projection in $\mathcal{H}_2$. F_2 commutes with all bounded operators that commute with U_2 since this is true for all operators $\chi_m(U_2)$, $m = 1, 2, \ldots$. Hence by (2.6) F_2 satisfies condition (i) with A_1 replaced by A_2.

We have

$$\sigma(U_2|F_2\mathcal{H}_2) \subset \sigma(U_2) \cap \overline{\gamma}. \tag{2.13}$$

Indeed, let $\mu \notin \overline{\gamma}$ and $g \in A_K^2$ equal to one on a neighbourhood of $\overline{\gamma}$ such that $h : z \mapsto (z-\mu)^{-1}g(z)$ belongs to A_K^2. Then the restriction of $h(U_2)$ to $F_2\mathcal{H}_2$ is the bounded inverse of $(U_2 - \mu)|F_2\mathcal{H}_2$. In a similar way one verifies that

$$\sigma(U_2|(1 - F_2)\mathcal{H}_2) \subset \sigma(U_2)\backslash\gamma. \tag{2.14}$$

We set

$$\Delta_2 := \varphi(\gamma).$$

Then the relations (2.13) and (2.14) imply (ii) and (iii) with A_1, F_1, Δ_1 replaced by A_2, F_2, Δ_2. Note that $\Delta_0 \subset \Delta_2$.

5. In order to prove that F_2 defined in part 4 of the proof satisfies condition (iv) we consider a function $\chi \in A_K^2$ with supp $\chi \subset \psi(\Delta_1)$ which is equal to one in some neighbourhood of $\overline{\gamma}$. It is not difficult to see that one can approximate χ in A_K^2 by a sequence of locally holomorphic functions on $\sigma(U_1)$ which uniformly converges to zero in some neighbourhood of $\sigma(U_1)\backslash\psi(\Delta_1)$. Then by (2.8) we have

$$\chi(U_1) = F_1\chi(U_1). \tag{2.15}$$

By (2.7)

$$(U_1 - \lambda)^{-1} - (U_2 - \lambda)^{-1} = (U_2 - \lambda)^{-1}(U_2 - U_1)(U_1 - \lambda)^{-1} \in \mathfrak{S}_\infty.$$

Hence for every function $\widetilde{\chi}$ which is locally holomorphic on $\sigma(U_1) \cup \sigma(U_2)$ we have $\widetilde{\chi}(U_1) - \widetilde{\chi}(U_2) \in \mathfrak{S}_\infty$. On account of the continuity of the A_K^2 functional calculus with respect to the operator norm we find

$$\chi(U_1) - \chi(U_2) \in \mathfrak{S}_\infty. \tag{2.16}$$

Then by condition (iv), (2.15) and (2.16),

$$\begin{aligned}(G_1 - G_2)F_2 &= (G_1 - G_2)\chi(U_2)F_2 = \\ &= (G_1 - G_2)(\chi(U_2) - \chi(U_1))F_2 + (G_1 - G_2)F_1\chi(U_1)F_2 \in \mathfrak{S}_\infty.\end{aligned}$$

6. It remains to prove that $(F_2\mathcal{H}_2, [\cdot,\cdot]_2)$ is a Pontryagin space with finite rank of negativity.

We choose $\alpha_j, \beta_j \in \mathbb{T}$, $j = 1, 2, \dots, n$, such that $\alpha_j \in (a_j^{(1)}, a_j)$, $\beta_j \in (b_j, b_j^{(2)})$, $j = 1, 2, \dots, n$. We set

$$\gamma' := \bigcup_{j=1}^{n} (\alpha_j, \beta_j)$$

and define a function f by

$$f(z) := \begin{cases} \prod_{j=1}^{n} (z-\alpha_j)^2 \left(\frac{1}{z} - \frac{1}{\alpha_j}\right)^2 (z-\beta_j)^2 \left(\frac{1}{z} - \frac{1}{\beta_j}\right)^2 & \text{if } z \in \gamma' \\ 0 & \text{if } z \in (K \cup \mathbb{T}) \backslash \gamma' \end{cases}.$$

This function is locally holomorphic on K. We have $f|\mathbb{T} \in C^3(\mathbb{T})$, and f is positive on γ'. Therefore, $f \in A_K^2$ and it follows as in part 5 of the proof that

$$f(U_1) - f(U_2) \in \mathfrak{S}_\infty. \tag{2.17}$$

The restriction $U_1|F_1\mathcal{H}_1$ is unitary in the Hilbert space $(F_1\mathcal{H}_1, [\cdot,\cdot]_1)$. Let $F_1(\gamma')$ be the spectral projection corresponding to $U_1|F_1\mathcal{H}_1$ and γ'. Since f can be approximated in A_K^2 by a sequence of functions locally holomorphic on $\mathbb{T} \cup K$ which on a neighbourhood of $\sigma(U_1)\backslash\psi(\Delta_1)$ uniformly converges to zero, we have $f(U_1) = f(U_1)F_1$. The restriction of $f(U_1)$ to $F_1\mathcal{H}_1$ coincides with $f(U_1|F_1\mathcal{H}_1)$; and by the functional calculus for unitary operators in Hilbert space we have

$$f(U_1|F_1\mathcal{H}) = f(U_1|F_1\mathcal{H})F_1(\gamma').$$

Therefore the operator $f(U_1)$ can be written as

$$f(U_1) = \begin{pmatrix} f(U_{1,\gamma'}) & 0 \\ 0 & 0 \end{pmatrix} \tag{2.18}$$

with respect to the decomposition $\mathcal{H}_1 = \mathcal{H}_1'[\dot{+}]\mathcal{H}_1''$, where

$$\mathcal{H}_1' := F_1(\gamma')F_1\mathcal{H}_1, \;\; \mathcal{H}_1'' := ((1 - F_1(\gamma'))F_1 + (1 - F_1))\mathcal{H}_1,$$

and $U_{1,\gamma'}$ is the restriction of U_1 to the Hilbert space $(\mathcal{H}_1', [\cdot,\cdot]_1)$.

If J_1'' is a fundamental symmetry of the Krein space $(\mathcal{H}_1'', [\cdot,\cdot]_1)$ and we define $J_1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & J_1'' \end{smallmatrix}\right)$, then

$$(x, y)^\sim := [J_1 x, y]_1, \qquad x, y \in \mathcal{H},$$

is a Hilbert scalar product on $\mathcal{H}$. The Gram operators of $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ with respect to $(\cdot,\cdot)^\sim$ are J_1 and $\widetilde{G}_2 := J_1 G_1^{-1} G_2$, respectively. Then by

$$(J_1 - \widetilde{G}_2)F_1 = -J_1 G_1^{-1}(G_2 - G_1)F_1 \in \mathfrak{S}_\infty,$$

$(J_1 - \widetilde{G}_2)F_1(\gamma')F_1$ is compact. If

$$\begin{pmatrix} \widetilde{G}_{2,11} & \widetilde{G}_{2,12} \\ \widetilde{G}_{2,21} & \widetilde{G}_{2,22} \end{pmatrix}$$

is the operator matrix corresponding to $\widetilde{G}_2$ with respect to the decomposition $\mathcal{H}_1 = \mathcal{H}_1'[\dot{+}]\mathcal{H}_1''$, then

$$(J_1 - \widetilde{G}_2)F_1(\gamma')F_1 = \begin{pmatrix} 1 - \widetilde{G}_{2,11} & 0 \\ -\widetilde{G}_{2,21} & 0 \end{pmatrix} \in \mathfrak{S}_\infty$$

and also $\widetilde{G}_{2,12}$, which is the adjoint of $\widetilde{G}_{2,21}$ with respect to $(\cdot,\cdot)^\sim$, is compact. Therefore,

$$\begin{pmatrix} \widetilde{G}_{2,11} & \widetilde{G}_{2,12} \\ \widetilde{G}_{2,21} & \widetilde{G}_{2,22} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G}_{2,22} \end{pmatrix} = \begin{pmatrix} \widetilde{G}_{2,11} - 1 & \widetilde{G}_{2,12} \\ \widetilde{G}_{2,21} & 0 \end{pmatrix} \in \mathfrak{S}_\infty \tag{2.19}$$

and, on account of $0 \in \rho(\widetilde{G}_2)$, we have $0 \in \rho(\widetilde{G}_{2,22}) \cup \sigma_{p,norm}(\widetilde{G}_{2,22})$. Let P_0 be the finite-rank orthogonal projection on ker $\widetilde{G}_{2,22}$ in $(\mathcal{H}_1'', (\cdot,\cdot)^\sim)$. We introduce a new inner product in $\mathcal{H}$ by

$$\left[\begin{pmatrix} x' \\ x'' \end{pmatrix}, \begin{pmatrix} y' \\ y'' \end{pmatrix}\right]_3 := \left(\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G}_{2,22} + P_0 \end{pmatrix}\begin{pmatrix} x' \\ x'' \end{pmatrix}, \begin{pmatrix} y' \\ y'' \end{pmatrix}\right)^\sim, \; x', y' \in \mathcal{H}_1'; x'', y'' \in \mathcal{H}_1''.$$

Since $0 \in \rho(\widetilde{G}_{2,22} + P_0)$, $[\cdot,\cdot]_3$ is a Krein space inner product. By (2.19), the difference of the Gram operators of $[\cdot,\cdot]_2$ and $[\cdot,\cdot]_3$ with respect to $(\cdot,\cdot)^\sim$ is compact. By (2.18) the operator $f(U_1)$ is selfadjoint in $(\mathcal{H}, [\cdot,\cdot]_3)$.

