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# Singular Schrödinger operators with prescribed spectral properties

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This paper deals with singular Schrödinger operators of the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\sum_{k\in\mathbb{Z}}\gamma_k\delta(\cdot-z_k),\quad \gamma_k\in\mathbb{R},$$

in $\mathsf{L}^2(\ell_-,\ell_+)$, where $\delta(\cdot - z_k)$ is the Dirac delta-function supported at $z_k \in (\ell_-,\ell_+)$ and $(\ell_-,\ell_+)$ is a bounded interval. It will be shown that the interaction strengths $\gamma_k$ and the points $z_k$ can be chosen in such a way that the essential spectrum and a bounded part of the discrete spectrum of this self-adjoint operator coincide with prescribed sets on the real line.



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## 1. Introduction

Self-adjoint Laplace and Schrödinger operators on bounded domains typically have purely discrete spectrum, since in many situations the operator or corresponding form domain is compactly embedded in the underlying $L^2$-space. In general, however, this is not true, and a well known example is the Neumann Laplacian on a bounded non-Lipschitz domain discussed by R. Hempel, L. Seco, and B. Simon in [19]. More precisely, for an arbitrary closed set $S \subset [0, \infty)$ a bounded domain $\Omega$ was constructed in [19] such that the essential spectrum of the Neumann Laplacian $-\Delta_\Omega^N$ on $\Omega$ coincides with the set $S$. In particular, in the case $0 \in S$ one can use a domain $\Omega$ consisting of a series of "rooms and passages", see Fig. 1. These results were further elaborated by R. Hempel, T. Kriecherbauer, and P. Plankensteiner in [18], where also a prescribed bounded part of the discrete spectrum was realized by constructing a domain with a certain "comb" structure. Furthermore, in [29] B. Simon found a bounded domain of "jelly roll" form such that the spectrum of $-\Delta_\Omega^N$ is purely absolutely continuous and covers $[0, \infty)$. Note that in the above situations the peculiar spectral properties of $-\Delta_\Omega^N$ are all caused by irregularity of $\partial\Omega$. Another approach to construct Laplace or Schrödinger operators on bounded domains with non-standard spectral properties is to choose "unusual" boundary conditions; e.g. 0 is always an eigenvalue of infinite multiplicity of the Krein-von Neumann realization of $-\Delta$. In the abstract setting S. Albeverio, J. Brasche, M. Malamud, H. Neidhardt, and J. Weidmann [1,2,8–14] consider a symmetric operator $S$ in a Hilbert space with infinite deficiency indices such that $\sigma(S)$ has a gap, and discuss the possible spectral properties of self-adjoint extensions of $S$ in the gap; cf. [2] for applications to the Laplacian. In this context we also refer the reader to the recent expository paper [6].

### *1.1. Setting of the problem and the main result*

In the present paper we consider (one-dimensional) Schrödinger operators with $\delta$-interactions defined by the formal expression

$$\mathcal{H}_\gamma = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{k\in K} \gamma_k \delta(\cdot - z_k); \tag{1.1}$$

here $\delta(\cdot - z_k)$ is the Dirac delta-function supported at $z_k$, $\gamma_k \in \mathbb{R}$, and $K$ is a countable set. Such operators can be regarded as so-called *solvable models* in quantum mechanics describing the motion of a particle in a potential supported by a discrete (finite or infinite) set of points; cf. the monograph [3] for more details. Now assume that all points $z_k$ are contained in a (bounded or unbounded) interval $(\ell_-, \ell_+)$. If the set $K$ is finite the formal expression (1.1) can be realized as a self-adjoint operator in $\mathsf{L}^2(\ell_-, \ell_+)$ with the action

$$-(u \restriction_{(\ell_-,\ell_+)\setminus \bigcup_{k\in K}\{z_k\}})''$$

defined for functions $u \in \mathsf{H}^2((\ell_-,\ell_+)\setminus \bigcup_{k\in K}\{z_k\})$ satisfying

$$u(z_k-0)=u(z_k+0) \quad \text{and} \quad u'(z_k+0)-u'(z_k-0)=\gamma_k u(z_k\pm 0) \tag{1.2}$$

at the points $z_k$, and suitable conditions at the endpoints of the interval $(\ell_-,\ell_+)$ (e.g., $u(\ell_-)=u(\ell_+)=0$). If $K$ is a countable infinite set, then the definition of $\mathcal{H}_\gamma$ is more subtle, in particular, if $|z_k-z_{k-1}|\to 0$ as $|k|\to\infty$; cf. [4,21]. If $(\ell_-,\ell_+)$ is bounded and $K$ is finite then the spectrum of $\mathcal{H}_\gamma$ is purely discrete, but if $K$ is infinite then the essential spectrum of $\mathcal{H}_\gamma$ may be non-empty (even if $(\ell_-,\ell_+)$ bounded).

The goal of the present paper is to show that for an arbitrary closed semibounded (from below) set $S_{\rm ess}$ one can construct an operator $\mathcal{H}_\gamma$ of the form (1.1) on a bounded interval such that $\sigma_{\rm ess}(\mathcal{H}_\gamma)=S_{\rm ess}$ and, in addition, a bounded part of the discrete spectrum can be controlled. More precisely, assume that we have a set $S_{\rm ess}\subset\mathbb{R}$, a sequence of real numbers $S_{\rm disc}=(s_k)_{k\in\mathbb{N}}$ and a bounded interval $(T_1,T_2)\subset\mathbb{R}$ such that

$$S_{\rm ess} \text{ is closed and bounded from below,} \tag{1.3}$$

$$S_{\rm ess}\cap[T_1,T_2]=\overline{\mathcal{O}}, \text{ where } \mathcal{O}\subset(T_1,T_2) \text{ is an open set,} \tag{1.4}$$

$$s_k\in(T_1,T_2)\setminus\overline{\mathcal{O}},\ \forall k\in\mathbb{N}, \tag{1.5}$$

$$s_k\neq s_l \text{ as } k\neq l, \tag{1.6}$$

$$\text{all accumulation points of } S_{\rm disc} \text{ are contained in } S_{\rm ess}. \tag{1.7}$$

From (1.3)–(1.7) we conclude that

$$\text{each } s_k \text{ has a punctured neighborhood containing no other points of } S_{\rm ess}\cup S_{\rm disc}. \tag{1.8}$$

Also note that in view of (1.4) $S_{\rm ess}$ has no isolated points in $[T_1,T_2]$.

The following theorem is the main result of this paper, see also Theorem 4.4 for a slightly more rigorous formulation with the formal operator $\mathcal{H}_\gamma$ replaced by the precisely defined operator $\mathcal{H}_{\alpha,\beta}$ from Section 4.

**Theorem 1.1.** *There exists a bounded interval $(\ell_-,\ell_+)\subset\mathbb{R}$, a sequence of points $(z_k)_{k\in\mathbb{Z}}$ with $z_k\in(\ell_-,\ell_+)$, and a sequence of real numbers $(\gamma_k)_{k\in\mathbb{Z}}$ such that the operator $\mathcal{H}_\gamma$ in $\mathsf{L}^2(\ell_-,\ell_+)$ defined by the formal expression (1.1) satisfies*

$$\sigma_{\rm ess}(\mathcal{H}_\gamma)=S_{\rm ess} \quad \textit{and} \quad \sigma_{\rm disc}(\mathcal{H}_\gamma)\cap(T_1,T_2)=S_{\rm disc}, \tag{1.9}$$

*and, moreover,*

$$\textit{the eigenvalues } \sigma_{\rm disc}(\mathcal{H}_\gamma)\cap(T_1,T_2) \textit{ are simple.} \tag{1.10}$$

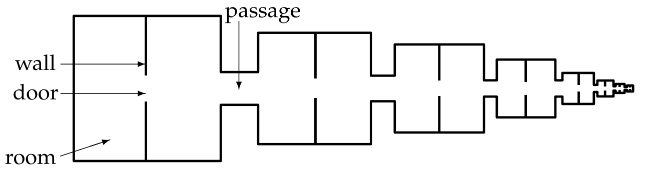


**Fig. 1.** Rooms-and-passages domain $\Omega$.

We mention that besides [18,19] our research is also inspired by a celebrated paper of Y. Colin de Verdiére [15], where a Riemannian metric $g$ on a given compact manifold $M$ is constructed such that the first $m$ eigenvalues of the Laplace-Beltrami operator on $(M,g)$ coincide with prescribed numbers; similar results were also obtained for the Neumann Laplacian and regular Schrödinger operators.

### *1.2. Sketch of the proof strategy*

To construct the operator $\mathcal{H}_\gamma$ satisfying (1.9)–(1.10) we utilize ideas of the aforementioned paper [19], where a bounded domain $\Omega$ was constructed such that the essential spectrum of the Neumann Laplacian $-\Delta_\Omega^N$ coincides with a predefined closed set $S\subset[0,\infty)$. If $0\in S$ the domain $\Omega$ consists of a sequence of *rooms* $R_k$ connected by *passages* $P_k$, each room has a *wall* dividing it in two subsets connected via a *door*, see Fig. 1. The diameters of $R_k$ and $P_k$ tend to zero as $k\to\infty$ in such a way that their union is a bounded domain.

The strategy of the proof in [19] is as follows: Choose a sequence $(s_k)_{k\in\mathbb{N}}$ such that $S=\{\text{accumulation points of }(s_k)_{k\in\mathbb{N}}\}$ and consider the "decoupled" operator

$$\mathcal{H}_{\rm dec}=\bigoplus_{k\in\mathbb{N}}\big((-\Delta_{R_k}^N)\oplus(-\Delta_{P_k}^{DN})\big)$$

in the space $\mathsf{L}^2(\Omega)=\bigoplus_{k\in\mathbb{N}}\big(\mathsf{L}^2(R_k)\oplus\mathsf{L}^2(P_k)\big)$. Here $\Delta_{R_k}^N$ is the Neumann Laplacian on $R_k$ and $\Delta_{P_k}^{DN}$ is the Laplacian on $P_k$ subject to the Dirichlet conditions on the parts of $\partial P_k$ touching the neighboring rooms and Neumann conditions on the remaining part of $\partial P_k$. Denote by $(\lambda_j(-\Delta_{R_k}^N))_{k\in\mathbb{N}}$ and $(\lambda_j(-\Delta_{P_k}^{DN}))_{k\in\mathbb{N}}$ the sequence of eigenvalues of $-\Delta_{R_k}^N$ and $-\Delta_{P_k}^{DN}$, respectively, numbered in ascending order with multiplicities taken into account. Then one has

$$\begin{aligned}\sigma_{\rm ess}(\mathcal{H}_{\rm dec})=&\big\{\text{accumulation points of }(\lambda_j(-\Delta_{R_k}^N))_{j,k\in\mathbb{N}}\big\}\\&\cup\big\{\text{accumulation points of }(\lambda_j(-\Delta_{P_k}^{DN}))_{j,k\in\mathbb{N}}\big\}.\end{aligned}\tag{1.11}$$

We have

$$\lambda_1(-\Delta_{R_k}^N)=0,\quad \lambda_1(-\Delta_{P_k}^{DN})=\pi^2/(\ell(P_k))^2,$$

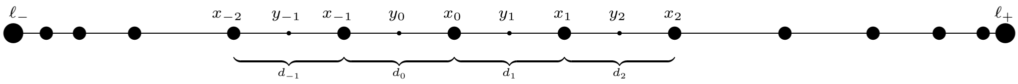


**Fig. 2.** Sequence of points supporting $\delta$-interactions.

where $\ell(P_k)$ is the length of the passage $P_k$. Next, the door in each $R_k$ is adjusted in such a way that

$$\lambda_2(-\Delta^N_{R_k}) = s_k$$

and it is not hard to show that

$$\lambda_3(-\Delta^N_{R_k}) \geq C/(\mathrm{diam}(R_k))^2, \tag{1.12}$$

where the constant $C > 0$ is the same for all rooms. Since $\ell(P_k) \to 0$ and $\mathrm{diam}(R_k) \to 0$ as $k \to \infty$ and $0 \in S$, one concludes from (1.11)–(1.12) that

$$\sigma_{\mathrm{ess}}(\mathcal{H}_{\mathrm{dec}}) = \{0\} \cup \{\text{accumulation points of } (s_k)_{k\in\mathbb{N}}\} = \{0\} \cup S = S.$$

Finally, if the thickness of the passages $P_k$ tends to zero sufficiently fast as $k \to \infty$, then the difference of the resolvents of $-\Delta^N_\Omega$ and $\mathcal{H}_{\mathrm{dec}}$ is a compact operator, and thus $\sigma_{\mathrm{ess}}(-\Delta^N_\Omega) = \sigma_{\mathrm{ess}}(\mathcal{H}_{\mathrm{dec}})$ by Weyl's theorem.

When constructing the operator $\mathcal{H}_\gamma$ in Theorem 1.1 we mimic the above idea. First of all the sequence $(z_k)_{k\in\mathbb{Z}}$ is split in two interlacing subsequences $(x_k)_{k\in\mathbb{Z}}$ and $(y_k)_{k\in\mathbb{Z}}$ (see Fig. 2) such that

$$\sum_{k\in\mathbb{Z}} d_k < \infty, \text{ where } d_k = x_k - x_{k-1}$$

is sufficiently small (see (3.3)) and $y_k$ is the center of the interval $\mathcal{I}_k := (x_{k-1}, x_k)$. We also set

$$\ell_- = -\sum_{k\in\mathbb{Z}\setminus\mathbb{N}} d_k \quad \text{and} \quad \ell_+ = \sum_{k\in\mathbb{N}} d_k. \tag{1.13}$$

For our purposes it is convenient to change the notation for the interaction strengths $\gamma_k$ as follows: at the points $y_k$ they will be denoted by $\alpha_k$, at the points $x_k$ they will be denoted by $\beta_k$, and instead of $\mathcal{H}_\gamma$ we will use the notation $\mathcal{H}_{\alpha,\beta}$ for the Schrödinger operator. Now, roughly speaking, the intervals $\mathcal{I}_k$ play the role of the rooms, the interactions at the points $x_k$ play the role of the passages, and the interactions at the points $y_k$ play the role of the doors. The desired operator is constructed in three steps.

**1) Decoupled operator.** We start from the case $\beta_k = \infty$ for all $k \in \mathbb{Z}$, which corresponds to Dirichlet decoupling at the points $x_k$. In other words, we treat the operator

$$\mathcal{H}_{\alpha,\infty} = \bigoplus_{k\in\mathbb{Z}} \mathbf{H}_{\alpha_k,\mathcal{I}_k} \quad \text{in} \quad \mathsf{L}^2(\ell_-,\ell_+) = \bigoplus_{k\in\mathbb{Z}} \mathsf{L}^2(\mathcal{I}_k),$$

where $\mathbf{H}_{\alpha_k,\mathcal{I}_k}$ is an operator in $\mathsf{L}^2(\mathcal{I}_k)$ (formally) defined by the differential expression

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha_k \delta(\cdot - y_k)$$

and Dirichlet boundary conditions at the endpoints of the interval $\mathcal{I}_k$. Recall that the sequence $S_{\mathrm{disc}} = (s_k)_{k\in\mathbb{N}}$ is already given and, in addition, we assign to $S_{\mathrm{ess}}$ a sequence $(s_k)_{k\in\mathbb{Z}\setminus\mathbb{N}}$ such that

$$S_{\mathrm{ess}} = \left\{\text{accumulation points of } (s_k)_{k\in\mathbb{Z}\setminus\mathbb{N}}\right\}. \tag{1.14}$$

If $d_k$ are sufficiently small (see the second condition in (3.3)), one can choose the constants $\alpha_k$ such that $\lambda_1(\mathbf{H}_{\alpha_k,\mathcal{I}_k}) = s_k$ and, moreover, $\lambda_2(\mathbf{H}_{\alpha_k,\mathcal{I}_k}) = (2\pi/d_k)^2$ for all $\alpha_k \in \mathbb{R}$. Therefore, since $d_k \to 0$ as $|k| \to \infty$ and (1.7) holds, we conclude

$$\begin{aligned} \sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\infty}) &= \{\text{accumulation points of } (s_k)_{k\in\mathbb{Z}}\} \\ &= \{\text{accumulation points of } (s_k)_{k\in\mathbb{Z}\setminus\mathbb{N}}\} = S_{\mathrm{ess}}. \end{aligned} \tag{1.15}$$

Similarly, if $\max_{k\in\mathbb{Z}} d_k$ is sufficiently small (see the first conditions in (3.3)), we obtain

$$\sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\infty}) \cap (T_1,T_2) = S_{\mathrm{disc}}.$$

Moreover, due to (1.6),

$$\text{all eigenvalues in } \sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\infty}) \cap (T_1,T_2) \text{ are simple.} \tag{1.16}$$

Thus, the decoupled operator $\mathcal{H}_{\alpha,\infty}$ satisfies (1.9) and (1.10) in Theorem 1.1. However, this is not the desired singular Schrödinger operator as we have Dirichlet conditions at the points $x_k$, $k \in \mathbb{Z}$.

**2) Partly coupled operator.** Let $\beta = (\beta_k)_{k\in\mathbb{N}}$ be a sequence of real numbers. For $n \in \mathbb{N}$ we denote by $\mathcal{H}^n_{\alpha,\beta}$ the operator, which is obtained from $\mathcal{H}_{\alpha,\infty}$ by "inserting" $\delta$-interactions of strengths $\beta_k$ at *finitely many* points $x_k$, $k \in \mathbb{Z} \cap [-n+1, n-1]$. Then one has

$$\forall n \in \mathbb{N}: \quad \sigma_{\mathrm{ess}}(\mathcal{H}^n_{\alpha,\beta}) = \sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\infty}), \tag{1.17}$$

one can also guarantee that the discrete spectrum of $\mathcal{H}^n_{\alpha,\beta}$ within $(T_1,T_2)$ changes slightly provided $\beta_k$ are sufficiently large. More precisely, for an arbitrary sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that the neighborhoods $[s_k - \delta_k, s_k + \delta_k]$ are pairwise disjoint and belong to $(T_1,T_2) \setminus \overline{\mathcal{O}}$ one has

$$\begin{aligned}&\sigma_{\mathrm{disc}}(\mathcal{H}^n_{\alpha,\beta})\cap(T_1,T_2)\ \subset\ \bigcup_{k\in\mathbb{N}}[s_k-\delta_k,s_k+\delta_k]\\ &\text{and each } [s_k-\delta_k,s_k+\delta_k] \text{ contains precisely one simple eigenvalue,}\end{aligned}\tag{1.18}$$

provided the entries of the sequence $\beta$ are large enough (independent of $n$). It is important that the properties (1.15) and (1.18) remain valid if the coefficients $\alpha_k$, $k\in\mathbb{N}$, chosen in the first step are slightly perturbed.