If $\mathcal{L}_-$ is a maximal uniformly negative subspace of $\big(\mathcal{H}_1'', ((\widetilde{G}_{2,22} + P_0)\ \cdot, \cdot)\big)$, then

$$\left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \;\middle|\; x \in \mathcal{L}_- \right\}$$

is a maximal uniformly negative invariant subspace of $f(U_1)$ in $(\mathcal{H}, [\cdot,\cdot]_3)$. Then, in view of (2.17) and (2.18), we can apply the invariant subspace result [16, Theorem 6] of H. Langer: The selfadjoint operator $f(U_2)$ in $\mathcal{H}_2$ has a maximal nonpositive invariant subspace $\mathcal{M}_-$, such that

$$\sigma_{ess}(f(U_2)|\mathcal{M}_-) = \{0\}. \tag{2.20}$$

By (2.18) the spectrum of the selfadjoint operator $f(U_1)$ in $(\mathcal{H}, [\cdot,\cdot]_3)$ is contained in the real interval $[0, M_1]$, $M_1 := \max_{z \in \mathbb{T}} f(z)$. Moreover,

$$(0, M_1] \subset \sigma_{++}(f(U_1)) \cup \rho(f(U_1)) \tag{2.21}$$

with respect to $[\cdot,\cdot]_3$. As in the proof of relation (2.13) we see that the spectrum of the selfadjoint operator $f(U_2)$ in $\mathcal{H}_2$ is also contained in $[0, M_1]$, and that

$$\sigma(f(U_2)|F_2\mathcal{H}) \subset [M_0, M_1], \tag{2.22}$$

where $M_0 := \inf\{f(z) : z \in \gamma\}$. By the definition of γ' and f we have $M_0 > 0$. Since the difference of the Gram operators of $[\cdot,\cdot]_3$ and $[\cdot,\cdot]_2$ is compact and the relations (2.21) and (2.17) hold, we find, as in parts 2 and 3 of the proof, that there is a $t_0 \in (0, M_0)$ with $t_0 \in \sigma_{++}(f(U_2)) \cup \rho(f(U_2))$. Let E_0 be the spectral projection corresponding to $f(U_2)$ and the interval $(t_0, M_1 + 1)$, which can be constructed in the same way as F_2 (see part 4).

We claim that $F_2\mathcal{H}_2 \subset E_0\mathcal{H}_2$. Indeed, since F_2 and E_0 both commute with U_2, F_2 and E_0 commute. Therefore,

$$F_2\mathcal{H}_2 = E_0F_2\mathcal{H}_2[\dot{+}](1 - E_0)F_2\mathcal{H}_2.$$

Writing $1 - E_0$ as a strong limit of Riesz-Dunford integrals,

$$\begin{aligned} 1 - E_0 = s-\lim_{\delta\searrow 0}\ s-\lim_{\epsilon\searrow 0} \frac{1}{2\pi i}\int_{-1}^{t_0+\delta} &\{(f(U_2) - (t + i\epsilon))^{-1} \\ &- (f(U_2) - (t - i\epsilon))^{-1}\}\, dt, \end{aligned} \tag{2.23}$$

and making use of (2.22) we see that $(1 - E_0)F_2 = 0$, that is $F_2\mathcal{H}_2 \subset E_0\mathcal{H}_2$.

To prove that $(F_2\mathcal{H}_2, [\cdot,\cdot]_2)$ is a Pontryagin space with finite rank of negativity, it is sufficient to show this for $(E_0\mathcal{H}_2, [\cdot,\cdot]_2)$. We make use of the maximal nonpositive $f(U_2)$-invariant subspace $\mathcal{M}_-$. Since, in view of (2.23), E_0 maps $\mathcal{M}_-$ into itself, we have

$$\mathcal{M}_- = E_0\mathcal{M}_-[\dot{+}](1 - E_0)\mathcal{M}_-.$$

This implies that $E_0\mathcal{M}_-$ is a maximal nonpositive subspace of $E_0\mathcal{H}_2$.

Let $E(t_0)$ be the orthogonal projection in $\mathcal{H}_2$ on the eigenspace of $f(U_2)$ corresponding to t_0. $E(t_0)$ can be constructed in a similar way as E_0 and F_2. Therefore, $E(t_0)$ maps $\mathcal{M}_-$ into itself. Since t_0 is not an eigenvalue of $f(U_2)|\mathcal{M}_-$, we have $E(t_0)\mathcal{M}_- = \{0\}$. It follows that, for $x \in \mathcal{M}_-$, E_0x can be written in the form

$$E_0x = -\frac{1}{2\pi i}\int_{\mathcal{C}} (f(U_2) - \lambda)^{-1}\, d\lambda\, x,$$

where $\mathcal{C}$ is the boundary of

$$\{s_1 + is_2 : s_1 \in (t_0, M_1 + 1), s_2 \in (-1, 1)\}$$

and the integral is understood in the sense of principal value. Since for $\lambda \neq \overline{\lambda}$ the operator $(f(U_2) - \lambda)^{-1}|\mathcal{M}_-$ coincides with $((f(U_2)|\mathcal{M}_-) - \lambda)^{-1}$ and t_0 belongs to $\rho(f(U_2)|\mathcal{M}_-)$, E_0 restricted to $\mathcal{M}_-$ coincides with the Riesz-Dunford projection corresponding to $f(U_2)|\mathcal{M}_-$ and the set $(t_0, M_1 + 1) \cap \sigma(f(U_2)|\mathcal{M}_-)$. By (2.20) this Riesz-Dunford projection is of finite rank: its range $E_0\mathcal{M}_-$ is the span of a finite number of finite-dimensional algebraic eigenspaces of $f(U_2)|\mathcal{M}_-$, that is, $\dim E_0\mathcal{M}_- < \infty$. It follows that $E_0\mathcal{H}_2$ is a Pontryagin space with finite rank of negativity. This completes the proof of Theorem 2.4.

□

In the case when A_1 and A_2 are unbounded operators we have the following corollary of Theorem 2.4.

Corollary 2.6. *Let A_1 and A_2 be selfadjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that condition* (I) *holds. Assume that there exists a selfadjoint projection E in $\mathcal{H}_1$ such that $E\mathcal{H}_1$ is a Pontryagin space with finite rank of negativity, and E reduces A_1, i.e., $EA_1 \subset A_1E$, and let the following conditions* (i′) *and* (ii′) *be fulfilled.*

(i′) $\widetilde{\sigma}(A_1|(1-E)\mathcal{H}_1) \cap \Omega = \emptyset$.
(ii′) $(G_1 - G_2)E \in \mathfrak{S}_\infty$.

Then, for every $\mathbb{R}$-symmetric domain Ω' with $\Omega' \cap \overline{\mathbb{R}} \neq \emptyset$ and $\overline{\Omega}' \subset \Omega$, there exists a selfadjoint projection F in $\mathcal{H}_2$ such that $F\mathcal{H}_2$ is a Pontryagin space with finite rank of negativity, F reduces A_2, and the following holds.

(a) $\widetilde{\sigma}(A_2|F\mathcal{H}_2) \subset \Omega$.
(b) $\widetilde{\sigma}(A_2|(1-F)\mathcal{H}_2) \cap \Omega' = \emptyset$.
(c) $(G_1 - G_2)F \in \mathfrak{S}_\infty$.

Proof. Since $A_1|E\mathcal{H}_1$ is a selfadjoint operator in the Pontryagin space $(E\mathcal{H}_1, [\cdot,\cdot]_1)$, $\widetilde{\sigma}(A_1|E\mathcal{H}_1) \cap (\Omega\backslash\overline{\mathbb{R}})$ consists of at most finitely many normal eigenvalues of A_1. By this fact and (i′) condition (II) of Theorem 2.4 is fulfilled. For every finite union Δ_1 of open connected subsets of $\Omega \cap \overline{\mathbb{R}}$, $\overline{\Delta_1} \subset \Omega \cap \overline{\mathbb{R}}$, such that the boundary points of Δ_1 in $\overline{\mathbb{R}}$ are no critical points of $A_1|E\mathcal{H}_1$ the spectral projection

$$E(\Delta_1, A_1|E\mathcal{H}_1) \in \mathcal{L}(E\mathcal{H}_1)$$

is defined, and the selfadjoint projection $E(\Delta_1, A_1|E\mathcal{H}_1)E$ in $\mathcal{H}_1$ fulfils the conditions of Theorem 2.4. In particular, the assumption (ii′) implies

$$(G_1 - G_2)E(\Delta_1, A|E\mathcal{H}_1)E \in \mathfrak{S}_\infty.$$

Then by Theorem 2.4 there exists a finite union Δ_2 of open connected subsets of $\Omega \cap \overline{\mathbb{R}}$ such that $\Omega' \cap \overline{\mathbb{R}} \subset \Delta_2$, $\overline{\Delta}_2 \subset \Omega \cap \overline{\mathbb{R}}$, and a selfadjoint projection F_2 in $\mathcal{H}_2$ such that the conclusion of Theorem 2.4 holds. By Theorem 2.4 and the remark following condition (II) the set $\widetilde{\sigma}(A_2) \cap (\Omega'\backslash\overline{\mathbb{R}})$ consists of at most finitely many normal eigenvalues of A_2. The Riesz-Dunford projection F_0 corresponding to A_2 and $\widetilde{\sigma}(A_2) \cap (\Omega'\backslash\overline{\mathbb{R}})$ has finite rank. Then the range of

$$F := F_2 + F_0$$

is a Pontryagin space with finite rank of negativity and F fulfils (a), (b) and (c). □

3. Perturbations preserving the nonnegativity with respect to the indefinite inner product over a neighbourhood of infinity

3.1. Selfadjoint operators nonnegative over a neighbourhood of ∞ and associated forms

In this section we consider a selfadjoint operator A in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ which is an orthogonal direct sum of some bounded selfadjoint operator A_0 and some nonnegative selfadjoint operator A_∞ with $\rho(A_\infty) \neq \emptyset$, and a class of, in general, unbounded perturbations of A which preserve this decomposition property. For such an operator A we have

$$(r_\sigma(A_0), \infty) \subset \sigma_{++}(A) \cup \rho(A) \quad \text{and} \quad (-\infty, -r_\sigma(A_0)) \subset \sigma_{--}(A) \cup \rho(A),$$

where $r_\sigma(A_0)$ denotes the spectral radius of A_0, and a domain $\Omega \subset \overline{\mathbb{C}}$ with $\infty \in \Omega$ which satisfies the assumptions of Theorem 2.4 may not exist. Moreover, simple examples show that the perturbations considered in Theorem 2.4, in general, do not preserve this decomposition property of A.

We recall that a selfadjoint operator B in $(\mathcal{H}, [\cdot,\cdot])$ is said to have l *negative squares* if the the form $[B\cdot,\cdot]$ on $\mathcal{D}(B)$ has l negative squares.

Definition 3.1. Let $\mathcal{U}_\infty$ be an $\mathbb{R}$-symmetric simply connected domain of $\overline{\mathbb{C}}$ with $\infty \in \mathcal{U}_\infty$ and $0 \notin \overline{\mathcal{U}}_\infty$. We say that the selfadjoint operator A in $(\mathcal{H}, [\cdot,\cdot])$ is *nonnegative* (has a *finite number of negative squares*) *over* $\mathcal{U}_\infty$ if there exists a selfadjoint projection E_∞ such that A can be written as a diagonal operator matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_\infty \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = (1-E_\infty)\mathcal{H}[\dot{+}]E_\infty\mathcal{H}$, where A_0 is a bounded selfadjoint operator in $((1-E_\infty)\mathcal{H}, [\cdot,\cdot])$ with $\sigma(A_0) \subset \overline{\mathbb{C}}\backslash\mathcal{U}_\infty$ and A_∞ is a nonnegative operator (resp. an operator with a finite number of negative squares) in $E_\infty\mathcal{H}$ with $0 \in \rho(A_\infty)$.

For A nonnegative over $\mathcal{U}_\infty$, and E_∞, A_∞ as in Definition 3.1 and every bounded interval $\Delta \subset \mathcal{U}_\infty \cap \mathbb{R}$, we define

$$E(\Delta, A) := \begin{pmatrix} 0 & 0 \\ 0 & E(\Delta, A_\infty) \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = (1-E_\infty)\mathcal{H}[\dot{+}]E_\infty\mathcal{H}$, where $E(\cdot, A_\infty)$ is the spectral function of A_∞. It is easy to see that $E(\cdot, A)$ is the uniquely determined local spectral function of A on $\mathcal{U}_\infty \cap \mathbb{R}$ with the usual properties (see e.g. [11]). We shall say that $E(\cdot, A)$ is *bounded at* ∞ if

$$\sup\{\|E(\Delta, A)\| : \Delta \text{ compact interval}, \Delta \subset \mathcal{U}_\infty \cap \mathbb{R}\} < \infty. \tag{3.1}$$

This holds if and only if there is a Hilbert scalar product $(\cdot,\cdot)_\infty$ on $E_\infty\mathcal{H}$ equivalent to $(\cdot,\cdot)$ such that A_∞ is selfadjoint in $(E_\infty\mathcal{H}, (\cdot,\cdot)_\infty)$. In the notation of [9] and other papers the property (3.1) is expressed by saying that ∞ is not a singular critical point of A, $\infty \notin c_s(A)$. Here we do not introduce the set of critical points of a locally definitizable operator.

If A has a finite number of negative squares over $\mathcal{U}_\infty$ and if E_∞, A_0 and A_∞ are as in Definition 3.1, then A_∞ is definitizable (see e.g. [17]) and, hence, has a spectral function $E(\cdot, A_\infty)$. Moreover, there exists an $s > 0$ such that

$$\{z \in \mathbb{C} : |z| \geq s\} \subset \mathcal{U}_\infty$$

and $E(\overline{\mathbb{R}}\backslash[-s, s], A_\infty)$ is defined, $A_\infty|E(\overline{\mathbb{R}}\backslash[-s, s], A_\infty)E_\infty\mathcal{H}$ is nonnegative and the spectrum of $A_\infty|(1 - E(\overline{\mathbb{R}}\backslash[-s, s], A_\infty))E_\infty\mathcal{H}$ is contained in $\{z \in \mathbb{C} : |z| \leq s\}$. Therefore, A is nonnegative over

$$\mathcal{U}_\infty(s) := \{z \in \mathbb{C} : |z| > s\} \cup \{\infty\}. \tag{3.2}$$

Let A be nonnegative over some neighbourhood of ∞ and let G be the Gram operator of the Krein inner product $[\cdot, \cdot]$ with respect to the fixed Hilbert scalar product $(\cdot, \cdot)$, $(Gx, y) = [x, y]$, $x, y \in \mathcal{H}$. Then $H := GA$ is selfadjoint in $(\mathcal{H}, (\cdot, \cdot))$ and, since there is a decomposition $A = AE_\infty + A(1 - E_\infty)$ as in Definition 3.1, bounded from below. Then, for

$$c > c(H) := \inf\, \{c \in \mathbb{R} : ((H + c)x, x) \geq 0 \text{ for all } x \in \mathcal{D}(H)\},$$

we have

$$((H + c)x, x) \geq (c - c(H))(x, x)$$

for all $x \in \mathcal{D}(H)$. Evidently, for two different $c_1, c_2 > c(H)$ the corresponding scalar products

$$(x, y)_{\frac{1}{2},c_j} := ((H + c_j)x, y), \quad x, y \in \mathcal{D}(H), \;\; j = 1, 2,$$

are equivalent. We denote by $\mathcal{D}[H]$ the completion of $\mathcal{D}(H)$ with respect to $(\cdot, \cdot)_{\frac{1}{2},c}$ for some $c > c(H)$. As the scalar products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\frac{1}{2},c}$ are coordinated, $\mathcal{D}[H]$ can be considered as a linear subspace of $\mathcal{H}$. $\mathcal{D}[H]$ equipped with the extension of the scalar product $(\cdot, \cdot)_{\frac{1}{2},c}$ is a Hilbert space. If we regard $\mathcal{D}[H]$ as a Hilbertable topological linear space, then for given A and $[\cdot, \cdot]$ the space $\mathcal{D}[H]$ does not depend on the choice of the Hilbert scalar product $(\cdot, \cdot)$. We define $\mathcal{D}[A] := \mathcal{D}[H]$.

We associate with A the extension $\mathfrak{a}$ of the form $[A\cdot, \cdot]$ to $\mathcal{D}[A]$. $\mathfrak{a}$ is a densely defined closed symmetric sesquilinear form bounded from below, and, evidently, it coincides with the form usually associated with the semibounded operator H in the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$. We have

$$c(H) = \inf\, \{c \in \mathbb{R} : \mathfrak{a}[x, x] + c(x, x) \geq 0 \text{ for all } x \in \mathcal{D}[A]\}.$$

On the other hand, let $\mathfrak{t}$ be a densely defined closed symmetric sesquilinear form bounded from below with domain $\mathcal{D}(\mathfrak{t})$ in the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$. Then there exists a uniquely determined selfadjoint operator T in $(\mathcal{H}, (\cdot, \cdot))$ such that $\mathcal{D}(T) \subset \mathcal{D}(\mathfrak{t})$ and $(Tu, v) = \mathfrak{t}[u, v]$ for every $u \in \mathcal{D}(T)$ and every $v \in \mathcal{D}(\mathfrak{t})$ (see [13, Theorem VI.2.1]). Let again G be the Gram operator of $[\cdot, \cdot]$ with respect to $(\cdot, \cdot)$. Then $S = G^{-1}T$ is the uniquely determined selfadjoint operator in $(\mathcal{H}, [\cdot, \cdot])$ such that $\mathcal{D}(S) \subset \mathcal{D}(\mathfrak{t})$ and $[Su, v] = \mathfrak{t}[u, v]$ for every $u \in \mathcal{D}(S)$ and every $v \in \mathcal{D}(\mathfrak{t})$. S will be called the selfadjoint operator *associated with* $\mathfrak{t}$ *in* $(\mathcal{H}, [\cdot, \cdot])$ (cf. [2, §2]).