Now, we *fix* a sequence $\beta$ for which (1.18) and some additional conditions for $\beta_k$ as $|k|\to\infty$ hold; see the next step. Then one can show that for each $n\in\mathbb{N}$ there exist $\alpha_k$, $k\in\mathbb{N}$, that in fact

$$\text{for } k\in\mathbb{Z}\cap[1,n] \text{ the eigenvalue of } \mathcal{H}^n_{\alpha,\beta} \text{ in } [s_k-\delta_k,s_k+\delta_k]\ \textit{coincides} \text{ with } s_k.\tag{1.19}$$

The proof of this fact is based on a multi-dimensional version of the intermediate value theorem proved in [18]. We denote the sequence $\alpha$ for which (1.19) holds by $\alpha^n$.

**3) Fully coupled operator.** Let $\alpha^n=(\alpha^n_k)_{k\in\mathbb{Z}}$, $n\in\mathbb{N}$, be the sequences from above (see the end of the previous step). Using a standard diagonal process one concludes that there exists a sequence $\alpha=(\alpha_k)_{k\in\mathbb{Z}}$ such that for each $k\in\mathbb{Z}$ one has $\alpha^n_k\to\alpha_k$ as $n\to\infty$ (probably, up to a subsequence which is independent $k$). As a result we get the operator $\mathcal{H}_{\alpha,\beta}$; it is obtained from $\mathcal{H}_{\alpha,\infty}$ by "inserting" $\delta$-interactions of the strengths $\beta_k$ at *all* points $x_k$, $k\in\mathbb{Z}$. We prove that, if $\lim_{k\to\infty}\beta_k=\infty$ and this convergence is fast enough, then also $\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\beta})=\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\infty})$. Hence, due to (1.15), we get

$$\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\beta})=S_{\mathrm{ess}}.$$

Moreover, we show that

$$\mathcal{H}^n_{\alpha^n,\beta} \text{ converges to } \mathcal{H}_{\alpha,\beta} \text{ in the norm resolvent sense as } n\to\infty.\tag{1.20}$$

Using (1.17)–(1.20) we arrive at

$$\sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\beta})\cap(T_1,T_2)=S_{\mathrm{disc}}.$$

Thus the operator $\mathcal{H}_{\alpha,\beta}$ satisfies (1.9) and (1.10); we have proved Theorem 1.1.

We note that for the rigorous definition of the singular Schrödinger operators above we will use the form approach, which is particularly convenient for our purposes. Alternatively, one can use extension theory methods as in [21] or regard a sequence of $\delta$-couplings as a distributional $\mathsf{W}^{-1,2}_{\mathrm{loc}}$-potential; cf. [25,26].

### *1.3. Schrödinger operators with $\delta'$-interactions*

We mention that a result similar to Theorem 1.1 can also be proved for singular Schrödinger operators with $\delta'$-interactions of the form

$$\mathcal{H}'_\gamma = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \sum_{k\in K} \gamma_k \langle \cdot\,,\, \delta'_{z_k}\rangle \delta'_{z_k}, \tag{1.21}$$

where $K \subset \mathbb{Z}$, $\delta'_{z_k}$ is the distributional derivative of the delta-function supported at $z_k \in \mathbb{R}$, $\langle \phi, \delta'_{z_k}\rangle$ denotes its action on the test function $\phi$, and $\gamma_k \in \mathbb{R} \cup \{\infty\}$. In contrast to (1.2) here the functions $u$ in the operator domain satisfy

$$u'(z_k - 0) = u'(z_k + 0) \quad \text{and} \quad u(z_k + 0) - u(z_k - 0) = \gamma_k u'(z_k \pm 0).$$

For the rigorous mathematical treatment of $\delta'$-interactions we refer to the standard monograph [3]. Among the subsequent contributions we mention the papers [21,22] dealing with the more subtle case $|z_k - z_{k-1}| \to 0$ as $|k| \to \infty$.

One has the following counterpart of Theorem 1.1.

**Theorem 1.2.** *Assume that the set $S_{\mathrm{ess}} \subset \mathbb{R}$, the sequence of real numbers $S_{\mathrm{disc}} = (s_k)_{k\in\mathbb{N}}$ and the interval $(T_1, T_2) \subset \mathbb{R}$ satisfy the conditions (1.3)–(1.7). Moreover, let $0 \in S_{\mathrm{ess}}$. Then there exists a bounded interval $(\ell_-, \ell_+) \subset \mathbb{R}$, a sequence of points $(z_k)_{k\in\mathbb{Z}}$ with $z_k \in (\ell_-, \ell_+)$, and a sequence of real numbers $(\gamma_k)_{k\in\mathbb{Z}}$ such that the operator $\mathcal{H}'_\gamma$ in $\mathsf{L}^2(\ell_-, \ell_+)$ defined by the formal expression (1.21) satisfies (1.9)–(1.10).*

The proof of the above theorem is similar (except for some technical details) to the proof of Theorem 1.1, therefore we only sketch it briefly. Note that in [6, Section 3] the "$\sigma_{\mathrm{ess}}$-part" of Theorem 1.2 was already shown (if $S \subset [0, \infty)$). Again we split the sequence $(z_k)_{k\in\mathbb{Z}}$ in (1.21) in two interlacing subsequences $(x_k)_{k\in\mathbb{Z}}$ and $(y_k)_{k\in\mathbb{Z}}$, where $y_k$ is in the center of $\mathcal{I}_k = (x_{k-1}, x_k)$. Instead of $\gamma_k$ we denote the interaction strengths at the points $x_k$ by $\beta_k$ and at the points $y_k$ by $\alpha_k$. We shall write $\mathcal{H}'_{\alpha,\beta}$ instead of $\mathcal{H}'_\gamma$ for the corresponding Schrödinger operator.

In the first step we set $\beta_k = \infty$, which corresponds to Neumann decoupling at the points $x_k$, and we consider

$$\mathcal{H}'_{\alpha,\infty} = \bigoplus_{k\in\mathbb{Z}} \mathbf{H}'_{\alpha_k,\mathcal{I}_k} \quad \text{in} \quad \mathsf{L}^2(\ell_-, \ell_+) = \bigoplus_{k\in\mathbb{Z}} \mathsf{L}^2(\mathcal{I}_k).$$

Here $\mathbf{H}'_{\alpha_k,\mathcal{I}_k}$ is the operator in $\mathsf{L}^2(\mathcal{I}_k)$ (formally) defined by the differential expression

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha_k \langle \cdot\,,\, \delta'_{z_k}\rangle \delta'_{z_k},$$

and Neumann boundary conditions at the endpoints of $\mathcal{I}_k$. In contrast to the operator $\mathbf{H}_{\alpha_k,\mathcal{I}_k}$, the spectrum of $\mathbf{H}'_{\alpha_k,\mathcal{I}_k}$ *always* contains the eigenvalue 0 (and leads to the additional condition $0 \in S$ in Theorem 1.2). Moreover, if $d_k = x_k - x_{k-1}$ is sufficiently small, one can choose $\alpha_k$ such that *the first nonzero eigenvalue* of $\mathbf{H}'_{\alpha_k,\mathcal{I}_k}$ coincides with $s_k$ (recall that $s_k$ are the elements of the sequence $S_{\mathrm{disc}}$ as $k \in \mathbb{N}$, while for $k \in \mathbb{Z} \setminus \mathbb{N}$ the numbers $s_k$ are defined in (1.14)). *The second nonzero eigenvalue* of $\mathbf{H}'_{\alpha_k,\mathcal{I}_k}$ is larger or equal to $\pi^2 d_k^{-2}$. Thus the properties (1.15)–(1.16) hold for the decoupled operator $\mathcal{H}'_{\alpha,\infty}$.

In the second step one perturbs $\mathcal{H}'_{\alpha,\infty}$ by "inserting" $\delta'$-interactions of strengths $\beta_k$ at finitely many points $x_k$, $k \in \mathbb{Z} \cap [-n+1, n-1]$. This perturbation does not change the essential spectrum, while the discrete spectrum will change slightly provided the constants $\beta_k$ are sufficiently large. Moreover, varying $\alpha_k$ one can even achieve a *precise coincidence* of the discrete spectrum within $(T_1, T_2)$ with a prescribed sequence $S_{\rm disc}$.

In the last step we pass to the limit $n \to \infty$ and prove that the above properties remain valid if $\beta_k \to \infty$ as $|k| \to \infty$ sufficiently fast (see the condition (3.18) in [6]).

### *1.4. Structure of the paper*

The paper is organized as follows. In Section 2 we recall the definition and some spectral properties of Schrödinger operators with a single $\delta$-interaction on a bounded interval. The decoupled operator $\mathcal{H}_{\alpha,\infty}$ is treated in Section 3 and the rigorous definition of the coupled operator $\mathcal{H}_{\alpha,\beta}$ and a precise formulation of our main result are contained in Section 4. In Section 5 we describe the essential spectrum of $\mathcal{H}_{\alpha,\beta}$. Section 6 is devoted to the partly coupled operator $\mathcal{H}^n_{\alpha,\beta}$ and its spectral properties. In Section 7 we describe the discrete spectrum of $\mathcal{H}_{\alpha,\beta}$ and complete the proof of our main result. Finally, in Appendix A we collect some useful material on the direct sum of semibounded closed forms and associated self-adjoint operators.

## 2. Single $\delta$-interaction on a bounded interval

In this section we recall the definition of Schrödinger operators on a bounded interval with a $\delta$-interaction supported at an internal point of the interval and either Dirichlet (this is the most important case for our constructions), Neumann or Robin boundary conditions; we also establish some spectral properties of these operators. For more details on $\delta$-interactions we refer to [3, Section I.3].

Throughout this paper $\lambda_j(\mathcal{H})$ denotes the $j$th eigenvalue of a self-adjoint operator $\mathcal{H}$ with purely discrete spectrum bounded from below and accumulating at $\infty$; as usual the eigenvalues are counted with multiplicities and ordered as a nondecreasing sequence. In the following let $\mathcal{I} = (x_-, x_+) \subset \mathbb{R}$ be a bounded interval of length $d(\mathcal{I}) = x_+ - x_-$ and middle point $y = \frac{x_- + x_+}{2}$. For $\mathbf{u}$, $\mathbf{v} \in \mathsf{W}^{1,2}(\mathcal{I})$, $\alpha$, $\beta_-$, $\beta_+ \in \mathbb{R}$, we consider

$$
\begin{aligned}
\mathbf{h}_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{v}] &= (\mathbf{u}',\mathbf{v}')_{\mathsf{L}^2(\mathcal{I})} + \alpha\,\mathbf{u}(y)\overline{\mathbf{v}(y)},\\
\mathbf{h}_{\mathcal{I},\alpha,\beta_-,\beta_+}[\mathbf{u},\mathbf{v}] &= \mathbf{h}_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{v}] + \frac{1}{2}\left(\beta_-\mathbf{u}(x_-)\overline{\mathbf{v}(x_-)} + \beta_+\mathbf{u}(x_+)\overline{\mathbf{v}(x_+)}\right)
\end{aligned}
\tag{2.1}
$$

(recall that $\mathsf{W}^{1,2}(\mathcal{I}) \subset \mathsf{C}(\overline{\mathcal{I}})$, that is, the values of $\mathbf{u}$, $\mathbf{v}$ at $x_-$, $x_+$, $y$ are well-defined).

### 2.1. Endpoints with Dirichlet boundary conditions

In the space $\mathsf{L}^2(\mathcal{I})$ we introduce the densely defined, symmetric sesquilinear form $\mathbf{h}^D_{\mathcal{I},\alpha}$ by

$$\mathbf{h}^D_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{v}] = \mathbf{h}_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{v}], \qquad \mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha}) = \mathsf{W}^{1,2}_0(\mathcal{I}).$$

A standard form perturbation argument shows that this form is bounded from below and closed in $\mathsf{L}^2(\mathcal{I})$, and the induced norm

$$\|\mathbf{u}\|_{\mathbf{h}^D_{\mathcal{I},\alpha}} := \left(\mathbf{h}^D_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{u}] - C\|\mathbf{u}\|^2_{\mathsf{L}^2(\mathcal{I})} + \|\mathbf{u}\|^2_{\mathsf{L}^2(\mathcal{I})}\right)^{1/2}, \text{ where } C = \inf_{\|\mathbf{u}\|_{\mathsf{L}^2(\mathcal{I})}=1} \mathbf{h}^D_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{u}],$$

on $\mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha})$ is equivalent to the usual norm on $\mathsf{W}^{1,2}_0(\mathcal{I})$. Therefore, by the first representation theorem (see, e.g. [20, Chapter 6, Theorem 2.1]) there exists a unique self-adjoint operator $\mathbf{H}^D_{\mathcal{I},\alpha}$ in $\mathsf{L}^2(\mathcal{I})$ such that $\mathrm{dom}(\mathbf{H}^D_{\mathcal{I},\alpha}) \subset \mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha})$ and

$$(\mathbf{H}^D_{\mathcal{I},\alpha}\mathbf{u},\mathbf{v})_{\mathsf{L}^2(\mathcal{I})} = \mathbf{h}^D_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{v}], \qquad \mathbf{u} \in \mathrm{dom}(\mathbf{H}^D_{\mathcal{I},\alpha}),\ \mathbf{v} \in \mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha}). \tag{2.2}$$

By inserting suitable test functions $\mathbf{v}$ in (2.2) one can easily derive the following explicit characterization of the domain and the action of $\mathbf{H}^D_{\mathcal{I},\alpha}$.

**Proposition 2.1.** *The self-adjoint operator $\mathbf{H}^D_{\mathcal{I},\alpha}$ associated to the form $\mathbf{h}^D_{\mathcal{I},\alpha}$ via (2.2) is given by*

$$\begin{aligned}
&(\mathbf{H}^D_{\mathcal{I},\alpha}\mathbf{u}) \restriction_{\mathcal{I}\setminus\{y\}} = -(\mathbf{u} \restriction_{\mathcal{I}\setminus\{y\}})'',\\
&\mathrm{dom}(\mathbf{H}^D_{\mathcal{I},\alpha}) = \left\{ \mathbf{u} \in \mathsf{W}^{2,2}(\mathcal{I}\setminus\{y\}) : \begin{array}{l} \mathbf{u}(x_-) = \mathbf{u}(x_+) = 0,\\ \mathbf{u}(y-0) = \mathbf{u}(y+0),\\ \mathbf{u}'(y+0) - \mathbf{u}'(y-0) = \alpha\mathbf{u}(y\pm 0) \end{array} \right\}.
\end{aligned}$$

By Rellich's theorem the space $(\mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha}), \|\cdot\|_{\mathbf{h}^D_{\mathcal{I},\alpha}})$ is compactly embedded in $\mathsf{L}^2(\mathcal{I})$; recall that the norm $\|\cdot\|_{\mathbf{h}^D_{\mathcal{I},\alpha}}$ is equivalent to the usual norm on $\mathsf{W}^{1,2}_0(\mathcal{I})$. Hence (see, e.g., [27, Proposition 10.6]) the spectrum of the operator $\mathbf{H}^D_{\mathcal{I},\alpha}$ is purely discrete. Using Proposition 2.1 one can easily calculate all eigenvalues of $\mathbf{H}^D_{\mathcal{I},\alpha}$. In order to formulate the related statement in Proposition 2.2 below we introduce for $d > 0$ the set

$$\Pi^D_d = \left\{ \left(\frac{2\pi k}{d}\right)^2 : k \in \mathbb{N} \right\}$$

and the function $\mathcal{F}^D_d : \mathbb{R} \setminus \Pi^D_d \to \mathbb{R}$,

$$\mathcal{F}_d^D(\lambda) = \begin{cases} -2\sqrt{\lambda}\cot\left(\dfrac{d\sqrt{\lambda}}{2}\right), & \lambda > 0, \\ -\dfrac{4}{d}, & \lambda = 0, \\ -2\sqrt{-\lambda}\coth\left(\dfrac{d\sqrt{-\lambda}}{2}\right), & \lambda < 0. \end{cases} \tag{2.3}$$

For any fixed $d > 0$ the function $\mathcal{F}_d^D$ is continuous and monotonically increasing on each connected component of $\mathbb{R} \setminus \Pi_d^D$, moreover we have

$$\lim_{\lambda\to-\infty} \mathcal{F}_d^D(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda\to\mu\mp 0} \mathcal{F}_d^D(\lambda) = \pm\infty \text{ for all } \mu \in \Pi_d^D. \tag{2.4}$$

In particular, for $d > 0$ fixed and any $\alpha \in \mathbb{R}$ the equation $\alpha = \mathcal{F}_d^D(\lambda)$ has a unique solution in $(-\infty, (2\pi/d)^2)$; this solution will be denoted by $\Lambda_{\alpha,d}^D$ in the following.