We fix some $\gamma > \inf\{\alpha \in \mathbb{R} : \mathfrak{t}[x, x] + \alpha(x, x) \geq 0 \text{ for all } x \in \mathcal{D}(\mathfrak{t})\}$. Then $\mathcal{D}(\mathfrak{t})$ equipped with the inner product

$$(x, y)_{\mathfrak{t}} := \mathfrak{t}[x, y] + \gamma(x, y), \quad x, y \in \mathcal{D}(\mathfrak{t}), \tag{3.3}$$

is a Hilbert space which will be denoted by $\mathcal{H}_{\mathfrak{t}}$ (see [13, VI.§1.3]). Let $\mathcal{H}_{\mathfrak{t},-}$ be the completion of $\mathcal{H}$ with respect to the quadratic norm

$$\|x\|_{\mathfrak{a},-} = \sup\{|[x, y]| : y \in \mathcal{H}_{\mathfrak{t}}, (y, y)_{\mathfrak{t}} \leq 1\}, \quad x \in \mathcal{H}. \tag{3.4}$$

The form $[\cdot, \cdot]$ can be extended by continuity to $\mathcal{H}_{\mathfrak{t}} \times \mathcal{H}_{\mathfrak{t},-}$ and to $\mathcal{H}_{\mathfrak{t},-} \times \mathcal{H}_{\mathfrak{t}}$. This extended form will also be denoted by $[\cdot, \cdot]$. Moreover, for every $y \in \mathcal{H}_{\mathfrak{t}}$ there is an element $z \in \mathcal{H}_{\mathfrak{t},-}$ such that

$$(x, y)_{\mathfrak{t}} = [x, z] \tag{3.5}$$

holds for all $x \in \mathcal{H}_{\mathfrak{t}}$ (see [10]). That is, $\mathcal{H}_{\mathfrak{t},-}$ is the dual space of $\mathcal{H}_{\mathfrak{t}}$ with respect to the duality $[\cdot, \cdot]$. The linear mapping

$$\iota : \mathcal{H}_{\mathfrak{t}} \ni y \mapsto z \in \mathcal{H}_{\mathfrak{t},-} \tag{3.6}$$

defined by (3.5) is an isometric isomorphism of $\mathcal{H}_{\mathfrak{t}}$ onto $\mathcal{H}_{\mathfrak{t},-}$. If S is the selfadjoint operator associated with $\mathfrak{t}$ in $(\mathcal{H}, [\cdot, \cdot])$, then S can be extended by continuity to an operator $\widetilde{S} \in \mathcal{L}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$ such that the relation $[\widetilde{S}x, y] = \mathfrak{t}[x, y]$ holds for all $x, y \in \mathcal{H}_{\mathfrak{t}}$ (see [10]).

If A and $\mathfrak{a}$ are as above, then A is the selfadjoint operator in $(\mathcal{H}, [\cdot, \cdot])$ associated with $\mathfrak{a}$, $\mathcal{D}(\mathfrak{a}) = \mathcal{D}[A]$ and $(\cdot, \cdot)_{\mathfrak{a}}$, defined as in (3.3) with $\gamma > \inf\{\alpha \in \mathbb{R} : \mathfrak{a}[x, x] + \alpha(x, x) \geq 0 \text{ for all } x \in \mathcal{D}(\mathfrak{a})\}$ fixed, coincides with the extension of $(\cdot, \cdot)_{\frac{1}{2},\gamma}$ to $\mathcal{D}(\mathfrak{a})$.

Definition 3.2. A sesquilinear form $\mathfrak{v}$ (not necessarily symmetric) in $\mathcal{H}$ is said to be *relatively compact* (*relatively* $\mathfrak{S}_p$, $1 \leq p < \infty$) *with respect to* $\mathfrak{t}$ if $\mathcal{D}(\mathfrak{v}) \supset \mathcal{D}(\mathfrak{t})$, $\mathfrak{v}$ is continuous on $\mathcal{H}_{\mathfrak{t}}$ and the operator $\mathcal{V}$ defined by

$$\mathfrak{v}[x, y] = (\mathcal{V}x, y)_{\mathfrak{t}}, \quad x, y \in \mathcal{H}_{\mathfrak{t}},$$

is compact (resp. belongs to the class $\mathfrak{S}_p(\mathcal{H}_{\mathfrak{t}})$).

Let $\mathfrak{v}$ and $\mathfrak{t}$ be as in Definition 3.2. We remark that the condition $\mathcal{V} \in \mathfrak{S}_\infty(\mathcal{H}_{\mathfrak{t}})$ (resp. $\mathcal{V} \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{t}})$) does not depend on the choice of the constant γ in the definition of $(\cdot, \cdot)_{\mathfrak{t}}$. If ι is the mapping defined in (3.6), $V := \iota\mathcal{V}$ belongs to the class $\mathfrak{S}_\infty(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$ (resp. $\mathfrak{S}_p(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$). We have $\mathfrak{v}[x, y] = [Vx, y]$ for all $x, y \in \mathcal{H}_{\mathfrak{t}}$. Since V can be approximated in $\mathcal{L}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$ by operators of finite rank, and $\mathcal{H}$ is dense in $\mathcal{H}_{\mathfrak{t},-}$ the $\mathfrak{t}$-bound of $\mathfrak{v}$ is zero ([13, VI.§1.6]). Therefore $\mathfrak{t}+\mathfrak{v}$, $\mathcal{D}(\mathfrak{t}+\mathfrak{v}) = \mathcal{D}(\mathfrak{t})$, is a closed sectorial sesquilinear form (see [13, Theorem VI.1.33]), and we have

$$(\mathfrak{t} + \mathfrak{v})[x, y] = [(\widetilde{S} + V)x, y], \quad x, y \in \mathcal{H}_{\mathfrak{t}}.$$

By [13, Theorem VI.2.1] there exists a uniquely determined closed operator Q in $(\mathcal{H}, [\cdot,\cdot])$ such that GQ is sectorial and maximal quasi-accretive in $(\mathcal{H}, (\cdot,\cdot))$, $\mathcal{D}(Q) \subset \mathcal{D}(\mathfrak{t}+\mathfrak{v})$ and

$$[Qx, y] = (GQx, y) = (\mathfrak{t}+\mathfrak{v})[x, y]$$

for every $x \in \mathcal{D}(Q)$ and $y \in \mathcal{H}_{\mathfrak{t}}$. Therefore,

$$[Qx, y] = [(\widetilde{S} + V)x, y], \quad x \in \mathcal{D}(Q), y \in \mathcal{H}_{\mathfrak{t}},$$

and, hence, $Qx = (\widetilde{S}+V)x$ for all $x \in \mathcal{D}(Q)$. If $S \stackrel{+}{\smile} V$ denotes the range restriction of $\widetilde{S} + V \in \mathcal{L}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$ to $\mathcal{H}$, i.e.

$$\begin{aligned} \mathcal{D}(S \stackrel{+}{\smile} V) &= \{x \in \mathcal{H}_{\mathfrak{t}} | (\widetilde{S} + V)x \in \mathcal{H}\}, \\ S \stackrel{+}{\smile} V &= (\widetilde{S} + V)|\mathcal{D}(S \stackrel{+}{\smile} V), \end{aligned} \tag{3.7}$$

we have $Q \subset S \stackrel{+}{\smile} V$. Since $G(S \stackrel{+}{\smile} V)$ is quasi-accretive and GQ is maximal quasi-accretive, we find $Q = S \stackrel{+}{\smile} V$.

In the case of a nonempty resolvent set of S the notation (3.7) for the range restriction was used, e.g., in [9] for a more general class of perturbations of S.

3.2. A consequence of Krein's lemma

The following lemma will be used in Section 3.3. It is a simple consequence of Krein's lemma (see [14]).

Lemma 3.3. *Let $(\mathcal{B}, (\cdot,\cdot))$ be a Hilbert space which is densely and continuously embedded into a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that there exists a positive bounded and boundedly invertible operator W in $\mathcal{H}$ such that $W\mathcal{B} \subset \mathcal{B}$. If $T \in \mathfrak{S}_p(\mathcal{B})$ for some $p \in [1,\infty)$ and*

$$[Tx, y] = [x, Ty], \quad x, y \in \mathcal{B}, \tag{3.8}$$

then T can be extended to an operator $\widetilde{T} \in \mathfrak{S}_p(\mathcal{H})$.

Proof. If we define $\langle x, y\rangle := [W^{-1}x, y]$, $x, y \in \mathcal{H}$, then, by the assumptions on W, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Since by the closed graph theorem $W|\mathcal{B}$ is a bounded operator in $\mathcal{B}$, we have $WT \in \mathfrak{S}_p(\mathcal{B})$. Moreover, by (3.8)

$$\langle (WT)x, y\rangle = \langle x, (WT)y\rangle, \quad x, y \in \mathcal{B}.$$

Then by Krein's Lemma ([14, Theorem 1]) WT can be extended by continuity to a bounded selfadjoint operator $(WT)^\sim$ in $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. By [14, Theorem 3] the operator $(WT)^\sim$ belongs to $\mathfrak{S}_p(\mathcal{H})$, and $\sigma(WT) = \sigma((WT)^\sim)$ holds. Moreover, every $\lambda \in \sigma(WT)\backslash\{0\}$ is a semisimple eigenvalue of WT and $\ker(WT-\lambda)$ and $\ker((WT)^\sim - \lambda)$ coincide.

If $\lambda_{+,i}$ $(\lambda_{-,j})$ denote the positive (resp. negative) eigenvalues of WT or $(WT)^\sim$ and we assume that $\lambda_{+,i+1} \le \lambda_{+,i}$ and $\lambda_{-,j} \le \lambda_{-,j+1}$ where multiplicity of the eigenvalues is counted, then by [6, III.§7.3] we have

$$\sum_i |\lambda_{+,i}|^p + \sum_j |\lambda_{-,j}|^p \leq \|WT\|^p_{\mathfrak{S}_p(\mathcal{B})} < \infty.$$

In view of the selfadjointness of $(WT)^\sim$ this implies $(WT)^\sim = W\widetilde{T} \in \mathfrak{S}_p(\mathcal{H})$ and, therefore, $\widetilde{T} \in \mathfrak{S}_p(\mathcal{H})$.