**Proposition 2.2.** *The spectrum of the self-adjoint operator $\mathbf{H}_{\mathcal{I},\alpha}^D$ is given by*

$$\sigma(\mathbf{H}_{\mathcal{I},\alpha}^D) = \Pi_{d(\mathcal{I})}^D \cup \left\{\lambda \in \mathbb{R} \setminus \Pi_{d(\mathcal{I})}^D : \mathcal{F}_{d(\mathcal{I})}^D(\lambda) = \alpha\right\}$$

*and one has*

$$\lambda_1(\mathbf{H}_{\mathcal{I},\alpha}^D) = \Lambda_{\alpha,d(\mathcal{I})}^D, \quad \lambda_2(\mathbf{H}_{\mathcal{I},\alpha}^D) = \left(\frac{2\pi}{d(\mathcal{I})}\right)^2.$$

*2.2. Endpoints with Neumann boundary conditions*

Besides the form $\mathbf{h}_{\mathcal{I},\alpha}^D$ we also consider the densely defined, symmetric sesquilinear form

$$\mathbf{h}_{\mathcal{I},\alpha}^N[\mathbf{u},\mathbf{v}] = \mathbf{h}_{\mathcal{I},\alpha}[\mathbf{u},\mathbf{v}], \qquad \mathrm{dom}(\mathbf{h}_{\mathcal{I},\alpha}^D) = \mathsf{W}^{1,2}(\mathcal{I}),$$

in $\mathsf{L}^2(\mathcal{I})$. This form is also bounded from below and closed, and hence there exists a unique self-adjoint operator $\mathbf{H}_{\mathcal{I},\alpha}^N$ in $\mathsf{L}^2(\mathcal{I})$ such that $\mathrm{dom}(\mathbf{H}_{\mathcal{I},\alpha}^N) \subset \mathrm{dom}(\mathbf{h}_{\mathcal{I},\alpha}^N)$ and

$$(\mathbf{H}_{\mathcal{I},\alpha}^N\mathbf{u},\mathbf{v})_{\mathsf{L}^2(\mathcal{I})} = \mathbf{h}_{\mathcal{I},\alpha}^N[\mathbf{u},\mathbf{v}], \qquad \mathbf{u} \in \mathrm{dom}(\mathbf{H}_{\mathcal{I},\alpha}^N),\ \mathbf{v} \in \mathrm{dom}(\mathbf{h}_{\mathcal{I},\alpha}^N). \tag{2.5}$$

The counterpart of Proposition 2.1 in the present situation reads as follows.

**Proposition 2.3.** *The self-adjoint operator $\mathbf{H}_{\mathcal{I},\alpha}^D$ associated to the form $\mathbf{h}_{\mathcal{I},\alpha}^N$ via (2.5) is given by*

$$(\mathbf{H}_{\mathcal{I},\alpha}^N\mathbf{u}) \restriction_{\mathcal{I}\setminus\{y\}} = -(\mathbf{u} \restriction_{\mathcal{I}\setminus\{y\}})'',$$

$$\mathrm{dom}(\mathbf{H}_{\mathcal{I},\alpha}^N) = \left\{ \mathbf{u} \in \mathsf{W}^{2,2}(\mathcal{I}\setminus\{y\}) : \begin{array}{l} \mathbf{u}'(x_-) = \mathbf{u}'(x_+) = 0, \\ \mathbf{u}(y-0) = \mathbf{u}(y+0), \\ \mathbf{u}'(y+0) - \mathbf{u}'(y-0) = \alpha\mathbf{u}(y\pm 0) \end{array} \right\}.$$

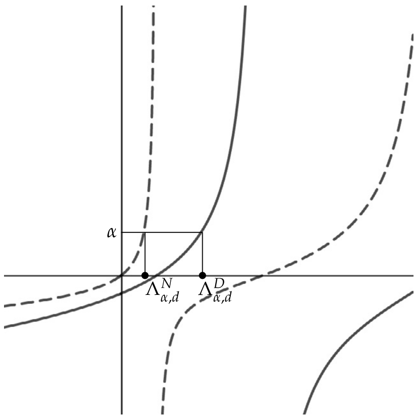


**Fig. 3.** The functions $\mathcal{F}_d^D$ (solid plot) and $\mathcal{F}_d^N$ (dashed plot), and the unique solutions $\Lambda^D_{\alpha,d}$ of $\alpha = \mathcal{F}_d^D(\lambda)$ in $(-\infty,\, (2\pi/d)^2)$ and $\Lambda^N_{\alpha,d}$ of $\alpha = \mathcal{F}_d^N(\lambda)$ in $(-\infty,\, (\pi/d)^2)$.

It follows that the spectrum of $\mathbf{H}^N_{\mathcal{I},\alpha}$ is purely discrete. In a similar way as in the previous subsection we consider for $d > 0$ the set

$$\Pi_d^N = \left\{ \left( \frac{\pi(2k-1)}{d} \right)^2 : k \in \mathbb{N} \right\}$$

and the function

$$\mathcal{F}_d^N(\lambda) = \begin{cases} 2\sqrt{\lambda}\tan\left(\dfrac{d\sqrt{\lambda}}{2}\right), & \lambda > 0, \\ 0, & \lambda = 0, \\ -2\sqrt{-\lambda}\tanh\left(\dfrac{d\sqrt{-\lambda}}{2}\right), & \lambda < 0. \end{cases}$$

The function $\mathcal{F}_d^N$ is continuous and monotonically increasing on each connected component of $\mathbb{R} \setminus \Pi_d^N$, and the properties in (2.4) hold also for $\mathcal{F}_d^N$. In particular, for $d > 0$ fixed and any $\alpha \in \mathbb{R}$ the equation $\alpha = \mathcal{F}_d^N(\lambda)$ has a unique solution $\Lambda^N_{\alpha,d}$ in $(-\infty,\, (\pi/d)^2)$; cf. Fig. 3.

**Proposition 2.4.** *The spectrum of the self-adjoint operator* $\mathbf{H}^N_{\mathcal{I},\alpha}$ *is given by*

$$\sigma(\mathbf{H}^N_{\mathcal{I},\alpha}) = \Pi^N_{d(\mathcal{I})} \cup \left\{ \lambda \in \mathbb{R} \setminus \Pi^N_{d(\mathcal{I})} : \mathcal{F}^N_{d(\mathcal{I})}(\lambda) = \alpha \right\}$$

*and one has*

$$\lambda_1(\mathbf{H}^N_{\mathcal{I},\alpha}) = \Lambda^N_{\alpha,d(\mathcal{I})}, \quad \lambda_2(\mathbf{H}^N_{\mathcal{I},\alpha}) = \left( \frac{\pi}{d(\mathcal{I})} \right)^2.$$

### 2.3. Endpoints with Robin boundary conditions

In this subsection we consider the densely defined, symmetric sesquilinear form

$$\mathbf{h}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}[\mathbf{u},\mathbf{v}] = \mathbf{h}_{\mathcal{I},\alpha,\beta_-,\beta_+}[\mathbf{u},\mathbf{v}], \qquad \mathrm{dom}(\mathbf{h}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}) = \mathsf{W}^{1,2}(\mathcal{I}),$$

which is again bounded from below and closed in $\mathsf{L}^2(\mathcal{I})$; we shall use later that the induced norm on $\mathrm{dom}(\mathbf{h}^R_{\mathcal{I},\alpha,\beta_-,\beta_+})$ is equivalent to the usual norm on $\mathsf{W}^{1,2}(\mathcal{I})$. The corresponding self-adjoint operator $\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}$ in $\mathsf{L}^2(\mathcal{I})$ has the following form.

**Proposition 2.5.** *The self-adjoint operator $\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}$ associated to the form $\mathbf{h}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}$ is given by*

$$(\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}\mathbf{u}) \restriction_{\mathcal{I}\setminus\{y\}} = -(\mathbf{u} \restriction_{\mathcal{I}\setminus\{y\}})'',$$

$$\mathrm{dom}(\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}) = \left\{ \mathbf{u} \in \mathsf{W}^{2,2}(\mathcal{I}\setminus\{y\}) : \begin{array}{l} \mathbf{u}'(x_-) = \frac{1}{2}\beta_-\mathbf{u}(x_-), \\ \mathbf{u}'(x_+) = -\frac{1}{2}\beta_+\mathbf{u}(x_+), \\ \mathbf{u}(y-0) = \mathbf{u}(y+0), \\ \mathbf{u}'(y+0) - \mathbf{u}'(y-0) = \alpha\mathbf{u}(y\pm 0) \end{array} \right\}.$$

The spectrum of the operator $\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}$ is purely discrete. It can be determined in a similar way as in the previous subsections. However, the precise values are not needed for our purposes. Instead, we make use of the fact that for large $\beta_\pm$ the eigenvalues of $\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}$ are close to the eigenvalues of the self-adjoint operator $\mathbf{H}^D_{\mathcal{I},\alpha}$.

**Proposition 2.6.** *For the $j$-th eigenvalue of the self-adjoint operators $\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}$ and $\mathbf{H}^D_{\mathcal{I},\alpha}$ one has*

$$\lambda_j(\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+}) \to \lambda_j(\mathbf{H}^D_{\mathcal{I},\alpha}) \text{ as } \min\{\beta_-,\beta_+\} \to +\infty. \tag{2.6}$$

**Proof.** First we note that by the min-max principle (see, e.g., [17, Section 4.5]) the function

$$\mathbb{R}^2 \ni (\beta_-,\beta_+) \mapsto \lambda_j(\mathbf{H}^R_{\mathcal{I},\alpha,\beta_-,\beta_+})$$

is monotonically increasing in each of its arguments. Therefore it suffices to show that

$$\lambda_j(\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta}) \to \lambda_j(\mathbf{H}^D_{\mathcal{I},\alpha}) \text{ as } \beta \to +\infty. \tag{2.7}$$

Without loss of generality we may assume in the following that $\beta \geq 0$. Note first that for $\beta \leq \widetilde{\beta}$ we have $\mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta} \leq \mathbf{h}^R_{\mathcal{I},\alpha,\widetilde{\beta},\widetilde{\beta}}$ in the (usual) form sense. In the present situation this means

$$\mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta}[u,u] \leq \mathbf{h}^R_{\mathcal{I},\alpha,\widetilde{\beta},\widetilde{\beta}}[u,u], \qquad u \in \mathrm{dom}(\mathbf{h}^R_{\mathcal{I},\alpha,\widetilde{\beta},\widetilde{\beta}}) = \mathrm{dom}(\mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta}).$$

Moreover, it is easy to see that

$$\mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha}) = \left\{ u \in \bigcap_{\beta\geq 0} \mathrm{dom}(\mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta}) :\ \sup_{\beta\geq 0} \mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta}[u,u] < \infty \right\}$$

and that $\mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta}[u,u] = \mathbf{h}^D_{\mathcal{I},\alpha}[u,u]$ holds for all $u \in \mathrm{dom}(\mathbf{h}^D_{\mathcal{I},\alpha})$. Therefore, [28, Theorem 3.1] (see also [5, Theorem 4.2]) implies the strong resolvent convergence

$$\forall f \in \mathsf{L}^2(\mathcal{I}) : \quad \left\|(\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta} - \mu\mathrm{I})^{-1} f - (\mathbf{H}^D_{\mathcal{I},\alpha} - \mu\mathrm{I})^{-1} f\right\|_{\mathsf{L}^2(\mathcal{I})} \to 0 \text{ as } \beta \to +\infty, \tag{2.8}$$

where $\mu \in \rho(\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta}) \cap \rho(\mathbf{H}^D_{\mathcal{I},\alpha})$, and as usual I stands for the identity operator. Now observe that we can choose $\mu < \min\{\sigma(\mathbf{H}^D_{\mathcal{I},\alpha}), \sigma(\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta})\}$, $\beta \geq 0$. Then $\mathbf{h}^R_{\mathcal{I},\alpha,\beta,\beta} \leq \mathbf{h}^D_{\mathcal{I},\alpha}$ implies

$$(\mathbf{H}^D_{\mathcal{I},\alpha} - \mu\mathrm{I})^{-1} \leq (\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta} - \mu\mathrm{I})^{-1},$$

and since the resolvents $(\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta} - \mu\mathrm{I})^{-1}$ and $(\mathbf{H}^D_{\mathcal{I},\alpha} - \mu\mathrm{I})^{-1}$ are both compact in $\mathsf{L}^2(\mathcal{I})$, we conclude from [20, Theorem VIII-3.5] that the strong convergence in (2.8) becomes even convergence in the operator norm, i.e.

$$\left\|(\mathbf{H}^R_{\mathcal{I},\alpha,\beta,\beta} - \mu\mathrm{I})^{-1} - (\mathbf{H}^D_{\mathcal{I},\alpha} - \mu\mathrm{I})^{-1}\right\| \to 0 \text{ as } \beta \to +\infty. \tag{2.9}$$

It is well-known (see, e.g., [23, Corollary A.15]) that (2.9) implies (2.7), and hence (2.6). $\square$

## 3. The decoupled operator $\mathcal{H}_{\alpha,\infty}$

In this section we define and study the spectrum of the self-adjoint operator

$$\mathcal{H}_{\alpha,\infty} = \bigoplus_{k\in\mathbb{Z}} \mathbf{H}^D_{\mathcal{I}_k,\alpha_k} \quad \text{in} \quad \bigoplus_{k\in\mathbb{Z}} \mathsf{L}^2(\mathcal{I}_k)$$

with suitably chosen interaction strengths $\alpha_k$ and intervals $\mathcal{I}_k$ that are stacked in a row with finite total length.

### *3.1. Auxiliary sequence $\widehat{S}_{\mathrm{ess}}$ and the intervals $\mathcal{I}_k$*

Recall that the set $S_{\mathrm{ess}}$ and the sequence $S_{\mathrm{disc}} = (s_k)_{k\in\mathbb{N}}$ satisfying (1.3)–(1.7) are already given. It is easy to see that, due to (1.3)–(1.4), one can always find a sequence $\widehat{S}_{\mathrm{ess}} = (s_k)_{k\in\mathbb{Z}\setminus\mathbb{N}}$ such that

$$S_{\mathrm{ess}} = \left\{\text{accumulation points of } \widehat{S}_{\mathrm{ess}}\right\}, \tag{3.1}$$

$$\widehat{S}_{\mathrm{ess}} \cap [T_1, T_2] \subset \mathcal{O}. \tag{3.2}$$

Note that the elements of $S_{\rm disc}$ and $\widehat{S}_{\rm ess}$ are both denoted by $s_k$, but $s_k$ with index $k \in \mathbb{N}$ belongs to $S_{\rm disc}$, while $s_k$ with index $k \in \mathbb{Z} \setminus \mathbb{N}$ is an element of $\widehat{S}_{\rm ess}$. Recall from (1.5) that the sequence $S_{\rm disc} = (s_k)_{k\in\mathbb{N}}$ is contained in $(T_1, T_2)$. For all $k \in \mathbb{Z}$ we fix $d_k > 0$ such that

$$T_2 < \min_{k\in\mathbb{Z}} (\pi/d_k)^2 \qquad \text{and} \qquad s_k < (\pi/d_k)^2, \quad k \in \mathbb{Z}, \tag{3.3}$$

and we assume, in addition, that the numbers $d_k$ satisfy

$$\sum_{k\in\mathbb{Z}} d_k < \infty. \tag{3.4}$$

In particular, this implies

$$d_k \to 0 \text{ as } k \to \pm\infty. \tag{3.5}$$

Note that for $k \in \mathbb{N}$ the first condition in (3.3) implies the second condition since $s_k \in (T_1, T_2)$ for $k \in \mathbb{N}$, and hence the second condition is only needed for $d_k$ (and $s_k$) with index $k \in \mathbb{Z} \setminus \mathbb{N}$. Finally, we set (see Fig. 2)

$$\mathcal{I}_k = (x_{k-1}, x_k), \quad k \in \mathbb{Z},$$

where

$$x_0 = 0, \quad x_k = \begin{cases} x_{k-1} + d_k, & k \in \mathbb{N}, \\ x_{k+1} - d_{k+1}, & k \in \mathbb{Z} \setminus (\mathbb{N} \cup \{0\}). \end{cases}$$

The intervals $\mathcal{I}_k$ satisfy $\cup_{k\in\mathbb{Z}} \overline{\mathcal{I}_k} = [\ell_-, \ell_+]$ with $\ell_\pm$ in (1.13). Due to (3.4) the interval $[\ell_-, \ell_+]$ is compact.

### *3.2. Choice of the interaction strengths $\alpha_k$*

In what follows we denote by $B_\delta(s)$ the open $\delta$-neighborhood of $s \in \mathbb{R}$, i.e.

$$B_\delta(s) = (s - \delta, s + \delta).$$

Let us fix a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive numbers with the properties

$$\overline{B_{\delta_k}(s_k)} \subset (T_1, T_2) \setminus \overline{\mathcal{O}}, \quad k \in \mathbb{N}, \tag{3.6}$$

$$\overline{B_{\delta_k}(s_k)} \cap \overline{B_{\delta_k}(s_l)} = \varnothing, \quad k \neq l. \tag{3.7}$$

The above choice of $\delta_k$ is always possible due to (1.5) and (1.8). Moreover, we claim that $\delta_k$ can be chosen so small that

$$\alpha_k^+ - \alpha_k^- \le c_k d_k \text{ with } c_k \to 0 \text{ as } k \to \infty, \tag{3.8}$$

where

$$\alpha_k^\pm = \mathcal{F}_{d_k}^D(s_k \pm \frac{1}{2}\delta_k). \tag{3.9}$$

In fact, (3.8) holds for $\delta_k$ small enough since the function $\mathcal{F}_{d_k}^D$ in (2.3) is continuous on $(-\infty, (2\pi/d_k)^2)$, and

$$s_k + \frac{1}{2}\delta_k < s_k + \delta_k < T_2 < \min_{k\in\mathbb{Z}}(\pi/d_k)^2 \tag{3.10}$$

(cf. (3.3), (3.6)). Condition (3.8) will be required only in the last step of our construction; cf. Lemma 7.1. Note, that by (3.10)

$$\alpha_k^\pm < 0, \qquad k \in \mathbb{N}, \tag{3.11}$$

since $\mathcal{F}_{d_k}^D((\pi/d_k)^2) = 0$ and $\mathcal{F}_{d_k}^D$ is strictly increasing on the interval $(-\infty, (2\pi/d_k)^2)$.

From now on we consider sequences $(\alpha_k)_{k\in\mathbb{Z}}$ that satisfy the following hypothesis.

**Hypothesis 3.1.** The notation $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ is used for a sequence of real numbers with the properties

$$\begin{aligned} &\alpha_k \in [\alpha_k^-, \alpha_k^+], \quad k \in \mathbb{N}, \\ &\alpha_k = \mathcal{F}_{d_k}^D(s_k), \quad k \in \mathbb{Z}\setminus\mathbb{N}. \end{aligned} \tag{3.12}$$

*3.3. The decoupled operator $\mathcal{H}_{\alpha,\infty}$*

Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ be a sequence satisfying Hypothesis 3.1. For each $k \in \mathbb{Z}$ we consider the sesquilinear form $\mathbf{h}_{\mathcal{I}_k,\alpha_k}^D$ and the associated self-adjoint operator $\mathbf{H}_{\mathcal{I}_k,\alpha_k}^D$ in Section 2.1 with $\alpha$, $\mathcal{I}$, $x_-$, $x_+$, and $y$ replaced by $\alpha_k$, $\mathcal{I}_k$, $x_{k-1}$, $x_k$, and $y_k = \frac{x_{k-1}+x_k}{2}$, respectively. The spectrum of $\mathbf{H}_{\mathcal{I}_k,\alpha_k}^D$ is discrete and by Proposition 2.2 the first eigenvalue $\lambda_1(\mathbf{H}_{\mathcal{I}_k,\alpha_k}^D)$ is the unique solution of the equation $\alpha_k = \mathcal{F}_{d_k}^D(\lambda)$ in $(0, (2\pi/d_k)^2)$. Therefore, taking into account the monotonicity and continuity of the function $\mathcal{F}_{d_k}^D$, (3.9), and (3.10) we conclude that

$$\lambda_1(\mathbf{H}_{\mathcal{I}_k,\alpha_k}^D) \in \overline{B_{\delta_k/2}(s_k)}, \qquad k \in \mathbb{N}, \tag{3.13}$$

$$\lambda_1(\mathbf{H}_{\mathcal{I}_k,\alpha_k}^D) = s_k, \qquad k \in \mathbb{Z}\setminus\mathbb{N}, \tag{3.14}$$

and Proposition 2.2 also gives

$$\lambda_2(\mathbf{H}_{\mathcal{I}_k,\alpha_k}^D) = \left(\frac{2\pi}{d_k}\right)^2, \qquad k \in \mathbb{Z}. \tag{3.15}$$

It is clear from (3.13) and (3.14) that the forms $\mathbf{h}^D_{\mathcal{I}_k,\alpha_k}$ are bounded from below by

$$s_{\inf} = \inf\left\{ \inf_{k\in\mathbb{N}} (s_k - \delta_k/2);\ \inf_{k\in\mathbb{Z}\setminus\mathbb{N}} s_k \right\}, \tag{3.16}$$

and from (1.3) and (3.6) we conclude $s_{\inf} \geq \min\{T_1; \min S_{\mathrm{ess}}\} > -\infty$.

Following Appendix A.3 we consider the densely defined, semibounded, closed form

$$\mathfrak{h}_{\alpha,\infty} = \bigoplus_{k\in\mathbb{Z}} \mathbf{h}^D_{\mathcal{I}_k,\alpha_k} \quad \text{in} \quad \mathsf{L}^2(\ell_-,\ell_+) = \bigoplus_{k\in\mathbb{Z}} \mathsf{L}^2(\mathcal{I}_k)$$

and the corresponding self-adjoint operator $\mathcal{H}_{\alpha,\infty}$. In the present situation Proposition A.4 and Theorem A.5 lead to the next statement.

**Theorem 3.2.** *Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ be a sequence satisfying Hypothesis 3.1. Then the self-adjoint operator $\mathcal{H}_{\alpha,\infty}$ associated to the form $\mathfrak{h}_{\alpha,\infty}$ in $\mathsf{L}^2(\ell_-,\ell_+)$ is given by*

$$(\mathcal{H}_{\alpha,\infty} u)\restriction_{\mathcal{I}_k} = \mathbf{H}^D_{\mathcal{I}_k,\alpha_k}\mathbf{u}_k = -(\mathbf{u}_k\restriction_{\mathcal{I}_k\setminus\{y_k\}})'',$$

$$\mathrm{dom}(\mathcal{H}_{\alpha,\infty}) = \left\{ u \in \mathsf{L}^2(\ell_-,\ell_+):\ \mathbf{u}_k \in \mathrm{dom}(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}) \text{ and } \sum_{k\in\mathbb{Z}} \|\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} < \infty \right\},$$

*where $\mathbf{u}_k$ stands for the restriction of $u$ on $\mathcal{I}_k$. Furthermore, one has*

$$\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\infty}) = S_{\mathrm{ess}}. \tag{3.17}$$

**Proof.** It is clear from Proposition A.4 and Proposition 2.1 that the self-adjoint operator $\mathcal{H}_{\alpha,\infty}$ is given as in the theorem. Hence it remains to verify (3.17). In fact, by Theorem A.5 we have

$$\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\infty}) = \{\text{accumulation points of } S\}, \tag{3.18}$$

where $S = (\lambda_j(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}))_{j\in\mathbb{N},k\in\mathbb{Z}}$ is the sequence of all eigenvalues of the operators $\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}$. By (3.5) and (3.15) one has

$$\lambda_j(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}) \to \infty \text{ as } |k| \to \infty, \qquad j \geq 2,$$

and using (3.1) and (3.14) we get

$$\{\text{accumulation points of } (\lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}))_{k\in\mathbb{Z}\setminus\mathbb{N}}\} = S_{\mathrm{ess}}.$$

Finally, it follows easily from (3.6), (3.7), and (3.13) that

$$\{\text{accumulation points of } (\lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}))_{k\in\mathbb{N}}\} = \{\text{accumulation points of } (s_k)_{k\in\mathbb{N}}\}.$$

Therefore, taking into account (1.7), we get

$$\big\{\text{accumulation points of } (\lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}))_{k\in\mathbb{N}}\big\} \subset S_{\mathrm{ess}}. \tag{3.19}$$

Combining (3.18)–(3.19) we arrive at (3.17). □

**Remark 3.3.** If $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ is a sequence satisfying Hypothesis 3.1 and $\mathcal{H}_{\alpha,\infty}$ is the self-adjoint operator in the previous theorem then one also has

$$\sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\infty}) \cap (T_1,T_2) = \big\{\lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}) :\ k\in\mathbb{N}\big\}$$

and each of these eigenvalues is simple, i.e. $\dim\ker(\mathcal{H}_{\alpha,\infty} - \lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k})\mathrm{I}) = 1$; these facts follow again from Proposition 2.2, (3.6), (3.7), (3.13), (3.15), and Theorem A.5. In particular, if $\alpha_k = \mathcal{F}^D_{d_k}(s_k)$ for **all** $k\in\mathbb{Z}$, then

$$\sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\infty}) \cap (T_1,T_2) = S_{\mathrm{disc}}.$$

The following lemma on the semiboundedness of the forms $\mathbf{h}_{\mathcal{I}_k,\alpha_k}$ will be used later.

**Lemma 3.4.** *Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ be a sequence satisfying Hypothesis 3.1. Then there exists a constant $C > 0$ which depends only on the quantity $s_{\inf}$ in (3.16) and the interval $(\ell_-,\ell_+)$ such that for all $k\in\mathbb{Z}$*

$$\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{u},\mathbf{u}] + \frac{C}{d_k^2}\|\mathbf{u}\|^2_{\mathsf{L}^2(\mathcal{I}_k)} \ge 0, \qquad \mathbf{u}\in\mathsf{W}^{1,2}(\mathcal{I}_k). \tag{3.20}$$

**Proof.** Consider the function $\mathcal{F}^D_{d_k}$ defined by (2.3) and recall that it is monotonically increasing on $(-\infty,(2\pi/d_k)^2)$ and $\mathcal{F}^D_{d_k}((\pi/d_k)^2) = 0$. It follows from Hypothesis 3.1, the choice of $d_k$ in (3.3), and (3.16) that

$$0 > \alpha_k \ge \mathcal{F}^D_{d_k}(s_{\inf}). \tag{3.21}$$

In the case $s_{\inf} \ge 0$ we have the estimate

$$\mathcal{F}^D_{d_k}(s_{\inf}) \ge \mathcal{F}^D_{d_k}(0) = -\frac{4}{d_k},$$

and in the case $s_{\inf} < 0$ we make use of the fact that the function $x \mapsto x\coth(x)$ is increasing on $[0,\infty)$ and $d_k < \ell_+ - \ell_-$ to derive

$$\begin{aligned}\mathcal{F}^D_{d_k}(s_{\inf}) &= -\frac{4}{d_k}\cdot\frac{d_k\sqrt{-s_{\inf}}}{2}\coth\frac{d_k\sqrt{-s_{\inf}}}{2}\\ &\ge -\frac{4}{d_k}\cdot\frac{(\ell_+-\ell_-)\sqrt{-s_{\inf}}}{2}\coth\frac{(\ell_+-\ell_-)\sqrt{-s_{\inf}}}{2}.\end{aligned} \tag{3.22}$$

Combining (3.21)–(3.22) we conclude that there exists a constant $\widehat{C} > 0$ which depends only on $s_{\inf}$ and $(\ell_-, \ell_+)$ such that

$$\alpha_k \geq -\frac{\widehat{C}}{d_k}, \qquad k \in \mathbb{Z}. \tag{3.23}$$

By Proposition 2.4 we have

$$\lambda_1(\mathbf{H}^N_{\mathcal{I}_k,\alpha_k}) = \Lambda^N_{\alpha_k,d_k},$$

where $\Lambda^N_{\alpha_k,d_k}$ is the unique solution of $\alpha_k = \mathcal{F}^N_{d_k}(\lambda)$ in $(-\infty, (\pi/d_k)^2)$. Using (3.23) and the monotonicity of the function $\mathcal{F}^N_d$ we obtain

$$\lambda_1(\mathbf{H}^N_{\mathcal{I}_k,\alpha_k}) \geq \Lambda^N_{-\frac{\widehat{C}}{d_k},d_k} = -\frac{4\widehat{\lambda}^2}{d_k^2}, \qquad \text{where } \widehat{\lambda} = \frac{d_k}{2}\sqrt{-\Lambda^N_{-\frac{\widehat{C}}{d_k},d_k}} > 0$$

is the unique solution of $4\widehat{\lambda}\tanh\widehat{\lambda} = \widehat{C}$. From this we finally conclude that there exists $C > 0$ such that (3.20) holds. $\square$

## 4. The coupled operator $\boldsymbol{\mathcal{H}_{\alpha,\beta}}$

In this short section we introduce the self-adjoint operator $\mathcal{H}_{\alpha,\beta}$ in $\mathsf{L}^2(\ell_-, \ell_+)$ and reformulate our main result Theorem 1.1 in a more rigorous form. The operator $\mathcal{H}_{\alpha,\beta}$ corresponding to the form $\mathfrak{h}_{\alpha,\beta}$ below will be our main object of interest in this paper, it can be viewed as perturbation of the decoupled operator $\mathcal{H}_{\alpha,\infty}$ in the sense that all neighboring intervals $\mathcal{I}_k$ and $\mathcal{I}_{k+1}$ are glued together via $\delta$-couplings of sufficiently large interaction strengths.

Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ be a sequence which satisfies Hypothesis 3.1 and let $\beta = (\beta_k)_{k\in\mathbb{Z}}$ be another sequence of real numbers. In the space $\mathsf{L}^2(\ell_-, \ell_+)$ we introduce the symmetric sesquilinear form $\mathfrak{h}_{\alpha,\beta}$ by

$$\begin{aligned} \mathfrak{h}_{\alpha,\beta}[u,v] &= \sum_{k\in\mathbb{Z}} \mathbf{h}_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}[\mathbf{u}_k, \mathbf{v}_k], \\ \mathrm{dom}(\mathfrak{h}_{\alpha,\beta}) &= \left\{ u \in \mathsf{W}^{1,2}_{\mathrm{loc}}(\ell_-, \ell_+) \cap \mathsf{L}^2(\ell_-, \ell_+) : \sum_{k\in\mathbb{Z}} |\mathbf{h}_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}[\mathbf{u}_k, \mathbf{u}_k]| < \infty \right\}, \end{aligned} \tag{4.1}$$

where $\mathbf{u}_k = u \restriction_{\mathcal{I}_k}$, $\mathbf{v}_k = v \restriction_{\mathcal{I}_k}$, and the forms $\mathbf{h}_{\mathcal{I}_k,\alpha,\beta_{k-1},\beta_k}$ are defined as in (2.1).

**Lemma 4.1.** *There exists a sequence $\beta^{\inf} = (\beta^{\inf}_k)_{k\in\mathbb{Z}}$ of real numbers such that for any sequence $\beta = (\beta_k)_{k\in\mathbb{Z}}$ satisfying*

$$\beta_k^{\inf} \le \beta_k < \infty \tag{4.2}$$

*and any sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ satisfying Hypothesis 3.1 the form $\mathfrak{h}_{\alpha,\beta}$ is densely defined, closed, and bounded from below by $s_{\inf} - 1$, where $s_{\inf}$ is the quantity specified in (3.16).*

**Proof.** For $k \in \mathbb{Z}$ consider the densely defined, closed, semibounded form $\mathbf{h}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}$ and the associated self-adjoint operator $\mathbf{H}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}$ as in Section 2.3, with $\alpha$, $\mathcal{I}$, $\beta_-$, and $\beta_+$ replaced by $\alpha_k$, $\mathcal{I}_k$, $\beta_{k-1}$, and $\beta_k$, respectively. By Proposition 2.6 we have

$$\lambda_1(\mathbf{H}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}) \nearrow \lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}) \text{ as } \min\{\beta_{k-1},\beta_k\} \to \infty \tag{4.3}$$

for any $k \in \mathbb{Z}$. Hence we conclude from (4.3) that there exists a sequence $\beta^{\inf} = (\beta_k^{\inf})_{k\in\mathbb{Z}}$ such that

$$\lambda_1(\mathbf{H}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}) \ge \lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}) - 1.$$

Taking into account (3.13)–(3.14), we get

$$\mathbf{h}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}[\mathbf{u},\mathbf{u}] \ge (s_{\inf} - 1)\|\mathbf{u}\|^2_{\mathsf{L}^2(\mathcal{I}_k)}, \qquad \mathbf{u} \in \mathsf{W}^{1,2}(\mathcal{I}_k),\ k \in \mathbb{Z},$$

for any sequence $\beta = (\beta_k)_{k\in\mathbb{Z}}$ satisfying (4.2) and any sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ satisfying Hypothesis 3.1.

Following Appendix A.3 we introduce in $\mathsf{L}^2(\ell_-,\ell_+)$ the densely defined, closed form

$$\widetilde{\mathfrak{h}}_{\alpha,\beta} = \bigoplus_{k\in\mathbb{Z}} \mathbf{h}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k},$$

that is,

$$\widetilde{\mathfrak{h}}_{\alpha,\beta}[u,v] = \sum_{k\in\mathbb{Z}} \mathbf{h}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}[\mathbf{u}_k,\mathbf{v}_k],$$

$$\mathrm{dom}(\widetilde{\mathfrak{h}}_{\alpha,\beta}) = \left\{ u \in \mathsf{L}^2(\ell_-,\ell_+) :\ \mathbf{u}_k \in \mathsf{W}^{1,2}(\mathcal{I}_k),\ \sum_{k\in\mathbb{Z}} |\mathbf{h}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}[\mathbf{u}_k,\mathbf{u}_k]| < \infty \right\},$$

which is bounded from below by $s_{\inf} - 1$. Observe that the form $\mathfrak{h}_{\alpha,\beta}$ in (4.1) is the restriction of $\widetilde{\mathfrak{h}}_{\alpha,\beta}$ onto

$$\mathrm{dom}(\mathfrak{h}_{\alpha,\beta}) = \mathrm{dom}(\widetilde{\mathfrak{h}}_{\alpha,\beta}) \cap \mathsf{W}^{1,2}_{\mathrm{loc}}(\ell_-,\ell_+). \tag{4.4}$$

This implies that $\mathfrak{h}_{\alpha,\beta}$ is also bounded from below by $s_{\inf} - 1$ and it is clear from (4.1) that $\mathfrak{h}_{\alpha,\beta}$ is densely defined in $\mathsf{L}^2(\ell_-,\ell_+)$. It remains to verify that $\mathfrak{h}_{\alpha,\beta}$ is closed. For this consider $\mathrm{dom}(\mathfrak{h}_{\alpha,\beta})$ equipped with the form norm

$$\|v\|_{\alpha,\beta} = \left(\mathfrak{h}_{\alpha,\beta}[v,v] - (s_{\inf}-1)\|v\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} + \|v\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}\right)^{1/2}$$

and let $(u^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{dom}(\mathfrak{h}_{\alpha,\beta})$ with respect to this norm. Then $(u^n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathrm{dom}(\widetilde{\mathfrak{h}}_{\alpha,\beta})$ equipped with the form norm

$$\|v\|_{\widetilde{\alpha,\beta}} = \left(\widetilde{\mathfrak{h}}_{\alpha,\beta}[v,v] - (s_{\inf}-1)\|v\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} + \|v\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}\right)^{1/2}$$

and as $\widetilde{\mathfrak{h}}_{\alpha,\beta}$ is closed there exists a limit $u\in\mathrm{dom}(\widetilde{\mathfrak{h}}_{\alpha,\beta})$. It is clear that for each $k\in\mathbb{Z}$ the restrictions $(\mathbf{u}_k^n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathrm{dom}(\mathbf{h}^R_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k})$ equipped with the corresponding form norm, which is equivalent to the usual $\mathsf{W}^{1,2}(\mathcal{I}_k)$-norm, and we have $\mathbf{u}_k^n\to\mathbf{u}_k$ as $n\to\infty$ in $\mathsf{W}^{1,2}(\mathcal{I}_k)$, where $\mathbf{u}_k = u\restriction_{\mathcal{I}_k}$. Then by the trace theorem

$$\mathbf{u}_k^n(x_k-0)\to\mathbf{u}_k(x_k-0) \quad\text{and}\quad \mathbf{u}_k^n(x_{k-1}+0)\to\mathbf{u}_k(x_{k-1}+0) \text{ as } n\to\infty. \tag{4.5}$$

Since $u^n\in\mathsf{W}^{1,2}_{\mathrm{loc}}(\ell_-,\ell_+)$ we have the continuity condition

$$\mathbf{u}_k^n(x_{k-1}+0) = \mathbf{u}_{k-1}^n(x_{k-1}-0),\qquad k\in\mathbb{Z},$$

and together with (4.5) and $\mathbf{u}_k\in\mathsf{W}^{1,2}(\mathcal{I}_k)$ for each $k\in\mathbb{Z}$, we conclude

$$u\in\mathsf{W}^{1,2}_{\mathrm{loc}}(\ell_-,\ell_+). \tag{4.6}$$

It follows from (4.4) and (4.6) that $u\in\mathrm{dom}(\mathfrak{h}_{\alpha,\beta})$, thus the form $\mathfrak{h}_{\alpha,\beta}$ is closed. □

Besides Hypothesis 3.1 we will also assume that the next hypothesis is satisfied.