□

3.3. Perturbations of selfadjoint operators nonnegative over a neighbourhood of ∞

Let in the following $\mathcal{H}_1$, $\mathcal{H}_2$, G_1 and G_2 be defined as at the beginning of Section 2. The following theorem is the main result of Section 3.

Theorem 3.4. *Let A_1 be a selfadjoint operator in $\mathcal{H}_1$ nonnegative over $\mathcal{U}_\infty(r_1)$ for some $r_1 > 0$ (see (3.2)), and let $\mathfrak{a}$ be the closed symmetric form corresponding to A_1. Assume that for the form domain of A_1 equipped with the Hilbert scalar product (3.3) where $\gamma > c(G_1A_1)$, i.e. for $\mathcal{H}_\mathfrak{a}$, the following conditions are fulfilled.*

(a) *$G_2^{-1}G_1$ maps $\mathcal{H}_\mathfrak{a}$ onto itself.*
(b) *The restriction of $1 - G_2^{-1}G_1$ to $\mathcal{H}_\mathfrak{a}$ belongs to $\mathfrak{S}_p(\mathcal{H}_\mathfrak{a})$ for some $p \in [1, \infty)$.*

Let $\mathfrak{v}$ be a symmetric form which is relatively $\mathfrak{S}_p$ with respect to $\mathfrak{a}$, and let A_2 be the selfadjoint operator in $\mathcal{H}_2$ associated with the form $\mathfrak{a} + \mathfrak{v}$.

Then there exists an $r_2 > 0$ such that A_2 is nonnegative over $\mathcal{U}_\infty(r_2)$. Moreover, $E(\cdot, A_1)$ is bounded at ∞ if and only if $E(\cdot, A_2)$ is bounded at ∞.

Proof. **1.** In this part of the proof we verify that each of the forms $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2$ leads to the same "negative" space $\mathcal{H}_{\mathfrak{a},-}$ and that $1 - G_2^{-1}G_1$ can be extended to an operator belonging to $\mathfrak{S}_p(\mathcal{H}_{\mathfrak{a},-})$.

By the closed graph theorem the restriction $G_{21} := G_2^{-1}G_1|\mathcal{H}_\mathfrak{a}$ regarded as an operator in $\mathcal{H}_\mathfrak{a}$ is an isomorphism. Hence there exist $m_1, m_2 > 0$ such that

$$m_1\|y\|_\mathfrak{a} \leq \|G_{21}y\|_\mathfrak{a} \leq m_2\|y\|_\mathfrak{a}, \quad y \in \mathcal{H}_\mathfrak{a}, \tag{3.9}$$

where $\|y\|^2_\mathfrak{a} = (y, y)_\mathfrak{a}$. On $\mathcal{H}$ we introduce the "negative norms" $\|\cdot\|_{\mathfrak{a},-,j}$, $j = 1, 2$, (see (3.4)), by

$$\|x\|_{\mathfrak{a},-,j} = \sup\{|[x, y]_j| : y \in \mathcal{H}_\mathfrak{a},\ \|y\|_\mathfrak{a} \leq 1\}, \quad x \in \mathcal{H}.$$

Then the relations

$$\begin{aligned} \|x\|_{\mathfrak{a},-,1} &= \sup\{|[x, y]_1| : y \in \mathcal{H}_\mathfrak{a},\ \|y\|_\mathfrak{a} = 1\} \\ &= \sup\{|[G_2^{-1}G_1x, y]_2| : y \in \mathcal{H}_\mathfrak{a},\ \|y\|_\mathfrak{a} = 1\} \\ &= \sup\{|[x, G_{21}y]_2| : y \in \mathcal{H}_\mathfrak{a},\ \|y\|_\mathfrak{a} = 1\} \end{aligned}$$

and (3.9) show that the norms $\|\cdot\|_{\mathfrak{a},-,1}$ and $\|\cdot\|_{\mathfrak{a},-,2}$ are equivalent on $\mathcal{H}$. The completion of $\mathcal{H}$ with respect to one of the quadratic norms $\|\cdot\|_{\mathfrak{a},-,1}$ or $\|\cdot\|_{\mathfrak{a},-,2}$ equipped with the extension of the scalar product corresponding to $\|\cdot\|_{\mathfrak{a},-,1}$ will be denoted by $\mathcal{H}_{\mathfrak{a},-}$. Each of the forms $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ can be extended by continuity to $\mathcal{H}_{\mathfrak{a}} \times \mathcal{H}_{\mathfrak{a},-}$ and to $\mathcal{H}_{\mathfrak{a},-} \times \mathcal{H}_{\mathfrak{a}}$. The linear mappings

$$\iota_k : \mathcal{H}_{\mathfrak{a}} \ni y \mapsto u_k \in \mathcal{H}_{\mathfrak{a},-},\ k = 1,2, \text{ where } (x,y)_{\mathfrak{a}} = [x,u_k]_k \text{ for all } x \in \mathcal{H}_{\mathfrak{a}}, \quad (3.10)$$

are isomorphisms of $\mathcal{H}_{\mathfrak{a}}$ onto $\mathcal{H}_{\mathfrak{a},-}$.

The operator $G_2^{-1}G_1$ is an isometry of $(\mathcal{H}, \|\cdot\|_{\mathfrak{a},-,1})$ onto $(\mathcal{H}, \|\cdot\|_{\mathfrak{a},-,2})$. The extension by continuity of $G_2^{-1}G_1$ to an operator in $\mathcal{H}_{\mathfrak{a},-}$ is denoted by $\widetilde{G_{21}}$. We have

$$[G_{21}x,y]_2 = [x,y]_1 = [x,\widetilde{G_{21}}y]_2, \quad x \in \mathcal{H}_{\mathfrak{a}}, y \in \mathcal{H}_{\mathfrak{a},-}, \quad (3.11)$$

and

$$[x,(\widetilde{G_{21}}-1)y]_2 = [x,y]_1 - [x,y]_2 = [(G_{21}-1)x,y]_2, \quad x \in \mathcal{H}_{\mathfrak{a}}, y \in \mathcal{H}_{\mathfrak{a},-}.$$

Therefore, the adjoint of $G_{21} - 1 \in \mathcal{L}(\mathcal{H}_{\mathfrak{a}})$ is given by $\iota_2^{-1}(\widetilde{G_{21}}-1)\iota_2$. Hence by condition (b) we have

$$\widetilde{G_{21}} - 1 \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a},-}). \quad (3.12)$$

2. Now we show that A_1 and A_2 satisfy the assumptions of [9, Theorem 3.10] as well as the assumptions of Theorem 2.4 for certain domains Ω, with the exception of the conditions on G_1 and G_2 in Theorem 2.4.

Let $\widetilde{A}_1$ be the extension of A_1 to an operator in $\mathcal{L}(\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-})$. Then

$$[\widetilde{A}_1 x, y]_1 = \mathfrak{a}[x,y], \quad x,y \in \mathcal{H}_{\mathfrak{a}}.$$

Define an operator $V_1 \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-})$ by

$$[V_1 x, y]_1 = \mathfrak{v}[x,y], \quad x,y \in \mathcal{H}_{\mathfrak{a}}.$$

$\widetilde{A}_1$, V_1 and $\widetilde{A}_1 + V_1$ are $[\cdot,\cdot]_1$-symmetric. Let

$$\widetilde{R} := \widetilde{G_{21}}(\widetilde{A}_1 + V_1) = \widetilde{A}_1 + V_1 + (\widetilde{G_{21}}-1)(\widetilde{A}_1 + V_1). \quad (3.13)$$

We have $V_1 + (\widetilde{G_{21}}-1)(\widetilde{A}_1 + V_1) \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-})$. By (3.11) $\widetilde{R}$ is $[\cdot,\cdot]_2$-symmetric,

$$[\widetilde{R}x,y]_2 = [x,\widetilde{R}y]_2, \quad x,y \in \mathcal{H}_{\mathfrak{a}}. \quad (3.14)$$

Let R be the range restriction of $\widetilde{R}$ considered as operator in $\mathcal{H}_2$, i.e.

$$R = \widetilde{R}|\{x \in \mathcal{H}_{\mathfrak{a}} : \widetilde{R}x \in \mathcal{H}\} = \widetilde{A}_1 \stackrel{+}{\smile} (V_1 + (\widetilde{G_{21}}-1)(\widetilde{A}_1 + V_1)) \quad (3.15)$$

(see Section 3.1). By [9, Proposition 3.1] there exists an $\eta_0 > r_1$ such that

$$\{i\eta : \eta \in \mathbb{R}, |\eta| > \eta_0\} \subset \rho(R). \quad (3.16)$$

Then (3.14) implies

$$[(R - i\eta)x, y]_2 = [x, (R + i\eta)y]_2, \quad \eta \in \mathbb{R}, |\eta| > \eta_0, x,y \in \mathcal{D}(R),$$

and, hence, we find

$$[(R - i\eta)^{-1}u, v]_2 = [u, (R + i\eta)^{-1}v]_2, \quad \eta \in \mathbb{R}, |\eta| > \eta_0, u,v \in \mathcal{H}.$$