**Hypothesis 4.2.** The sequence $\beta=(\beta_k)_{k\in\mathbb{Z}}$ satisfies the condition $\beta_k^{\inf}\le\beta_k<\infty$ in (4.2) and moreover, without loss of generality, we assume that

$$\beta_k>0,\quad k\in\mathbb{Z}. \tag{4.7}$$

It is clear that any sequence $\widehat{\beta}=(\widehat{\beta}_k)_{k\in\mathbb{Z}}$ such that $\beta_k\le\widehat{\beta}_k$ for all $k\in\mathbb{Z}$ also satisfies (4.2) and (4.7).

The self-adjoint operator associated to the form $\mathfrak{h}_{\alpha,\beta}$ is denoted by $\mathcal{H}_{\alpha,\beta}$. It is not difficult to verify the next statement.

**Proposition 4.3.** *Let $\alpha=(\alpha_k)_{k\in\mathbb{Z}}$ and $\beta=(\beta_k)_{k\in\mathbb{Z}}$ be sequences satisfying Hypothesis 3.1 and Hypothesis 4.2. Then the self-adjoint operator $\mathcal{H}_{\alpha,\beta}$ in $\mathsf{L}^2(\ell_-,\ell_+)$ associated to the form $\mathfrak{h}_{\alpha,\beta}$ is bounded from below by $s_{\inf}-1$ and is given by*

$$(\mathcal{H}_{\alpha,\beta}u)\restriction_{\mathcal{I}_k\setminus\{y_k\}} = (-\mathbf{u}_k\restriction_{\mathcal{I}_k\setminus\{y_k\}})'',$$

$$\mathrm{dom}(\mathcal{H}_{\alpha,\beta}) = \left\{\begin{array}{c} u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in \mathsf{L}^2(\ell_-,\ell_+), \\ \mathbf{u}_k \in \mathsf{W}^{2,2}(\mathcal{I}_k\setminus\{y_k\}) \end{array} : \begin{array}{c} u(y_k+0) = u(y_k-0), \\ u(x_k+0) = u(x_k-0), \\ u'(y_k+0) - u'(y_k-0) = \alpha_k u(y_k\pm 0), \\ u'(x_k+0) - u'(x_k-0) = \beta_k u(x_k\pm 0) \end{array}\right\}.$$

Now we are ready to formulate a rigorous version of our main result Theorem 1.1.

**Theorem 4.4** *(Main Theorem). There exist sequences $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ and $\beta = (\beta_k)_{k\in\mathbb{Z}}$ satisfying Hypothesis 3.1 and Hypothesis 4.2 such that the essential spectrum of the self-adjoint operator $\mathcal{H}_{\alpha,\beta}$ coincides with $S_{\mathrm{ess}}$,*

$$\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\beta}) = S_{\mathrm{ess}}, \tag{4.8}$$

*the discrete spectrum in $(T_1,T_2)$ coincides with $S_{\mathrm{disc}}$,*

$$\sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\beta}) \cap (T_1,T_2) = S_{\mathrm{disc}}, \tag{4.9}$$

*and each eigenvalue $\lambda \in \sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\beta}) \cap (T_1,T_2)$ is simple, i.e.* $\dim(\ker(\mathcal{H}_{\alpha,\beta} - \lambda \mathrm{I})) = 1$.

In the next section we will prove (4.8). The assertion (4.9) and the multiplicity statement will be shown in Section 7.

## 5. Essential spectrum of $\mathcal{H}_{\alpha,\beta}$

In this section we prove the identity (4.8) in Theorem 4.4. More precisely, we will show that for any sequence $\alpha$ satisfying Hypothesis 3.1 the essential spectra of $\mathcal{H}_{\alpha,\beta}$ and $\mathcal{H}_{\alpha,\infty}$ coincide provided $(\beta_k)^{-1}$ decays sufficiently fast as $|k| \to \infty$. Then (3.17) in Theorem 3.2 implies (4.8).

**Theorem 5.1.** *Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ and $\beta = (\beta_k)_{k\in\mathbb{Z}}$ be sequences satisfying Hypothesis 3.1 and Hypothesis 4.2, and assume that for*

$$\rho_k = \frac{1}{D_k^2}\max\left\{\frac{1}{\beta_k D_k^3},\ \frac{1}{\beta_{k-1}D_{k-1}^3}\right\},\quad \textit{where } D_k = \min\{d_k, d_{k+1}\},\ k\in\mathbb{Z},$$

*one has*

$$\rho_k \to 0 \ \textit{as}\ |k| \to \infty. \tag{5.1}$$

*Then the essential spectrum of the self-adjoint operator $\mathcal{H}_{\alpha,\beta}$ is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\beta}) = \sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha,\infty}) = S_{\mathrm{ess}}.$$

**Remark 5.2.** Condition (5.1) can be easily achieved by taking $\beta_k$ converging fast enough to $\infty$ as $|k| \to \infty$. For example, (5.1) holds if we choose $\beta_k$ such that

$$\beta_k \geq \frac{1}{\min\{D_k^{5+\epsilon},\, D_{k+1}^2 D_k^{3+\epsilon}\}} \quad \text{with } \epsilon > 0.$$

**Proof of Theorem 5.1.** We set

$$\mu = s_{\mathrm{inf}} - 2, \tag{5.2}$$

where $s_{\mathrm{inf}}$ is the quantity specified in (3.16). Recall that the form $\mathfrak{h}_{\alpha,\infty}$ is bounded from below by $s_{\mathrm{inf}}$ and the form $\mathfrak{h}_{\alpha,\beta}$ is bounded from below by $s_{\mathrm{inf}} - 1$. Hence $\mu$ belongs to the resolvent set of both operators $\mathcal{H}_{\alpha,\beta}$ and $\mathcal{H}_{\alpha,\infty}$, and we have

$$1 \leq \operatorname{dist}(\mu, \sigma(\mathcal{H}_{\alpha,\beta})) \quad \text{and} \quad 1 \leq \operatorname{dist}(\mu, \sigma(\mathcal{H}_{\alpha,\infty})). \tag{5.3}$$

We introduce the resolvent difference

$$T_{\alpha,\beta} = (\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1} - (\mathcal{H}_{\alpha,\infty} - \mu \mathrm{I})^{-1}.$$

Our goal is to prove that $T_{\alpha,\beta}$ is a compact operator; then by virtue of Weyl's theorem (see, e.g., [24, Theorem XIII.14]) and Theorem 3.2 we immediately obtain the statement of Theorem 5.1.

For $f, g \in \mathsf{L}^2(\ell_-, \ell_+)$ we set

$$u = (\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1} f \quad \text{and} \quad v = (\mathcal{H}_{\alpha,\infty} - \mu \mathrm{I})^{-1} g.$$

Then one has

$$\begin{aligned}
(T_{\alpha,\beta} f, g)_{\mathsf{L}^2(\ell_-,\ell_+)} &= \big((\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1} f, g\big)_{\mathsf{L}^2(\ell_-,\ell_+)} - \big(f, (\mathcal{H}_{\alpha,\infty} - \mu \mathrm{I})^{-1} g\big)_{\mathsf{L}^2(\ell_-,\ell_+)} \\
&= \big(u, (\mathcal{H}_{\alpha,\infty} - \mu \mathrm{I}) v\big)_{\mathsf{L}^2(\ell_-,\ell_+)} - \big((\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I}) u, v\big)_{\mathsf{L}^2(\ell_-,\ell_+)} \\
&= (u, \mathcal{H}_{\alpha,\infty} v)_{\mathsf{L}^2(\ell_-,\ell_+)} - (\mathcal{H}_{\alpha,\beta} u, v)_{\mathsf{L}^2(\ell_-,\ell_+)} \\
&= \sum_{k \in \mathbb{Z}} \left( - \int_{x_{k-1}}^{y_k} u \overline{v''} \, \mathrm{d}x - \int_{y_k}^{x_k} u \overline{v''} \, \mathrm{d}x + \int_{x_{k-1}}^{y_k} u'' \overline{v} \, \mathrm{d}x + \int_{y_k}^{x_k} u'' \overline{v} \, \mathrm{d}x \right)
\end{aligned}$$

and integration by parts leads to

$$
\begin{aligned}
(T_{\alpha,\beta}f,g)_{\mathsf{L}^2(\ell_-,\ell_+)} = &-\sum_{k\in\mathbb{Z}}\Big(u(y_k-0)\overline{v'(y_k-0)}-u(x_{k-1}+0)\overline{v'(x_{k-1}+0)}\Big)\\
&-\sum_{k\in\mathbb{Z}}\Big(u(x_k-0)\overline{v'(x_k-0)}-u(y_k+0)\overline{v'(y_k+0)}\Big)\\
&+\sum_{k\in\mathbb{Z}}\Big(u'(y_k-0)\overline{v(y_k-0)}-u'(x_{k-1}+0)\overline{v(x_{k-1}+0)}\Big)\\
&+\sum_{k\in\mathbb{Z}}\Big(u'(x_k-0)\overline{v(x_k-0)}-u'(y_k+0)\overline{v(y_k+0)}\Big).
\end{aligned}
$$

Since both $u$ and $v$ satisfy the conditions

$$
\begin{aligned}
u(y_k+0)=u(y_k-0),\quad u'(y_k+0)-u'(y_k-0)=\alpha_k u(y_k\pm 0),\quad k\in\mathbb{Z},\\
v(y_k+0)=v(y_k-0),\quad v'(y_k+0)-v'(y_k-0)=\alpha_k v(y_k\pm 0),\quad k\in\mathbb{Z},
\end{aligned}
$$

$v(x_k)=0$ and $u(x_k-0)=u(x_k+0)$ for all $k\in\mathbb{Z}$, the expression for $(T_{\alpha,\beta}f,g)_{\mathsf{L}^2(\ell_-,\ell_+)}$ reduces to

$$
(T_{\alpha,\beta}f,g)_{\mathsf{L}^2(\ell_-,\ell_+)}=\sum_{k\in\mathbb{Z}}u(x_k)\overline{v'(x_k+0)-v'(x_k-0)}. \tag{5.4}
$$

We introduce the operators $\Gamma_{\alpha,\beta}:\mathsf{L}^2(\ell_-,\ell_+)\to l^2(\mathbb{Z})$, $\Gamma_{\alpha,\infty}:\mathsf{L}^2(\ell_-,\ell_+)\to l^2(\mathbb{Z})$, by

$$
\begin{aligned}
(\Gamma_{\alpha,\beta}f)_k &= D_k^{-3/2}\big[\big((\mathcal{H}_{\alpha,\beta}-\mu\mathrm{I})^{-1}f\big)(x_k)\big], && k\in\mathbb{Z},\\
(\Gamma_{\alpha,\infty}g)_k &= D_k^{3/2}\big[\big((\mathcal{H}_{\alpha,\infty}-\mu\mathrm{I})^{-1}g\big)'(x_k+0)-\big((\mathcal{H}_{\alpha,\infty}-\mu\mathrm{I})^{-1}g\big)'(x_k-0)\big], && k\in\mathbb{Z},
\end{aligned}
$$

on their natural domains

$$
\begin{aligned}
\mathrm{dom}(\Gamma_{\alpha,\beta}) &= \big\{f\in\mathsf{L}^2(\ell_-,\ell_+):\ \Gamma_{\alpha,\beta}f\in l^2(\mathbb{Z})\big\},\\
\mathrm{dom}(\Gamma_{\alpha,\infty}) &= \big\{g\in\mathsf{L}^2(\ell_-,\ell_+):\ \Gamma_{\alpha,\infty}g\in l^2(\mathbb{Z})\big\}.
\end{aligned}
$$

Below we prove in Lemma 5.3 and Lemma 5.4 that both operators $\Gamma_{\alpha,\beta}$ and $\Gamma_{\alpha,\infty}$ are bounded and everywhere defined, and moreover $\Gamma_{\alpha,\beta}$ is compact. Hence (5.4) can be rewritten as

$$
(T_{\alpha,\beta}f,g)_{\mathsf{L}^2(\ell_-,\ell_+)}=(\Gamma_{\alpha,\beta}f,\Gamma_{\alpha,\infty}g)_{l^2(\mathbb{Z})}=(\Gamma_{\alpha,\infty}^*\Gamma_{\alpha,\beta}f,g)_{\mathsf{L}^2(\ell_-,\ell_+)},\quad f,g\in\mathsf{L}^2(\ell_-,\ell_+),
$$

and we conclude that the resolvent difference $T_{\alpha,\beta}=\Gamma_{\alpha,\infty}^*\Gamma_{\alpha,\beta}$ is compact.

Thus, to finish the proof of Theorem 5.1 we have to prove the lemmata below.

**Lemma 5.3.** *The operator $\Gamma_{\alpha,\beta}$ is bounded, defined on the whole space $\mathsf{L}^2(\ell_-,\ell_+)$, and compact.*

**Proof.** First we prove that $\Gamma_{\alpha,\beta} f$ is well-defined for any $f \in \mathsf{L}^2(\ell_-, \ell_+)$. As before we consider $u = (\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1} f \in \mathrm{dom}(\mathcal{H}_{\alpha,\beta}) \subset \mathrm{dom}(\mathfrak{h}_{\alpha,\beta})$. Then one has for each $n \in \mathbb{N}$

$$
\begin{aligned}
&\sum_{k:\,|k|\le n} |(\Gamma_{\alpha,\beta} f)_k|^2 = \sum_{k:\,|k|\le n} \frac{|u(x_k)|^2}{D_k^3} \\
&= \sum_{k=-n}^{n+1} \frac{D_k^{-3}|u(x_k)|^2 + D_{k-1}^{-3}|u(x_{k-1})|^2}{2} - \frac{D_{n+1}^{-3}|u(x_{n+1})|^2}{2} - \frac{D_{-n-1}^{-3}|u(x_{-n-1})|^2}{2} \\
&\le \sum_{k=-n}^{n+1} \rho_k D_k^2 \left( \frac{\beta_k |u(x_k)|^2 + \beta_{k-1}|u(x_{k-1})|^2}{2} \right) \\
&\le \sum_{k=-n}^{n+1} \rho_k D_k^2 \left( \mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{u}_k, \mathbf{u}_k] + \frac{C}{d_k^2} \|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} + \frac{\beta_k |u(x_k)|^2 + \beta_{k-1}|u(x_{k-1})|^2}{2} \right),
\end{aligned} \tag{5.5}
$$

where we have used Lemma 3.4 and the corresponding constant $C > 0$ from there in the last estimate. Taking into account the definition of the forms $\mathbf{h}_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}$ and $\mathfrak{h}_{\alpha,\beta}$ we continue the above estimates by

$$
\begin{aligned}
&= \sum_{k=-n}^{n+1} \rho_k D_k^2 \left( \mathbf{h}_{\mathcal{I}_k,\alpha_k,\beta_{k-1},\beta_k}[\mathbf{u}_k, \mathbf{u}_k] - \mu \|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} + \left( \frac{C}{d_k^2} + \mu \right) \|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} \right) \\
&\le \max_{k\in\mathbb{Z}} \left( \rho_k D_k^2 \right) \left( \mathfrak{h}_{\alpha,\beta}[u,u] - \mu \|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} \right) \\
&\qquad + \max_{k\in\mathbb{Z}} \left( C\rho_k \frac{D_k^2}{d_k^2} + |\mu| \rho_k D_k^2 \right) \|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}
\end{aligned} \tag{5.6}
$$

and using $\max_{k\in\mathbb{Z}} \rho_k < \infty$ (this follows from (5.1)) and

$$D_k \le d_k < \ell_+ - \ell_-$$

we conclude

$$
\sum_{k:\,|k|\le n} |(\Gamma_{\alpha,\beta} f)_k|^2 \le C_1 \left( \mathfrak{h}_{\alpha,\beta}[u,u] - \mu \|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} \right) + C_2 \|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}, \tag{5.7}
$$

where

$$
C_1 = \max_{k\in\mathbb{Z}} \left( \rho_k D_k^2 \right) \quad \text{and} \quad C_2 = \max_{k\in\mathbb{Z}} \left( C\rho_k \frac{D_k^2}{d_k^2} + |\mu| \rho_k D_k^2 \right).
$$

Since $(\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})u = f$ we have

$$
\mathfrak{h}_{\alpha,\beta}[u,u] - \mu \|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} = (f, u)_{\mathsf{L}^2(\ell_-,\ell_+)} \tag{5.8}
$$

and (5.3) implies

$$\begin{aligned}\|u\|_{\mathsf{L}^2(\ell_-,\ell_+)} &= \|(\mathcal{H}_{\alpha,\beta}-\mu\mathrm{I})^{-1}f\|_{\mathsf{L}^2(\ell_-,\ell_+)}\\ &\le \frac{1}{\operatorname{dist}(\mu,\sigma(\mathcal{H}_{\alpha,\beta}))}\|f\|_{\mathsf{L}^2(\ell_-,\ell_+)}\\ &\le \|f\|_{\mathsf{L}^2(\ell_-,\ell_+)},\end{aligned}\tag{5.9}$$

so that (5.7) leads to

$$\sum_{k\colon |k|\le n}|(\Gamma_{\alpha,\beta}f)_k|^2\le (C_1+C_2)\|f\|^2_{\mathsf{L}^2(\ell_-,\ell_+)},$$

and hence $\|\Gamma_{\alpha,\beta}f\|_{l^2(\mathbb{Z})}\le\sqrt{C_1+C_2}\|f\|_{\mathsf{L}^2(\ell_-,\ell_+)}$. Therefore, the operator $\Gamma_{\alpha,\beta}$ is bounded and well-defined on the whole space $\mathsf{L}^2(\ell_-,\ell_+)$.

In order to prove the compactness of $\Gamma_{\alpha,\beta}$ we consider for $n\in\mathbb{N}$ the finite rank operators

$$\Gamma^n_{\alpha,\beta}:\mathsf{L}^2(\ell_-,\ell_+)\to l^2(\mathbb{Z}),\quad (\Gamma^n_{\alpha,\beta}f)_k=\begin{cases}(\Gamma_{\alpha,\beta}f)_k, & |k|\le n,\\ 0, & |k|\ge n+1.\end{cases}$$

Then for $f\in\mathsf{L}^2(\ell_-,\ell_+)$ one has

$$\|\Gamma^n_{\alpha,\beta}f-\Gamma_{\alpha,\beta}f\|^2_{l^2(\mathbb{Z})}=\sum_{|k|\ge n+1}D_k^{-3}|u(x_k)|^2,\tag{5.10}$$

where as before $u=(\mathcal{H}_{\alpha,\beta}-\mu\mathrm{I})^{-1}f$. Repeating verbatim the arguments in (5.5) and (5.6) we obtain

$$\begin{aligned}\sum_{|k|\ge n+1}D_k^{-3}|u(x_k)|^2 &\le \max_{|k|\ge n+1}\left(\rho_k D_k^2\right)\left(\mathfrak{h}_{\alpha,\beta}[u,u]-\mu\|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}\right)\\ &\quad+\max_{|k|\ge n+1}\left(C\rho_k\frac{D_k^2}{d_k^2}+|\mu|\rho_k D_k^2\right)\|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}.\end{aligned}\tag{5.11}$$

From (5.1) it is clear that $\max_{|k|\ge n+1}\rho_k\to 0$ as $n\to\infty$ and hence

$$\max_{|k|\ge n+1}\left(\rho_k D_k^2\right)\to 0\quad\text{and}\quad \max_{|k|\ge n+1}\left(C\rho_k\frac{D_k^2}{d_k^2}+|\mu|\rho_k D_k^2\right)\to 0\tag{5.12}$$

as $n\to\infty$. Using (5.8) and (5.9) we conclude for (5.10) from (5.11)–(5.12)

$$\|\Gamma^n_{\alpha,\beta}f-\Gamma_{\alpha,\beta}f\|^2_{l^2(\mathbb{Z})}=\sum_{|k|\ge n+1}D_k^{-3}|u(x_k)|^2\le C_n\|f\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}\tag{5.13}$$

with $C_n \to 0$ as $n \to \infty$. Thus $\Gamma^n_{\alpha,\beta} \to \Gamma_{\alpha,\beta}$ in the operator norm as $n \to \infty$. Since $\Gamma^n_{\alpha,\beta}$ are finite rank operators, the operator $\Gamma_{\alpha,\beta}$ is compact. $\square$

**Lemma 5.4.** *The operator $\Gamma_{\alpha,\infty}$ is bounded and defined on the whole space $\mathsf{L}^2(\ell_-,\ell_+)$.*

**Proof.** For $g \in \mathsf{L}^2(\ell_-,\ell_+)$ we consider $v = (\mathcal{H}_{\alpha,\infty} - \mu \mathrm{I})^{-1} g \in \mathrm{dom}(\mathcal{H}_{\alpha,\infty}) \subset \mathrm{dom}(\mathfrak{h}_{\alpha,\infty})$. For $k \in \mathbb{Z}$ we denote by $\mathbf{w}_k$ the linear function defined on $\mathcal{I}_k$ which has the value 0 at the left endpoint $x_{k-1}$ and the value 1 at the right endpoint $x_k$, i.e.

$$\mathbf{w}_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}}, \qquad x \in \mathcal{I}_k.$$

Integrating by parts, taking into account that $v$ satisfies $v(y_k+0) = v(y_k-0)$, and using the notation $\mathbf{v}_k = v \restriction_{\mathcal{I}_k}$ we get

$$\begin{aligned}\big((\mathcal{H}_{\alpha,\infty} v) \restriction_{\mathcal{I}_k}, \mathbf{w}_k\big)_{\mathsf{L}^2(\mathcal{I}_k)} &= -\int_{x_{k-1}}^{y_k} \mathbf{v}_k'' \mathbf{w}_k \,\mathrm{d}x - \int_{y_k}^{x_k} \mathbf{v}_k'' \mathbf{w}_k \,\mathrm{d}x \\ &= \int_{x_{k-1}}^{x_k} \mathbf{v}_k' \mathbf{w}_k' \,\mathrm{d}x + \alpha_k \mathbf{v}_k(y_k)\mathbf{w}_k(y_k) - \mathbf{v}_k'(x_k - 0),\end{aligned}$$

which leads to

$$\begin{aligned}\mathbf{v}_k'(x_k-0) &= \int_{x_{k-1}}^{x_k} \mathbf{v}_k'\mathbf{w}_k' \,\mathrm{d}x + \alpha_k \mathbf{v}_k(y_k)\mathbf{w}_k(y_k) - \big((\mathcal{H}_{\alpha,\infty} v) \restriction_{\mathcal{I}_k}, \mathbf{w}_k\big)_{\mathsf{L}^2(\mathcal{I}_k)} \\ &= \left[\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{v}_k,\mathbf{w}_k] + \frac{C}{d_k^2}(\mathbf{v}_k,\mathbf{w}_k)_{\mathsf{L}^2(\mathcal{I}_k)}\right] - \big((\mathcal{H}_{\alpha,\infty} v) \restriction_{\mathcal{I}_k}, \mathbf{w}_k\big)_{\mathsf{L}^2(\mathcal{I}_k)} \\ &\quad - \frac{C}{d_k^2}(\mathbf{v}_k,\mathbf{w}_k)_{\mathsf{L}^2(\mathcal{I}_k)},\end{aligned} \tag{5.14}$$

where $C$ is the positive constant in Lemma 3.4. It is easy to compute

$$\|\mathbf{w}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} = \frac{d_k}{3} \quad \text{and} \quad \mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{w}_k,\mathbf{w}_k] = \frac{1}{d_k} + \frac{\alpha_k}{4} \le \frac{1}{d_k},$$

where (3.11) was used in the last estimate. Due to Lemma 3.4 we then obtain the following Cauchy-Schwarz inequality and corresponding estimate

$$\begin{aligned}&\left|\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{v}_k,\mathbf{w}_k] + \frac{C}{d_k^2}(\mathbf{v}_k,\mathbf{w}_k)_{\mathsf{L}^2(\mathcal{I}_k)}\right|^2 \\ &\quad \le \left(\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{v}_k,\mathbf{v}_k] + \frac{C}{d_k^2}\|\mathbf{v}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)}\right)\left(\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{w}_k,\mathbf{w}_k] + \frac{C}{d_k^2}\|\mathbf{w}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)}\right) \\ &\quad \le \frac{C'}{d_k}\left(\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{v}_k,\mathbf{v}_k] + \frac{C}{d_k^2}\|\mathbf{v}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)}\right)\end{aligned} \tag{5.15}$$

with $C' > 0$. Combining (5.14) and (5.15), and taking into account that $d_k < \ell_+ - \ell_-$ we arrive at the estimate

$$|\mathbf{v}_k'(x_k - 0)|^2 \le C'' \left( d_k^{-1} \mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{v}_k, \mathbf{v}_k] + \|(\mathcal{H}_{\alpha,\infty} v) \restriction_{\mathcal{I}_k} \|^2_{\mathsf{L}^2(\mathcal{I}_k)} + d_k^{-3} \|\mathbf{v}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} \right). \tag{5.16}$$

With $\mathbf{w}_k$ replaced by $1 - \mathbf{w}_k$ we obtain with the same arguments the estimate

$$|\mathbf{v}_k'(x_{k-1} + 0)|^2 \le C''' \left( d_k^{-1} \mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{v}_k, \mathbf{v}_k] + \|(\mathcal{H}_{\alpha,\infty} v) \restriction_{\mathcal{I}_k} \|^2_{\mathsf{L}^2(\mathcal{I}_k)} + d_k^{-3} \|\mathbf{v}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)} \right).$$

Now we obtain

$$\begin{aligned} \|\Gamma_{\alpha,\infty} g\|^2_{l^2(\mathbb{Z})} &= \sum_{k\in\mathbb{Z}} D_k^3 |v'(x_k - 0) - v'(x_k + 0)|^2 \\ &\le 2 \sum_{k\in\mathbb{Z}} D_k^3 |\mathbf{v}_k'(x_k - 0)|^2 + 2 \sum_{k\in\mathbb{Z}} D_{k-1}^3 |\mathbf{v}_k'(x_{k-1} + 0)|^2 \end{aligned}$$

and using $D_{k-1} \le d_k < \ell_+ - \ell_-$ and $D_k \le d_k < \ell_+ - \ell_-$, $k \in \mathbb{Z}$, we conclude

$$\|\Gamma_{\alpha,\infty} g\|^2_{l^2(\mathbb{Z})} \le \widetilde{C} \left( \mathfrak{h}_{\alpha,\infty}[v, v] + \|\mathcal{H}_{\alpha,\infty} v\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} + \|v\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} \right)$$

with some $\widetilde{C} > 0$. Since $(\mathcal{H}_{\alpha,\infty} - \mu \mathrm{I})v = g$,

$$\mathfrak{h}_{\alpha,\infty}[v, v] = (\mathcal{H}_{\alpha,\infty} v, v)_{\mathsf{L}^2(\ell_-,\ell_+)} = (g + \mu v, v)_{\mathsf{L}^2(\ell_-,\ell_+)},$$

and $\|v\|_{\mathsf{L}^2(\ell_-,\ell_+)} \le \|g\|_{\mathsf{L}^2(\ell_-,\ell_+)}$ by (5.3) (see also (5.9)), we obtain finally

$$\|\Gamma_{\alpha,\infty} g\|^2_{l^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} D_k^3 |v'(x_k + 0) - v'(x_k - 0)|^2 \le C \|g\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}. \tag{5.17}$$

This shows that $\Gamma_{\alpha,\infty}$ is bounded and well-defined on the whole space $\mathsf{L}^2(\ell_-, \ell_+)$. □

## 6. Partly coupled operators and their spectra

In this section we take another step towards the proof of the main result in this paper. Here our objective is to study a partly coupled operator $\mathcal{H}^n_{\alpha,\beta}$ for $n \in \mathbb{N}$ that is obtained from the decoupled operator $\mathcal{H}_{\alpha,\infty}$ by introducing finitely many $\delta$-couplings of strengths $\beta_k$ at the points $x_k$, $k = -n+1, \dots, n-1$. After some technical preparations in Section 6.1 it will be shown in Theorem 6.7 that one can pick sequences $\alpha^n$ and $\beta$ such that $\sigma_{\mathrm{ess}}(\mathcal{H}^n_{\alpha^n,\beta}) = S_{\mathrm{ess}}$ and $\sigma_{\mathrm{disc}}(\mathcal{H}^n_{\alpha^n,\beta}) \cap (T_1, T_2) = S_{\mathrm{disc}}$, that is, Theorem 4.4 holds for the partly coupled operator $\mathcal{H}^n_{\alpha,\beta}$.

In the following we use the notation

$$\mathcal{K} = \left\{ k \in \mathbb{Z} \setminus \mathbb{N} : \ s_k \in [T_1, T_2] \right\}$$

and for $n \in \mathbb{N}$ we set

$$\mathcal{K}^n = \big\{k \in \{-n+1, \dots, 0\} : \ s_k \in [T_1, T_2]\big\}.$$

Recall that we have already fixed a sequence $(\delta_k)_{k\in\mathbb{N}}$ which satisfies (3.6)–(3.8). In addition, we now introduce a sequence $(\delta_k)_{k\in\mathbb{Z}\setminus\mathbb{N}}$ such that

$$B_{\delta_k}(s_k) \subset \begin{cases} \mathcal{O}, & k \in \mathcal{K}, \\ \mathbb{R}\setminus[T_1, T_2], & k \in \mathbb{Z}\setminus(\mathbb{N}\cup\mathcal{K}), \end{cases} \tag{6.1}$$

which is possible since $s_k \in \mathcal{O}$ for $k \in \mathcal{K}$ (see (3.2)) and $s_k \in \mathbb{R}\setminus[T_1, T_2]$ for $k \in \mathbb{Z}\setminus(\mathbb{N}\cup\mathcal{K})$. To avoid technical difficulties below we sometimes discuss the situation $\mathcal{K} = \mathbb{Z}\setminus\mathbb{N}$, in which case $\mathcal{K}^n = \{-n+1, \dots, 0\}$; in other words we treat the situation $s_k \in (T_1, T_2)$ for all $k \in \mathbb{Z}$, which appears if $S_{\rm ess} \subset [T_1, T_2]$.

*6.1. The operator $\mathbf{H}^n_{\alpha,\beta}$ and its spectrum*

Let again $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ be a sequence satisfying Hypothesis 3.1, while $\beta = (\beta_k)_{k\in\mathbb{Z}}$ is a sequence of real numbers. For $n \in \mathbb{N}$ we consider the densely defined, closed, symmetric sesquilinear form

$$\begin{aligned} \mathbf{h}^n_{\alpha,\beta}[u, v] &= \sum_{k=-n+1}^{n} \mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{u}_k, \mathbf{v}_k] + \sum_{k=-n+1}^{n-1} \beta_k u(x_k)\overline{v(x_k)}, \\ \mathrm{dom}(\mathbf{h}^n_{\alpha,\beta}) &= \mathsf{W}^{1,2}_0(x_{-n}, x_n), \end{aligned}$$

in $\mathsf{L}^2(x_{-n}, x_n)$ and the corresponding self-adjoint operator $\mathbf{H}^n_{\alpha,\beta}$ in $\mathsf{L}^2(x_{-n}, x_n)$ with $\delta$-interactions of strengths $\beta_k$ and $\alpha_k$ on

$$\mathcal{Z}_k := \big\{x_k : -n+1 \le k \le n-1\big\} \cup \big\{y_k : -n+1 \le k \le n\big\},$$

which is given by

$$\begin{aligned} &(\mathbf{H}^n_{\alpha,\beta} u) \restriction_{\mathcal{I}_k\setminus\{y_k\}} = (-\mathbf{u}_k \restriction_{\mathcal{I}_k\setminus\{y_k\}})'', \qquad -n+1 \le k \le n, \\ &\mathrm{dom}(\mathbf{H}^n_{\alpha,\beta}) = \left\{ \begin{matrix} u \in \mathsf{W}^{2,2}((x_{-n}, x_n)\setminus\mathcal{Z}_k) \\ u(x_{-n}) = u(x_n) = 0 \end{matrix} : \begin{matrix} u(y_k+0) = u(y_k-0), \\ u(x_k+0) = u(x_k-0), \\ u'(y_k+0) - u'(y_k-0) = \alpha_k u(y_k \pm 0), \\ u'(x_k+0) - u'(x_k-0) = \beta_k u(x_k \pm 0), \\ \text{for all } x_k, y_k \in \mathcal{Z}_k \end{matrix} \right\}. \end{aligned}$$

It is clear that $\mathbf{H}^n_{\alpha,\beta}$ is independent of $\alpha_k$ with $k \notin \{-n+1, \dots, n\}$ and $\beta_k$ with $k \notin \{-n+1, \dots, n-1\}$. Furthermore, the spectrum of $\mathbf{H}^n_{\alpha,\beta}$ is purely discrete.

The main result of this subsection is Theorem 6.4 for which some preparatory statements are needed. The first lemma shows that the eigenvalues of $\mathbf{H}^n_{\alpha,\beta}$ are close to the eigenvalues of the self-adjoint operator

$$\bigoplus_{k=-n+1}^{n} \mathbf{H}^D_{\mathcal{I}_k,\alpha_k}, \tag{6.2}$$

provided $\beta_k$ are sufficiently large. Moreover, the eigenvalues satisfy the additional useful inequalities (6.5) below, where we consider sequences $\alpha = ((\underline{\alpha}^{n,k})_l)_{l\in\mathbb{Z}}$, $\alpha = ((\overline{\alpha}^{n,k})_l)_{l\in\mathbb{Z}}$, $k = 1,\dots,n$, that satisfy Hypothesis 3.1 and are of the particular form (6.4) with the numbers $\alpha_k^\pm$ defined by (3.9). The inequalities (6.5) will be needed later, when applying the intermediate value theorem from [18] in the proof of Lemma 6.2.

**Lemma 6.1.** *There exists a sequence $\beta' = (\beta'_k)_{k\in\mathbb{Z}}$ of real numbers such that the following holds: for any sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ satisfying Hypothesis 3.1, for any sequence $\beta = (\beta_k)_{k\in\mathbb{Z}}$ satisfying $\beta'_k \le \beta_k$, $k \in \mathbb{Z}$, and for any $n \in \mathbb{N}$, one has*

$$\sigma(\mathbf{H}^n_{\alpha,\beta}) \cap (T_1,T_2) = \left\{ s^n_{\alpha,\beta;\,k} : k \in \mathcal{K}^n \cup \{1,\dots,n\} \right\}, \tag{6.3}$$

*where $s^n_{\alpha,\beta;\,k} \in B_{\delta_k}(s_k)$ are simple eigenvalues of $\mathbf{H}^n_{\alpha,\beta}$. Moreover, for sequences $\underline{\alpha}^{n,k}$ and $\overline{\alpha}^{n,k}$, $k = 1,\dots,n$, that satisfy Hypothesis 3.1 and are of the particular form*

$$\begin{aligned} \underline{\alpha}^{n,k} &= \left(\dots,\alpha_{-1},\alpha_0,\alpha_1^+,\dots,\alpha_{k-1}^+,\alpha_k^-,\alpha_{k+1}^+,\dots,\alpha_n^+,\alpha_{n+1},\dots\right), \\ \overline{\alpha}^{n,k} &= \left(\dots,\alpha_{-1},\alpha_0,\alpha_1^-,\dots,\alpha_{k-1}^-,\alpha_k^+,\alpha_{k+1}^-,\dots,\alpha_n^-,\alpha_{n+1},\dots\right) \end{aligned} \tag{6.4}$$

*one has*

$$s^n_{\underline{\alpha}^{n,k},\beta;\,k} < s_k - \frac{1}{4}\delta_k \quad \textit{and} \quad s_k + \frac{1}{4}\delta_k < s^n_{\overline{\alpha}^{n,k},\beta;\,k}, \qquad k = 1,\dots,n. \tag{6.5}$$

**Proof.** To avoid further technical difficulties we assume that $\mathcal{K} = \mathbb{Z}\setminus\mathbb{N}$, which leads to $\mathcal{K}^n = \{-n+1,\dots,0\}$. Hence

$$s_k \in \mathcal{O} \subset (T_1,T_2), \qquad k \in \mathbb{Z}\setminus\mathbb{N}, \tag{6.6}$$

and, in particular, it follows from (6.6) that $S_{\mathrm{ess}} \subset [T_1,T_2]$. The general case needs only slight modifications.