Therefore $(R+i\eta)^{-1}$ is the adjoint of $(R-i\eta)^{-1}$ in the Krein space $\mathcal{H}_2$, and it follows that R is selfadjoint in $\mathcal{H}_2$. Since in view of (3.13), for every $x\in\mathcal{D}(R)$, $y\in\mathcal{D}[A_1]$,

$$\begin{aligned}[Rx,y]_2 &= [\widetilde{R}x,y]_2 = [\widetilde{G_{21}}(\widetilde{A}_1+V_1)x,y]_2 = [(\widetilde{A}_1+V_1)x,y]_1 \\ &= \mathfrak{a}[x,y]+\mathfrak{v}[x,y],\end{aligned} \tag{3.17}$$

R coincides with the semibounded selfadjoint operator A_2 in $\mathcal{H}_2$ associated with the form $\mathfrak{a}+\mathfrak{v}$. By [9, Lemma 2.3] we have

$$(A_1-i\eta)^{-1}-(A_2-i\eta)^{-1}\in\mathfrak{S}_p(\mathcal{H}). \tag{3.18}$$

Let

$$\Omega_+ := \mathcal{U}_\infty(r_1)\backslash\big((-\infty,-r_1)\cup\{\infty\}\big),\quad \Omega_- := \mathcal{U}_\infty(r_1)\backslash\big((r_1,\infty)\cup\{\infty\}\big). \tag{3.19}$$

Then, by the nonnegativity of A_1 over $\mathcal{U}_\infty(r_1)$ and the relations (3.16) and (3.18), the assumptions of Theorem 2.4, except condition (iv) on the difference of the Gram operators, are fulfilled with Ω replaced by Ω_+ or Ω_-.

3. In this part of the proof we assume that the local spectral function $E(\cdot,A_1)$ of A_1 is bounded at ∞, and we prove Theorem 3.4 under this assumption. First we show that the difference of the Gram operators G_1 and G_2 belongs to $\mathfrak{S}_p(\mathcal{H})$. On account of

$$G_1-G_2 = G_1(1-G_1^{-1}G_2) \tag{3.20}$$

it is sufficient to verify that $1-G_1^{-1}G_2\in\mathfrak{S}_p(\mathcal{H})$.

Let, for some $s>r_1$, $E_s := E(\overline{\mathbb{R}}\backslash(-s,s),A_1)$, $E_{s,+} := ([s,\infty),A_1)$ and $E_{s,-} := E((-\infty,-s],A_1)$. Then $E_s = E_{s,+}+E_{s,-}$ maps $\mathcal{H}_\mathfrak{a}$ continuously into itself as this is true for $1-E_s$. By

$$[A_1E_{s,\pm}x,E_{s,\pm}x]_1 \le [A_1E_sx,E_sx]_1,\quad x\in\mathcal{D}(A_1),$$

we have

$$\begin{aligned}\|E_{s,\pm}x\|_\mathfrak{a}^2 &= [A_1E_{s,\pm}x,E_{s,\pm}x]_1+\gamma(E_{s,\pm}x,E_{s,\pm}x)\\ &\le [A_1E_sx,E_sx]_1+\gamma\|E_{s,\pm}\|^2\|E_sx\|^2,\quad x\in\mathcal{D}(A_1),\end{aligned}$$

i.e. the projections $E_{s,\pm}$ map $(E_s\mathcal{H}_\mathfrak{a},\|\cdot\|_\mathfrak{a})$ continuously into itself. Therefore, if J_0 is some fundamental symmetry of the Krein space $((1-E_\infty)\mathcal{H},[\cdot,\cdot]_1)$,

$$W := J_0(1-E_s)+(E_{s,+}-E_{s,-})E_s\in\mathcal{L}(\mathcal{H})$$

maps $\mathcal{H}_\mathfrak{a}$ continuously into itself. Moreover $W^2=1$ and $[Wx,x]_1>0$ for all $x\in\mathcal{H}$, $x\neq 0$.

The operator $1-G_1^{-1}G_2\in\mathcal{L}(\mathcal{H})$ is the extension by continuity of

$$1-G_{21}^{-1} = -G_{21}^{-1}(1-G_{21})\in\mathfrak{S}_p(\mathcal{H}_\mathfrak{a}).$$

Since G_{21}^{-1} is the restriction of $G_1^{-1}G_2$ to $\mathcal{H}_{\mathfrak{a}}$ we have

$$[G_{21}^{-1}x, y]_1 = [x, G_{21}^{-1}y]_1, \qquad x, y \in \mathcal{H}_{\mathfrak{a}}.$$

Then Lemma 3.3 applied to $\mathcal{B} := \mathcal{H}_{\mathfrak{a}}$, W as above and $T := 1 - G_{21}^{-1}$ gives $1 - G_1^{-1}G_2 \in \mathfrak{S}_p(\mathcal{H})$. Hence, by (3.20), $G_1 - G_2 \in \mathfrak{S}_p(\mathcal{H})$.

We have shown that in the case when the local spectral function of A_1 is bounded at ∞, all assumptions of Theorem 2.4 with Ω replaced by Ω_+ or Ω_- (see (3.19)) are satisfied. Then it follows, in particular, that no point of $\Omega_+ \cap \overline{\mathbb{R}} = (r_1, \infty)$ and no point of $\Omega_- \cap \overline{\mathbb{R}} = (-\infty, -r_1)$ is an accumulation point of the nonreal spectrum of A_2.

Then the assumptions of [9, Theorem 3.10] are fulfilled with the exception of the condition that the nonreal spectrum of A_1 has no more than a finite number of nonreal accumulation points. It is easy to see that the latter assumption can be dropped in [9, Theorem 3.10]. It was proved in [9, proof of Theorem 3.10] (the text of that theorem does not completely describe what is shown in the proof, see [12, footnote p.103]) that, for some $r_2 \geq r_1$, A_2 is nonnegative over $\mathcal{U}_\infty(r_2)$ and that the local spectral function of A_2 is bounded at ∞.

4. In the rest of the proof we assume that the local spectral function of A_1 is not bounded at ∞. In this part of the proof we show that the conclusion of Theorem 3.4 holds for A_1 replaced by its "regularization" A_1' (see [9, §2.4]).

Let $\mathcal{G}$ be the Hilbertable topological linear space corresponding to the middle of the interpolation scale between $\mathcal{H}_{\mathfrak{a}}$ and $\mathcal{H}_{\mathfrak{a},-}$,

$$\mathcal{G} := [\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-}]_{\frac{1}{2}}$$

(see e.g. [20, chapter 1]), and let $(\cdot,\cdot)_{\mathcal{G}}$ be a Hilbert scalar product on $\mathcal{G}$ which induces the topology of $\mathcal{G}$. Then $[\cdot,\cdot]_1$ restricted to $\mathcal{H}_{\mathfrak{a}}$ is continuous with respect to the topology of $\mathcal{G}$ (see [9], [10]). The extension by continuity of $[\cdot,\cdot]_1$ to $\mathcal{G}$ will be denoted by $[\cdot,\cdot]_{(1)}$. It was shown in [9] and [10] that $\mathcal{G}_{(1)} := (\mathcal{G}, [\cdot,\cdot]_{(1)})$ is a Krein space. Since the isomorphism $\widetilde{G_{21}}$ of $\mathcal{H}_{\mathfrak{a},-}$ is the extension of the isomorphism G_{21} of $\mathcal{H}_{\mathfrak{a}}$, by interpolation G_{21} and G_{21}^{-1} are continuous with respect to the topology of $\mathcal{G}$, and the extension $G_{21,\mathcal{G}}$ of G_{21} to an operator in $\mathcal{G}$ is an isomorphism of $\mathcal{G}$. It follows that the continuous sesquilinear form $[\cdot,\cdot]_{(2)}$ on $\mathcal{G}$, defined by

$$[x, y]_{(2)} := [G_{21,\mathcal{G}}^{-1}x, y]_{(1)}, \quad x, y \in \mathcal{G}, \tag{3.21}$$

is the extension to $\mathcal{G}$ of $[\cdot,\cdot]_2$ restricted to $\mathcal{H}_{\mathfrak{a}}$, and it is a Krein space inner product in $\mathcal{G}$. We set $\mathcal{G}_{(2)} := (\mathcal{G}, [\cdot,\cdot]_{(2)})$. If $G_{(j)}$, $j = 1, 2$, denotes the Gram operator of $[\cdot,\cdot]_{(j)}$ with respect to $(\cdot,\cdot)_{\mathcal{G}}$, then (3.21) gives

$$G_{(1)} - G_{(2)} = G_{(1)}(1 - G_{21,\mathcal{G}}^{-1}). \tag{3.22}$$

We define an operator A_1' in $\mathcal{G}$ by range restriction of $\widetilde{A}_1$ to $\mathcal{G}$:

$$A_1' := \widetilde{A}_1 | \{x \in \mathcal{H}_{\mathfrak{a}} : \widetilde{A}_1 x \in \mathcal{G}\}$$

(cf. [9, §2.4]). As, for $u \in \mathcal{G}$, $v \in \mathcal{H}_{\mathfrak{a}}$, $[u,v]_{(1)} = [u,v]_1$, where $[\cdot,\cdot]_1$ is the inner product of $\mathcal{H}_1$ extended to $\mathcal{H}_{\mathfrak{a},-} \times \mathcal{H}_{\mathfrak{a}}$, we have

$$[A_1'x,y]_{(1)} = [\widetilde{A}_1 x,y]_1 = [x,\widetilde{A}_1 y]_1 = [x,A_1'y]_{(1)}, \quad x,y \in \mathcal{D}(A_1').$$

In view of $\rho(A_1) \subset \rho(A_1')$ (see [9, (2.7)]) it follows as in part 2 of the proof that A_1' is selfadjoint in $\mathcal{G}_{(1)}$. By [9, Lemma 2.1] we have $\mathcal{D}[A_1'] = \mathcal{D}[A_1] = \mathcal{H}_{\mathfrak{a}}$. Clearly, A_1' is the operator associated with $\mathfrak{a}$ in $\mathcal{G}_{(1)}$.