We prove (6.3) and (6.5) below by induction. For convenience, we restrict ourselves to the sequences $\beta = (\beta_k)_{k\in\mathbb{Z}}$ such that

$$\beta_k \ge \beta_k^{\inf} \quad \text{and} \quad \beta_k > 0 \tag{6.7}$$

(in other words, the sequence $\beta$ satisfies Hypothesis 4.2). This assumption implies, in particular, that for each $n \in \mathbb{N}$ the form $\mathbf{h}^n_{\alpha,\beta}$ is bounded from below by $s_{\inf} - 1$. In fact, for $u \in \mathrm{dom}(\mathbf{h}^n_{\alpha,\beta}) = \mathsf{W}^{1,2}_0(x_{-n}, x_n)$ and its extension $\widetilde{u}$ by zero on all of $(\ell_-, \ell_+)$ one has $\widetilde{u} \in \mathrm{dom}(\mathbf{h}_{\alpha,\beta})$ and Lemma 4.1 implies

$$\mathbf{h}^n_{\alpha,\beta}[u,u] = \mathbf{h}_{\alpha,\beta}[\widetilde{u},\widetilde{u}] \geq (s_{\inf} - 1)\|\widetilde{u}\|_{\mathsf{L}^2(\ell_-,\ell_+)} = (s_{\inf} - 1)\|u\|_{\mathsf{L}^2(x_{-n},x_n)}.$$

Therefore, $\mu = s_{\inf} - 2$ defined in (5.2) belongs to the resolvent set of the self-adjoint operator $\mathbf{H}^n_{\alpha,\beta}$ and we have

$$1 \leq \mathrm{dist}(\mu, \sigma(\mathbf{H}^n_{\alpha,\beta})), \qquad n \in \mathbb{N}. \tag{6.8}$$

It is also clear from (3.13)–(3.14) and (3.16) that $\mu$ belongs to the resolvent set of the operator in (6.2) and the same estimate holds.

**Base case ($n = 1$).** We consider the bounded operator

$$T_{\beta_0} = (\mathbf{H}^1_{\alpha,\beta} - \mu\mathrm{I})^{-1} - \big((\mathbf{H}^D_{\mathcal{I}_0,\alpha_0} \oplus \mathbf{H}^D_{\mathcal{I}_1,\alpha_1}) - \mu\mathrm{I}\big)^{-1}$$

in $\mathsf{L}^2(\mathcal{I}_0 \cup \mathcal{I}_1)$. For $f, g \in \mathsf{L}^2(\mathcal{I}_0 \cup \mathcal{I}_1)$ we set

$$u = (\mathbf{H}^1_{\alpha,\beta} - \mu\mathrm{I})^{-1} f \quad \text{and} \quad v = \big((\mathbf{H}^D_{\mathcal{I}_0,\alpha_0} \oplus \mathbf{H}^D_{\mathcal{I}_1,\alpha_1}) - \mu\mathrm{I}\big)^{-1} g.$$

Following the arguments that led to (5.4) one verifies in the present situation that

$$(T_{\beta_0} f, g)_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)} = u(x_0)\overline{v'(x_0 + 0) - v'(x_0 - 0)}. \tag{6.9}$$

Taking into account that $\beta_0 > 0$ (see (6.7)), using Lemma 3.4 with the constant $C$ from there, and the definition of the form $\mathbf{h}^1_{\alpha,\beta}$ we obtain

$$\begin{aligned}
|u(x_0)|^2 &\leq \frac{1}{\beta_0}\bigg[\mathbf{h}_{\mathcal{I}_0,\alpha_0}[\mathbf{u}_0,\mathbf{u}_0] + \frac{C}{d_0^2}\|\mathbf{u}_0\|^2_{\mathsf{L}^2(\mathcal{I}_0)} + \mathbf{h}_{\mathcal{I}_1,\alpha_1}[\mathbf{u}_1,\mathbf{u}_1] \\
&\qquad + \frac{C}{d_1^2}\|\mathbf{u}_1\|^2_{\mathsf{L}^2(\mathcal{I}_1)} + \beta_0|u(x_0)|^2\bigg] \\
&= \frac{1}{\beta_0}\bigg[\mathbf{h}^1_{\alpha,\beta}[u,u] + \frac{C}{d_0^2}\|\mathbf{u}_0\|^2_{\mathsf{L}^2(\mathcal{I}_0)} + \frac{C}{d_1^2}\|\mathbf{u}_1\|^2_{\mathsf{L}^2(\mathcal{I}_1)}\bigg] \\
&= \frac{1}{\beta_0}\bigg[(f + \mu u, u)_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)} + \frac{C}{d_0^2}\|\mathbf{u}_0\|^2_{\mathsf{L}^2(\mathcal{I}_0)} + \frac{C}{d_1^2}\|\mathbf{u}_1\|^2_{\mathsf{L}^2(\mathcal{I}_1)}\bigg].
\end{aligned}$$

Since $\|\mathbf{u}_k\|_{\mathsf{L}^2(\mathcal{I}_k)} \leq \|u\|_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)} \leq \|f\|_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)}$ by (6.8) for $k = 0, 1$, we conclude

$$|u(x_0)|^2 \leq \frac{C_0}{\beta_0}\|f\|^2_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)}, \tag{6.10}$$

where the constant $C_0$ depends on $d_0, d_1, \mu$, and $C$ from Lemma 3.4, but is independent of $\alpha_0, \alpha_1, \beta_0$. As in the proof of Lemma 5.4 (cf. (5.16)) one obtains the estimate

$$|v'(x_0-0)|^2 \le C''\left(d_0^{-1}\mathbf{h}_{\mathcal{I}_0,\alpha_0}[\mathbf{v}_0,\mathbf{v}_0] + \|\mathbf{H}^D_{\mathcal{I}_0,\alpha_0}\mathbf{v}_0\|^2_{\mathsf{L}^2(\mathcal{I}_0)} + d_0^{-3}\|\mathbf{v}_0\|^2_{\mathsf{L}^2(\mathcal{I}_0)}\right)$$

with the constant $C''$ being independent of $\alpha_0$. Using this estimate, and also taking into account that $\mathbf{H}^D_{\mathcal{I}_0,\alpha_0}\mathbf{v}_0 = \mathbf{g}_0 + \mu\mathbf{v}_0$, $\mathfrak{h}_{\mathcal{I}_0,\alpha_0}[\mathbf{v}_0,\mathbf{v}_0] = (\mathbf{g}_0+\mu\mathbf{v}_0,\mathbf{v}_0)_{\mathsf{L}^2(\mathcal{I}_0)}$, and

$$\|\mathbf{v}_0\|_{\mathsf{L}^2(\mathcal{I}_0)} \le \|\mathbf{g}_0\|_{\mathsf{L}^2(\mathcal{I}_0)} \le \|g\|_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)},$$

we conclude

$$|v'(x_0-0)|^2 \le C_0^-\|g\|^2_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)}, \tag{6.11}$$

and similarly

$$|v'(x_0+0)|^2 \le C_0^+\|g\|^2_{\mathsf{L}^2(\mathcal{I}_0\cup\mathcal{I}_1)}, \tag{6.12}$$

where the constants $C_0^-$ and $C_0^+$ depend, respectively, on $d_0$ and $d_1$, but are independent of $\alpha_0$, $\alpha_1$ and $\beta_0$. Using (6.10)–(6.12) we conclude from (6.9) that

$$\|(\mathbf{H}^1_{\alpha,\beta}-\mu\mathrm{I})^{-1} - ((\mathbf{H}^D_{\mathcal{I}_0,\alpha_0}\oplus\mathbf{H}^D_{\mathcal{I}_1,\alpha_1})-\mu\mathrm{I})^{-1}\| \le \widetilde{C}_0\beta_0^{-1/2}, \tag{6.13}$$

where again the constant $\widetilde{C}_0>0$ is independent of $\alpha_0,\alpha_1,\beta_0$. It follows from (6.13) that for each $j\in\mathbb{N}$

$$\sup_{\alpha_0,\alpha_1}|\lambda_j(\mathbf{H}^1_{\alpha,\beta}) - \lambda_j(\mathbf{H}^D_{\mathcal{I}_0,\alpha_0}\oplus\mathbf{H}^D_{\mathcal{I}_1,\alpha_1})| \to 0 \text{ as } \beta_0\to\infty. \tag{6.14}$$

Recall, that

$$\begin{aligned} \lambda_1(\mathbf{H}^D_{\mathcal{I}_0,\alpha_0}) &= s_0, & \lambda_1(\mathbf{H}^D_{\mathcal{I}_1,\alpha_1}) &\in \overline{B_{\delta_1/2}(s_1)},\\ \lambda_2(\mathbf{H}^D_{\mathcal{I}_0,\alpha_0}) &= \left(\frac{2\pi}{d_0}\right)^2, & \lambda_2(\mathbf{H}^D_{\mathcal{I}_1,\alpha_1}) &= \left(\frac{2\pi}{d_1}\right)^2; \end{aligned}$$

cf. (3.13)–(3.14). In particular (cf. Proposition 2.2 and (3.9)), we get

$$\lambda_1(\mathbf{H}^D_{\mathcal{I}_1,\alpha_1^-}) = s_1 - \frac{1}{2}\delta_1 \quad\text{and}\quad \lambda_1(\mathbf{H}^D_{\mathcal{I}_1,\alpha_1^+}) = s_1+\frac{1}{2}\delta_1. \tag{6.15}$$

Combining (6.14)–(6.15), and taking into account that $B_{\delta_0}(s_0)\cap B_{\delta_1}(s_1)=\emptyset$ (since $B_{\delta_1}(s_1)\subset(T_1,T_2)\setminus\overline{\mathcal{O}}$ and $B_{\delta_0}(s_0)\subset\mathcal{O}$; cf. (3.6), (6.1), (6.6)) we conclude that there exists a positive $\beta_0'\ge\beta_0^{\inf}$ such that for any $\beta_0\in[\beta_0',\infty)$ one has

- $\lambda_j(\mathbf{H}^1_{\alpha,\beta}) > T_2$ for $j\ge 3$,
- if $s_0<s_1$, then $\lambda_1(\mathbf{H}^1_{\alpha,\beta})\in B_{\delta_0}(s_0)$ and $\lambda_2(\mathbf{H}^1_{\alpha,\beta})\in B_{\delta_1}(s_1)$, and, moreover,

$$\lambda_2(\mathbf{H}^1_{\underline{\alpha}^{1,1},\beta}) < s_1 - \frac{1}{4}\delta_1 \quad\text{and}\quad s_1+\frac{1}{4}\delta_1 < \lambda_2(\mathbf{H}^1_{\overline{\alpha}^{1,1},\beta}).$$

- if $s_1 < s_0$, then $\lambda_1(\mathbf{H}^1_{\alpha,\beta}) \in B_{\delta_1}(s_1)$ and $\lambda_2(\mathbf{H}^1_{\alpha,\beta}) \in B_{\delta_0}(s_0)$, and, moreover,

$$\lambda_1(\mathbf{H}^1_{\underline{\alpha}^{1,1},\beta}) < s_1 - \frac{1}{4}\delta_1 \quad \text{and} \quad s_1 + \frac{1}{4}\delta_1 < \lambda_1(\mathbf{H}^1_{\overline{\alpha}^{1,1},\beta})$$

(recall from (6.4) that $\big((\underline{\alpha}^{1,1})_l\big)_{l\in\mathbb{Z}}$ and $\big((\overline{\alpha}^{1,1})_l\big)_{l\in\mathbb{Z}}$ are sequences satisfying Hypothesis 3.1 with the property $(\underline{\alpha}^{1,1})_1 = \alpha_1^-$, $(\overline{\alpha}^{1,1})_1 = \alpha_1^+$). Evidently, the above properties yield (6.3), (6.5) for $n = 1$ (recall, that the operator $\mathbf{H}^1_{\alpha,\beta}$ does not dependent on $\beta_k$ with $k \neq 0$).

**Induction step ($N \to N+1$).** Assume that (6.3) and (6.5) hold for some fixed $N \in \mathbb{N}$, that is, there exist $\beta'_k$, $k = -N+1, \ldots, N-1$, such that for $\beta'_k \le \beta_k$ the spectrum of $\mathbf{H}^N_{\alpha,\beta}$ in $(T_1, T_2)$ consists of $2N$ simple eigenvalues which are contained in $B_{\delta_k}(s_k)$, and, moreover, the inequalities (6.5) hold with $n = N$. It is no restriction to assume that $\beta'_k$, $k = -N+1, \ldots, N-1$, are positive and satisfy $\beta'_k \ge \beta_k^{\inf}$. Recall that $\mathbf{H}^N_{\alpha,\beta}$ does not dependent on $\beta_k$ with $|k| > N-1$.

Now let the sequence $\beta = (\beta_k)_{k\in\mathbb{Z}}$ (which of course satisfies Hypothesis 4.2) be chosen such that $\beta'_k \le \beta_k$ holds for $k = -N+1, \ldots, N-1$. We denote $\widetilde{\mathcal{I}}_{N+1} := (x_{-N-1}, x_{N+1})$. For $f, g \in \mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})$ we set

$$u = (\mathbf{H}^{N+1}_{\alpha,\beta} - \mu\mathrm{I})^{-1} f \quad \text{and} \quad v = \Big(\big(\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}} \oplus \mathbf{H}^N_{\alpha,\beta} \oplus \mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}\big) - \mu\mathrm{I}\Big)^{-1} g.$$

Following the arguments that led to (6.9) we get the similar equality

$$\begin{aligned}&\Big((\mathbf{H}^{N+1}_{\alpha,\beta} - \mu\mathrm{I})^{-1} f - \big((\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}} \oplus \mathbf{H}^N_{\alpha,\beta} \oplus \mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}) - \mu\mathrm{I}\big)^{-1} f, g\Big)_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})} \\ &\qquad = u(x_N)\overline{v'(x_N+0) - v'(x_N-0)} + u(x_{-N})\overline{v'(x_{-N}+0) - v'(x_{-N}-0)}.\end{aligned} \tag{6.16}$$

Let $C$ be the constant from Lemma 3.4 and let $\widehat{d}_N = \min\{d_{-N}, d_{-N+1}, \ldots, d_{N+1}\}$. Taking into account that $\beta_k > 0$ (cf. (4.7)) we get

$$\begin{aligned}|u(x_N)|^2 &\le \frac{1}{\beta_N}\left[\beta_N |u(x_N)|^2 + \sum_{k=-N}^{N+1}\left(\mathbf{h}_{\mathcal{I}_k,\alpha_k}[\mathbf{u}_k, \mathbf{u}_k] + \frac{C}{d_k^2}\|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)}\right)\right] \\ &\le \frac{1}{\beta_N}\left[\mathbf{h}^{N+1}_{\alpha,\beta}[u,u] + \frac{C}{(\widehat{d}_N)^2}\|u\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})}\right] \\ &= \frac{1}{\beta_N}\left[(f + \mu u, u)_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})} + \frac{C}{(\widehat{d}_N)^2}\|u\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})}\right].\end{aligned}$$

Using (6.8) we then arrive at the estimate

$$|u(x_N)|^2 \le \frac{C_N}{\beta_N}\|f\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})},$$

and, similarly,

$$|u(x_{-N})|^2 \le \frac{C_{-N}}{\beta_{-N}}\|f\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})},$$

with the constants $C_{-N}$, $C_N$ being independent of $\alpha$ and $\beta$ (however, they depend on $\widehat{d}_N$, and $C$ from Lemma 3.4). Similarly to (6.11)–(6.12) we get the estimates

$$|v'(x_{-N}-0)|^2 \le C^-_{-N}\|g\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})},$$
$$|v'(x_N+0)|^2 \le C^+_N\|g\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})},$$

with the constants $C^-_{-N}$, $C^+_N$, which depend on $d_{-N}$ and $d_{N+1}$, respectively, but are independent of $\alpha$ and $\beta$. Further, denote $\widetilde{v} = v\restriction_{(x_{-N},x_N)}$. As in the proof of Lemma 5.4 (cf. (5.16)) one obtains the estimate

$$|v'(x_N-0)|^2 \le C''\left(d_N^{-1}\mathbf{h}_{\mathcal{I}_N,\alpha_N}[\mathbf{v}_N,\mathbf{v}_N] + \|(\mathbf{H}^N_{\alpha,\beta}\widetilde{v})\restriction_{\mathcal{I}_N}\|^2_{\mathsf{L}^2(\mathcal{I}_N)} + d_N^{-3}\|\mathbf{v}_N\|^2_{\mathsf{L}^2(\mathcal{I}_N)}\right)$$

with the constant $C''$ being independent of $\alpha$ and $\beta$. Using Lemma 3.4 and taking into account that $\beta_k > 0$ we can extend the above estimate as follows,

$$|v'(x_N-0)|^2 \le C''\Bigg[d_N^{-1}\left(\mathbf{h}^{N+1}_{\alpha,\beta}[v,v] + \frac{C}{(\widehat{d}_N)^2}\|v\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})}\right) + \|\mathbf{H}^{N+1}_{\alpha,\beta}v\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})} + d_N^{-3}\|v\|^2_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})}\Bigg],$$

where again $C$ is the constant from Lemma 3.4. Using

$$\mathbf{H}^{N+1}_{\alpha,\beta}v = g+\mu v,\ \ \mathbf{h}^{N+1}_{\alpha,\beta}[v,v] = (g+\mu v, v)_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})},\ \text{ and } \|v\|_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})} \le \|g\|_{\mathsf{L}^2(\widetilde{\mathcal{I}}_{N+1})}$$

we conclude that

$$|v'(x_N-0)|^2 \le C^-_N\|g\|^2_{\mathsf{L}^2(x_{-N-1},x_{N+1})}$$

and, similarly,

$$|v'(x_{-N}+0)|^2 \le C^+_{-N}\|g\|^2_{\mathsf{L}^2(x_{-N-1},x_{N+1})}, \tag{6.17}$$

where the constants $C^-_N$ and $C^+_{-N}$ depend on $d_k$, $k=-N,\dots,N+1$, but are independent of the sequences $\alpha$ and $\beta$. Combining (6.16)–(6.17) we arrive at the estimate

$$\begin{aligned}\Big\|(\mathbf{H}^{N+1}_{\alpha,\beta}-\mu\mathrm{I})^{-1} - \big((\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}}\oplus\mathbf{H}^N_{\alpha,\beta}\oplus\mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}) - \mu\mathrm{I}\big)^{-1}\Big\| \\ \le \widetilde{C}_N\max\{\beta_{-N}^{-1/2},\beta_N^{-1/2}\},\end{aligned} \tag{6.18}$$

where $\widetilde{C}_N$ is independent of the sequences $\alpha$ and $\beta$. Consequently, for each $j \in \mathbb{N}$

$$\sup_{\alpha_{-N},\dots,\alpha_{N+1}} \left|\lambda_j(\mathbf{H}^{N+1}_{\alpha,\beta}) - \lambda_j\left(\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}} \oplus \mathbf{H}^N_{\alpha,\beta} \oplus \mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}\right)\right| \to 0 \text{ as } \beta_{-N}, \beta_N \to \infty. \tag{6.19}$$