If E_r, $r > r_1$, denotes the spectral projection $E(\overline{\mathbb{R}}\backslash(-r,r), A_1)$, the decomposition

$$\mathcal{H}_1 = (1-E_r)\mathcal{H}_1[\dot{+}]E_r\mathcal{H}_1$$

reduces A_1 and $A_1|E_r\mathcal{H}_1$ is nonnegative. If $|\eta| > \eta_0$ (see (3.16)), the latter fact is equivalent to

$$\begin{aligned} &\operatorname{Re}[(A_1 - i\eta)^{-1}E_r x, E_r x]_1 = \\ &\qquad = [A_1(A_1-i\eta)^{-1}E_r x, (A_1-i\eta)^{-1}E_r x]_1 \geq 0, \quad x \in \mathcal{H}. \end{aligned} \tag{3.23}$$

Since $1 - E_r$ maps $\mathcal{H}_{\mathfrak{a}}$ continuously into itself the same is true for E_r. We denote the restriction of E_r to $\mathcal{H}_{\mathfrak{a}}$ also by E_r, then

$$\mathcal{H}_{\mathfrak{a}} = (1-E_r)\mathcal{H}_{\mathfrak{a}} \dot{+} E_r\mathcal{H}_{\mathfrak{a}}.$$

The adjoint of $E_r \in \mathcal{L}(\mathcal{H}_{\mathfrak{a}})$ with respect to the $[\cdot,\cdot]_1$-duality is the extension by continuity of E_r to an operator in $\mathcal{H}_{\mathfrak{a},-}$ which will be denoted by $\widetilde{E}_r$. From the fact that the topologies of $\mathcal{H}_{\mathfrak{a}}$ and $\mathcal{H}_{\mathfrak{a},-}$ coincide on $(1-E_r)\mathcal{H}$ it follows that the spaces $(1-E_r)\mathcal{H}$, $(1-E_r)\mathcal{H}_{\mathfrak{a}}$ and $(1-\widetilde{E}_r)\mathcal{H}_{\mathfrak{a},-}$ coincide.

If $\widetilde{I}$ denotes the natural embedding of $\mathcal{H}_{\mathfrak{a}}$ into $\mathcal{H}_{\mathfrak{a},-}$, then $\widetilde{A}_1 - i\eta\widetilde{I}$, $|\eta| > \eta_0$, is an isomorphism from $\mathcal{H}_{\mathfrak{a}}$ onto $\mathcal{H}_{\mathfrak{a},-}$ (see [9]). The restriction of $(\widetilde{A}_1 - i\eta\widetilde{I})^{-1}\widetilde{E}_r$ to $\mathcal{H}$ coincides with $(A_1 - i\eta)^{-1}E_r$. Since $E_r\mathcal{H}$ is dense in $\widetilde{E}_r\mathcal{H}_{\mathfrak{a},-}$ the relation (3.23) implies

$$\operatorname{Re}[(\widetilde{A}_1 - i\eta\widetilde{I})^{-1}\widetilde{E}_r x, \widetilde{E}_r x]_1 \geq 0, \quad x \in \mathcal{H}_{\mathfrak{a},-}. \tag{3.24}$$

The restriction of $(\widetilde{A}_1 - i\eta\widetilde{I})^{-1}$ to $\mathcal{G}$ coincides with $(A_1' - i\eta)^{-1}$. By interpolation between $E_r \in \mathcal{L}(\mathcal{H}_{\mathfrak{a}})$ and $\widetilde{E}_r \in \mathcal{L}(\mathcal{H}_{\mathfrak{a},-})$ we obtain a projection $E_{r,\mathcal{G}}$ in $\mathcal{G}$. Since E_r is symmetric in $(\mathcal{H}_{\mathfrak{a}}, [\cdot,\cdot]_1)$, $E_{r,\mathcal{G}}$ is selfadjoint in $\mathcal{G}_{(1)}$. The operator E_r and the restriction of $(A_1 - i\eta)^{-1}$ to $\mathcal{H}_{\mathfrak{a}}$ commute, hence the operators $(A_1' - i\eta)^{-1}$ and $E_{r,\mathcal{G}}$ commute. Therefore the decomposition

$$\mathcal{G}_{(1)} = (1-E_{r,\mathcal{G}})\mathcal{G}_{(1)}[\dot{+}]E_{r,\mathcal{G}}\mathcal{G}_{(1)}$$

reduces A_1'. By $E_{r,\mathcal{G}}\mathcal{G} \subset \widetilde{E}_r\mathcal{H}_{\mathfrak{a},-}$ and (3.24) we have

$$\operatorname{Re}[(A_1' - i\eta)^{-1}E_{r,\mathcal{G}}x, E_{r,\mathcal{G}}x]_{(1)} \geq 0, \quad x \in \mathcal{G},$$

therefore A_1' is a nonnegative operator in $E_{r,\mathcal{G}}\mathcal{G}_{(1)}$. Since $(1-E_r)\mathcal{H} = (1-E_{r,\mathcal{G}})\mathcal{G}$ and

$$A_1|(1-E_r)\mathcal{H} = A_1'|(1-E_{r,\mathcal{G}})\mathcal{G},$$

the operator A_1' is nonnegative over $\mathcal{U}_\infty(r)$. By [9, Lemma 2.1] the local spectral function $E(\cdot, A_1')$ of A_1' is bounded at ∞.

We define A_2' by

$$A_2' := \widetilde{R}|\{x \in \mathcal{H}_\mathfrak{a} : \widetilde{R}x \in \mathcal{G}\}$$

(see (3.13)). As in part 2 of the proof one verifies that the operator A_2' is selfadjoint in $\mathcal{G}_{(2)}$, and by (3.17) we have

$$[A_2'x, y]_{(2)} = \mathfrak{a}[x, y] + \mathfrak{v}[x, y], \quad x \in \mathcal{D}(A_2'),\, y \in \mathcal{H}_\mathfrak{a}.$$

Hence A_2' is the operator associated to the form $\mathfrak{a} + \mathfrak{v}$ in $\mathcal{G}_{(2)}$. The relation (3.18) holds with A_1, A_2 replaced by A_1', A_2'.

Let, as in part 3 of the proof, for some $s > r$, $E_s' := E(\overline{\mathbb{R}}\backslash(-s, s), A_1')$, $E_{s,+}' := E([s, \infty), A_1')$, $E_{s,-}' := E((-\infty, -s], A_1')$ and let J_0' be a fundamental symmetry in $((1 - E_s')\mathcal{G}, [\cdot, \cdot]_{(1)})$. Lemma 3.3 applied in the case where $\mathcal{B} := \mathcal{H}_\mathfrak{a}$,

$$W' := J_0'(1 - E_s') + (E_{s,+}' - E_{s,-}')E_s' \in \mathcal{L}(\mathcal{G})$$

and $T := 1 - G_{21}^{-1}$ gives $1 - G_{21,\mathcal{G}}^{-1} \in \mathfrak{S}_p(\mathcal{G})$. By (3.22) we have

$$G_{(1)} - G_{(2)} \in \mathfrak{S}_p(\mathcal{G}).$$

Then, again making use of Theorem 2.4, we see that A_1', A_2', $G_{(1)}$, $G_{(2)}$ fulfil all conditions of [9, Theorem 3.10]. Therefore there is an $r' \geq r$ such that A_2' is nonnegative over $\mathcal{U}_\infty(r')$ in $\mathcal{G}_{(2)}$, and the local spectral function of A_2' is bounded at ∞.

5. Now we show that the nonnegativity of A_2' over $\mathcal{U}_\infty(r')$ implies the nonnegativity of A_2 over $\mathcal{U}_\infty(r_2)$ for any $r_2 > r'$.

We define the spectral projection

$$E_{r_2}' := E(\overline{\mathbb{R}}\backslash(-r_2, r_2), A_2')$$

in $\mathcal{G}_{(2)}$. The operator A_2' is nonnegative in $E_{r_2}'\mathcal{G}_{(2)}$. By [9, Lemma 2.7] we have

$$\mathcal{D}[A_1'] = \mathcal{D}[A_2'] = \mathcal{H}_\mathfrak{a}.$$

Therefore, the operator E_{r_2}', regarded as an operator in $\mathcal{H}_\mathfrak{a}$ is continuous. The extension by continuity of $E_{r_2}' \in \mathcal{L}(\mathcal{H}_\mathfrak{a})$ to a projection in $\mathcal{H}_{\mathfrak{a},-}$ will be denoted by $\widetilde{E}_{r_2}'$. From

$$\operatorname{Re}\,[(A_2' - i\eta)^{-1}E_{r_2}'x, E_{r_2}'x]_{(2)} \geq 0, \quad x \in \mathcal{G},$$

it follows, that

$$\operatorname{Re}\,[(\widetilde{R} - i\eta\widetilde{I})^{-1}\widetilde{E}_{r_2}'x, \widetilde{E}_{r_2}'x]_2 \geq 0, \quad x \in \mathcal{H}_{\mathfrak{a},-}. \tag{3.25}$$

Again, the topologies of $\mathcal{H}_\mathfrak{a}$ and $\mathcal{H}_{\mathfrak{a},-}$ coincide on $(1 - E_{r_2}')\mathcal{G}$, therefore the spaces $(1 - E_{r_2}')\mathcal{G}$, $(1 - E_{r_2}')\mathcal{H}_\mathfrak{a}$ and $(1 - \widetilde{E}_{r_2}')\mathcal{H}_{\mathfrak{a},-}$ coincide. Hence the operator $1 - \widetilde{E}_{r_2}'$ maps $\mathcal{H}_{\mathfrak{a},-}$ continuously into $\mathcal{H}_\mathfrak{a}$. Then $1 - \widetilde{E}_{r_2}'$ maps $\mathcal{H}$ continuously into itself and the same is true for $\widetilde{E}_{r_2}'$. We denote by F_{r_2} the operator $\widetilde{E}_{r_2}'|\mathcal{H}$ regarded as a bounded operator in $\mathcal{H}$. Since $E'_{r_2} \in \mathcal{L}(\mathcal{H}_\mathfrak{a})$ is a symmetric projection with respect to $[\cdot,\cdot]_2$, F_{r_2} is a selfadjoint projection in $\mathcal{H}_2$. By (3.15) and (3.17) the range restriction of $\widetilde{R}$ to $\mathcal{H}_2$ coincides with A_2. The projection F_{r_2} commutes with $(A_2 - i\eta)^{-1}$, $|\eta| > \eta_0$ (see (3.16)), since the same is true for E'_{r_2} and $(A'_2 - i\eta)^{-1}$ in $\mathcal{H}_\mathfrak{a}$. It follows from (3.25), that for $x \in \mathcal{H}$

$$\operatorname{Re}\,[(A_2 - i\eta)^{-1} F_{r_2} x, F_{r_2} x] \geq 0$$

holds. On the other hand, $A_2|(1 - F_{r_2})\mathcal{H}$ is bounded. Therefore A_2 is nonnegative over $\mathcal{U}_\infty(r_2)$.

It remains to show that the local spectral function of A_2 is unbounded at ∞. Assume that this does not hold. Then, if we regard A_2 as the unperturbed operator and A_1 as the perturbed operator, the assumptions of the theorem are fulfilled. By the first part of the proof we find that the spectral function of A_1 is bounded at ∞, a contradiction.

□

In the following corollary we make use of the notation introduced at the end of Section 3.1.

Corollary 3.5. *Let A_1, G_1 and G_2 be as in Theorem* 3.4 *and assume that the conditions* (a) *and* (b) *are fulfilled. Let $V \in \mathfrak{S}_p(\mathcal{H}_\mathfrak{a}, \mathcal{H}_{\mathfrak{a},-})$ and let $A_2 = A_1 \dot{+} V$ be selfadjoint in $\mathcal{H}_2$. Then the conclusions of Theorem* 3.4 *are true.*

Proof. Since A_2 is selfadjoint in $\mathcal{H}_2$, the extension of A_2 by continuity to an operator $\widetilde{A}_2 \in \mathcal{L}(\mathcal{H}_\mathfrak{a}, \mathcal{H}_{\mathfrak{a},-})$ is symmetric with respect to $[\cdot,\cdot]_2$. Let $\widetilde{A}_1 \in \mathcal{L}(\mathcal{H}_\mathfrak{a}, \mathcal{H}_{\mathfrak{a},-})$ and $\widetilde{G_{21}}$ be as in the proof of Theorem 3.4. We have $\widetilde{A}_2 = \widetilde{A}_1 + V$ and by (3.12)

$$\widetilde{G_{21}}^{-1} - 1 = -\widetilde{G_{21}}^{-1}(\widetilde{G_{21}} - 1) \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a},-}).$$

The operator

$$\widetilde{G_{21}}^{-1}\widetilde{A}_2 = \widetilde{A}_1 + V + (\widetilde{G_{21}}^{-1} - 1)(\widetilde{A}_1 + V)$$

is symmetric with respect to $[\cdot,\cdot]_1$. Since, by assumption, $\widetilde{A}_1$ is symmetric with respect to $[\cdot,\cdot]_1$, the same holds for

$$V + (\widetilde{G_{21}}^{-1} - 1)(\widetilde{A}_1 + V) \in \mathfrak{S}_p(\mathcal{H}_\mathfrak{a}, \mathcal{H}_{\mathfrak{a},-}).$$

Therefore the operator

$$\mathcal{V} := \iota_1^{-1}(V + (\widetilde{G_{21}}^{-1} - 1)(\widetilde{A}_1 + V)) \in \mathfrak{S}_p(\mathcal{H}_\mathfrak{a})$$

(see (3.10)) is symmetric in $\mathcal{H}_\mathfrak{a} = (\mathcal{D}[A_1], (\cdot,\cdot)_\mathfrak{a})$. Then

$$\mathfrak{v}[x,y] := (\mathcal{V}x, y)_\mathfrak{a}, \qquad x, y \in \mathcal{H}_\mathfrak{a},$$

is a continuous symmetric sesquilinear form which is relatively $\mathfrak{S}_p$ with respect to $\mathfrak{a}$, and A_2 is the selfadjoint operator in $\mathcal{H}_2$ associated with $\mathfrak{a} + \mathfrak{v}$. Hence, the assumptions of Theorem 3.4 are fulfilled.

□

References

[1] V. Adamjan, H. Langer, M. Möller: Compact Perturbation of Definite Type Spectra of Self-adjoint Quadratic Operator Pencils, Integral Equations Operator Theory **39** (2001), 127-152.

[2] B. Ćurgus, B. Najman: Preservation of the Range under Perturbations of an Operator, Proc. AMS **125** (1997), 2627-2631.

[3] A. Dijksma, H.S.V. de Snoo: Symmetric and Selfadjoint Relations in Krein Spaces I, Operator Theory: Advances and Applications **24** (1987), Birkhäuser Verlag Basel, 145-166.

[4] I.S. Iohvidov, M.G. Krein, H. Langer: Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric, Mathematical Research **9** (1982), Akademie-Verlag Berlin.

[5] I.C. Gohberg, M.G. Krein: The Basic Propositions on Defect Numbers, Root Numbers and Indices of Linear Operators, AMS Transl. (2) **13** (1960), 185-265.

[6] I.C. Gohberg, M.G. Krein: Introduction to the Theory of Linear Nonselfadjoint Operators (Russian), Moscow 1965; English Transl.: AMS Transl. of Math. Monographs **24** (1970).

[7] P. Jonas: On a Class of Unitary Operators in Krein Space. Advances in Invariant Subspaces and Other Results of Operator Theory. Operator Theory: Advances and Applications **17** (1986) Birkhäuser Verlag Basel, 151-172.

[8] P. Jonas: A Note on Perturbations of Selfadjoint Operators in Krein Spaces, Operator Theory: Advances and Applications **43** (1990), Birkhäuser Verlag Basel, 229-235.

[9] P. Jonas: On a Problem of the Perturbation Theory of Selfadjoint Operators in Krein Spaces, J. Operator Theory **25** (1991), 183-211.

[10] P. Jonas: Riggings and Relatively Form Bounded Perturbations of Nonnegative Operators in Krein Spaces, Operator Theory: Advances and Applications **106** (1998), Birkhäuser Verlag Basel, 259-273.

[11] P. Jonas: On Locally Definite Operators in Krein Spaces, *in*: "Spectral Theory and Applications", Ion Colojoara Anniversary Volume, Theta, Bucharest, 2003, 95-127.

[12] P. Jonas, C. Trunk: On a Class of Analytic Operator Functions and their Linearizations, Math. Nachrichten **243** (2002), 92-133.

[13] T. Kato: Perturbation Theory of Linear Operators, Springer Verlag Berlin-Heidelberg-New York, 1966.

[14] M.G. Krein: On Linear Compact Operators in Spaces with Two Norms (Ukrainian), Zbirnik Prac. Inst. Mat. Akad. Nauk. URSR **9** (1947), 104-129.

[15] P. Lancaster, A. Markus, V. Matsaev: Definitizable Operators and Quasihyperbolic Operator Polynomials, J. Funct. Anal. **131** (1995), 1-28.

[16] H. Langer: Factorization of Operator Pencils, Acta Sci. Math. (Szeged), **38** (1976), 83-96.

[17] H. Langer: Spectral Functions of Definitizable Operators in Krein Spaces, Functional Analysis Proceedings of a conference held at Dubrovnik, Yugoslavia, November 2-14, 1981, Lecture Notes in Mathematics **948**, Springer Verlag Berlin-Heidelberg-New York, 1982, 1-46.

[18] H. Langer, A. Markus, V. Matsaev: Locally Definite Operators in Indefinite Inner Product Spaces, Math. Annalen **308** (1997), 405-424.

[19] H. Langer, A. Markus, V. Matsaev: Linearization and Compact Perturbation of Self-adjoint Analytic Operator Functions, Operator Theory: Advances and Applications **118** (2000), Birkhäuser Verlag Basel, 255-285.

[20] J.-L. Lions, E. Magenes: Problèmes aux limites nonhomogènes et applications, Paris 1968.

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