By construction the set

$$\sigma\left(\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}} \oplus \mathbf{H}^N_{\alpha,\beta} \oplus \mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}\right) \cap (T_1, T_2)$$

consists of $2N + 2$ simple eigenvalues (we denote them by $\gamma_{\alpha,\beta;k}$, $k = -N, \dots, N + 1$) such that

$$\begin{aligned} &\gamma_{\alpha,\beta;-N} = \lambda_1(\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}}), \quad \gamma_{\alpha,\beta;N+1} = \lambda_1(\mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}) \\ &\gamma_{\alpha,\beta;k} \in B_{\delta_k}(s_k), \ \ k = -N + 1, \dots, N, \end{aligned} \tag{6.20}$$

where $\gamma_{\alpha,\beta;k}$, $k = -N + 1, \dots, N$, are the $2N$ simple eigenvalues of the operator $\mathbf{H}^N_{\alpha,\beta}$ inside $(T_1, T_2)$. Moreover, we then have

$$\lambda_1(\mathbf{H}^D_{\mathcal{I}_{-N},\alpha_{-N}}) = s_{-N} \quad \text{and} \quad \lambda_1(\mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}}) \in \overline{B_{\delta_{N+1}/2}(s_{N+1})}$$

(cf. (3.13)–(3.14)). By induction hypothesis, for $n = N$ one has

$$\gamma_{\underline{\alpha}^{n,k},\beta;k} < s_k - \frac{1}{4}\delta_k \quad \text{and} \quad s_k + \frac{1}{4}\delta_k < \gamma_{\overline{\alpha}^{n,k},\beta;k}, \quad k = 1, \dots, N. \tag{6.21}$$

Since the eigenvalues $\gamma_{\alpha,\beta;k}$, $k = 1, \dots, N$, of $\mathbf{H}^N_{\alpha,\beta}$ are independent of $\alpha_l$ with $l > N$ it is clear that

$$\text{(6.21) holds also with } n = N + 1. \tag{6.22}$$

Moreover, the property $\gamma_{\alpha,\beta;N+1} = \lambda_1(\mathbf{H}^D_{\mathcal{I}_{N+1},\alpha_{N+1}})$ in (6.20) shows that

$$\gamma_{\underline{\alpha}^{N+1,N+1},\beta;N+1} = s_{N+1} - \frac{1}{2}\delta_{N+1} \quad \text{and} \quad \gamma_{\overline{\alpha}^{N+1,N+1},\beta;N+1} = s_{N+1} + \frac{1}{2}\delta_{N+1}. \tag{6.23}$$

Hence, using (6.19) we conclude from (6.20), (6.22), (6.23) that there exist positive $\beta'_{-N} \geq \beta^{\inf}_{-N}$ and $\beta'_N \geq \beta^{\inf}_N$ such that for $\beta_{-N} \in [\beta'_{-N}, \infty)$ and $\beta_N \in [\beta'_N, \infty)$ the operator $\mathbf{H}^{N+1}_{\alpha,\beta}$ also has precisely $2N + 2$ simple eigenvalues $\widetilde{\gamma}_{\alpha,\beta;k}$, $k = -N, \dots, N + 1$, in the interval $(T_1, T_2)$ that satisfy $\widetilde{\gamma}_{\alpha,\beta;k} \in B_{\delta_k}(s_k)$ as $k = -N, \dots, N+1$, and, moreover,

$$\widetilde{\gamma}_{\underline{\alpha}^{n,k},\beta;k} < s_k - \frac{1}{4}\delta_k \quad \text{and} \quad s_k + \frac{1}{4}\delta_k < \widetilde{\gamma}_{\overline{\alpha}^{n,k},\beta;k}, \quad k = 1, \dots, N + 1, \ n = N + 1.$$

Consequently, (6.3) and (6.5) hold for $n = N + 1$ and $(\beta_k)_{k\in\mathbb{Z}}$ satisfying $\beta_k \in [\beta'_k, \infty)$ as $k = -N, \dots, N$. Note that $\mathbf{H}^{N+1}_{\alpha,\beta}$ is independent of $\beta_k$ with $|k| > N$. This completes the induction step and the proof of Lemma 6.1. □

Recall that the eigenvalues $s^n_{\alpha,\beta;\,k}$, $k = 1, \ldots, n$, of the operator $\mathbf{H}^n_{\alpha,\beta}$ are independent of $\alpha_k$ with $k \notin \{-n+1, \ldots, n\}$ and that the entries $\alpha_k$ with $k = -n+1, \ldots, 0$ are fixed; cf. Hypothesis 3.1. Therefore, for a fixed sequence $\beta$ the eigenvalues $s^n_{\alpha,\beta;\,k}$, $k = 1, \ldots, n$, can be regarded as functions of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Bearing this in mind, for the following considerations we shall denote the eigenvalues $s^n_{\alpha,\beta;\,k}$, $k = 1, \ldots, n$, by

$$s^\beta_k[\alpha_1, \alpha_2, \ldots, \alpha_n];$$

of course we assume here that the sequence $(\beta_k)_{k\in\mathbb{Z}}$ satisfies $\beta'_k \le \beta_k < \infty$. In particular, the property (6.5) now reads as follows:

$$\begin{aligned} s^\beta_k[\alpha^+_1, \ldots, \alpha^+_{k-1}, \alpha^-_k, \alpha^+_{k+1}, \ldots, \alpha^+_n] &\le s_k - \tfrac{1}{4}\delta_k, \quad k = 1, \ldots, n, \\ s^\beta_k[\alpha^-_1, \ldots, \alpha^-_{k-1}, \alpha^+_k, \alpha^-_{k+1}, \ldots, \alpha^-_n] &\ge s_k + \tfrac{1}{4}\delta_k, \quad k = 1, \ldots, n. \end{aligned} \tag{6.24}$$

It is easy to see that the function

$$f : (\alpha_1, \alpha_2, \ldots, \alpha_n) \mapsto \begin{pmatrix} s^\beta_1[\alpha_1, \alpha_2, \ldots, \alpha_n] \\ \vdots \\ s^\beta_n[\alpha_1, \alpha_2, \ldots, \alpha_n] \end{pmatrix} \tag{6.25}$$

is continuous and each coordinate function $f_k(\cdot) = s^\beta_k[\cdot]$ is monotonically increasing in each of its arguments. Indeed, using the same arguments as in the proof of (6.18) we get the following estimate for two sequences $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\widetilde{\alpha}_k)_{k\in\mathbb{Z}}$:

$$\left\|(\mathbf{H}^n_{\alpha,\beta} - \mu\mathrm{I})^{-1} - (\mathbf{H}^n_{\widetilde{\alpha},\beta} - \mu\mathrm{I})^{-1}\right\| \le \widetilde{C} \max_{k=1,\ldots,n} |\alpha_k - \widetilde{\alpha}_k|, \tag{6.26}$$

where $\widetilde{C}$ is independent of $\alpha_k$ and $\widetilde{\alpha}_k$ (but it depends on $d_k$, $k = -N+1, \ldots, N$). Taking into account that $s_k[\alpha_1, \ldots, \alpha_n] \in B_{\delta_k}(s_k)$, $k = 1, \ldots, n$, are simple eigenvalues we conclude from (6.26) the continuity of the function $f$. The monotonicity of $f_k$ in each of its arguments follows from the min-max principle (see, e.g., [17, Section 4.5]).

The next lemma is an important ingredient for Theorem 6.4.

**Lemma 6.2.** *Let $\beta = (\beta_k)_{k\in\mathbb{Z}}$ be such that $\beta'_k \le \beta_k$, $k \in \mathbb{Z}$, where $(\beta'_k)_{k\in\mathbb{Z}}$ is a sequence as in Lemma 6.1. Then for $n \in \mathbb{N}$ the entries $\alpha_1, \ldots, \alpha_n$ of the sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ (satisfying Hypothesis 3.1) can be chosen such that*

$$s^\beta_k[\alpha_1, \alpha_2, \ldots, \alpha_n] = s_k, \qquad k = 1, \ldots, n.$$

The proof of the above lemma is based on the following multi-dimensional version of the intermediate value theorem, which was established in [18, Lemma 3.5].

**Lemma 6.3.** *Let $\mathcal{D} = \Pi_{k=1}^n [a_k, b_k]$ with $a_k < b_k$, $k = 1, \dots, n$, assume that $f : \mathcal{D} \to \mathbb{R}^n$ is continuous and each coordinate function $f_k$ of $f$ is monotonically increasing in each of its arguments. If $F_k^- < F_k^+$, $k = 1, \dots, n$, where*

$$F_k^- = f_k(b_1, b_2, \dots, b_{k-1}, a_k, b_{k+1}, \dots, b_n), \quad F_k^+ = f_k(a_1, a_2, \dots, a_{k-1}, b_k, a_{k+1}, \dots, b_n),$$

*then for any $F \in \Pi_{k=1}^n [F_k^-, F_k^+]$ there exists $x \in \mathcal{D}$ such that $f(x) = F$.*

**Proof of Lemma 6.2.** We fix $n \in \mathbb{N}$ and set $\mathcal{D} = \Pi_{k=1}^n [\alpha_k^-, \alpha_k^+]$; the points in $\mathcal{D}$ will be denoted in the form $(\alpha_1, \alpha_2, \dots, \alpha_n)$. Now consider the function $f : \mathcal{D} \to \mathbb{R}^n$ given by (6.25). As noted above, this function is continuous and each coordinate function $f_k(\cdot)$ is monotonically increasing in each of its arguments. Moreover, according to (6.24) we have

$$F_k^- := s_k^\beta[\alpha_1^+, \dots, \alpha_{k-1}^+, \alpha_k^-, \alpha_{k+1}^+, \dots, \alpha_n^+] \leq s_k - \frac{1}{4}\delta_k$$

and

$$F_k^+ := s_k^\beta[\alpha_1^-, \dots, \alpha_{k-1}^-, \alpha_k^+, \alpha_{k+1}^-, \dots, \alpha_n^-] \geq s_k + \frac{1}{4}\delta_k$$

for $k = 1, \dots, n$, and hence, in particular, $F_k^- < s_k < F_k^+$ for $k = 1, \dots, n$. Therefore, by Lemma 6.3 there exists $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{D}$ such that

$$f(\alpha_1, \alpha_2, \dots, \alpha_n) = (s_1, s_2, \dots, s_n);$$

this completes the proof of Lemma 6.2. □

Combining Lemma 6.1 and Lemma 6.2 we immediately arrive at the main result of this subsection.

**Theorem 6.4.** *Let $\alpha^n = (\alpha_k^n)_{k \in \mathbb{Z}}$ be a sequence satisfying Hypothesis 3.1, and assume that $\beta = (\beta_k)_{k \in \mathbb{Z}}$ is such that $\beta'_k \leq \beta_k$, $k \in \mathbb{Z}$, where $(\beta'_k)_{k \in \mathbb{Z}}$ is a sequence as in Lemma 6.1. Then for $n \in \mathbb{N}$ the entries $\alpha_k^n \in [\alpha_k^-, \alpha_k^+]$, $k = 1, \dots, n$, can be chosen such that*

$$\sigma(\mathbf{H}^n_{\alpha^n,\beta}) \cap (T_1, T_2) = \left\{ s^n_{\alpha^n,\beta;\,k} : k \in \mathcal{K}^n \cup \{1, \dots, n\} \right\}, \tag{6.27}$$

*where $s^n_{\alpha^n,\beta;\,k}$ are simple eigenvalues of $\mathbf{H}^n_{\alpha^n,\beta}$ satisfying*

$$\begin{aligned} s^n_{\alpha^n,\beta;\,k} &\in B_{\delta_k}(s_k), \quad && k \in \mathcal{K}^n, \\ s^n_{\alpha^n,\beta;\,k} &= s_k, && k \in \{1, \dots, n\}. \end{aligned} \tag{6.28}$$

**Remark 6.5.** We mention that the entries $\alpha_k^n$, $k = 1, \dots, n$, in the sequence $\alpha^n = (\alpha_k^n)_{k \in \mathbb{Z}}$ chosen above depend on the choice of the sequence $\beta$.

### 6.2. Spectrum of the operator $\mathcal{H}^n_{\alpha^n,\beta}$

Let $n \in \mathbb{N}$ and let $\mathbf{H}^n_{\alpha,\beta}$ be the self-adjoint operator in $\mathsf{L}^2(x_{-n}, x_n)$ from the previous subsection. In this subsection we will investigate the self-adjoint operator

$$\mathcal{H}^n_{\alpha,\beta} = \left( \bigoplus_{k \le -n} \mathbf{H}^D_{\mathcal{I}_k,\alpha_k} \right) \oplus \mathbf{H}^n_{\alpha,\beta} \oplus \left( \bigoplus_{k \ge n+1} \mathbf{H}^D_{\mathcal{I}_k,\alpha_k} \right) \tag{6.29}$$

acting in

$$\mathsf{L}^2(\ell_-, \ell_+) = \left( \bigoplus_{k \le -n} \mathsf{L}^2(\mathcal{I}_k) \right) \oplus \mathsf{L}^2(x_{-n}, x_n) \oplus \left( \bigoplus_{k \ge n+1} \mathsf{L}^2(\mathcal{I}_k) \right).$$

Informally speaking the operator $\mathcal{H}^n_{\alpha,\beta}$ is obtained from the decoupled operator $\mathcal{H}_{\alpha,\infty}$ in Section 3.3 by adding $\delta$-couplings of the strengths $\beta_k$ at *finitely* many points $x_k$, $k = -n+1, \dots, n-1$. It is clear that $\mathcal{H}^n_{\alpha,\beta}$ (and $\mathbf{H}^n_{\alpha,\beta}$) is independent of $\beta_k$ with $k \notin \{-n+1, \dots, n-1\}$.

It is convenient to strengthen Hypothesis 4.2 and from now on to consider sequences $\beta = (\beta_k)_{k\in\mathbb{Z}}$ that satisfy the following condition.

**Hypothesis 6.6.** The sequence $\beta = (\beta_k)_{k\in\mathbb{Z}}$ satisfies Hypothesis 4.2 and, in addition, it is assumed that (5.1) holds and $\beta'_k \le \beta_k$, $k \in \mathbb{Z}$, where $(\beta'_k)_{k\in\mathbb{Z}}$ denotes the sequence in Lemma 6.1.

The following theorem is a consequence of Theorem 6.4 and the considerations in Section 3 and Section 5.

**Theorem 6.7.** *Let $\alpha^n = (\alpha^n_k)_{k\in\mathbb{Z}}$ and $\beta = (\beta_k)_{k\in\mathbb{Z}}$ be sequences satisfying Hypothesis 3.1 and Hypothesis 6.6, and assume that $\alpha^n = (\alpha^n_k)_{k\in\mathbb{Z}}$ is chosen such that (6.27)–(6.28) hold, and*

$$\alpha^n_k = \mathcal{F}^D_{d_k}(s_k), \quad k \in \mathbb{N} \setminus \{1, \dots, n\}. \tag{6.30}$$

*Then one has*

$$\sigma_{\mathrm{ess}}(\mathcal{H}^n_{\alpha^n,\beta}) = S_{\mathrm{ess}} \quad \text{and} \quad \sigma_{\mathrm{disc}}(\mathcal{H}^n_{\alpha^n,\beta}) \cap (T_1, T_2) = S_{\mathrm{disc}}, \tag{6.31}$$

*and, moreover, each $s_k$, $k \in \mathbb{N}$, is a simple eigenvalue of $\mathcal{H}^n_{\alpha^n,\beta}$.*

**Proof.** It is not difficult to see that the resolvent difference

$$(\mathcal{H}^n_{\alpha^n,\beta} - \lambda \mathrm{I})^{-1} - (\mathcal{H}_{\alpha^n,\infty} - \lambda \mathrm{I})^{-1}$$

is a finite rank operator for any $\lambda \in \rho(\mathcal{H}^n_{\alpha^n,\beta}) \cap \rho(\mathcal{H}_{\alpha^n,\infty})$, and hence, in particular, a compact operator in $\mathsf{L}^2(\ell_-,\ell_+)$; this follows, e.g., by observing that both operators $\mathcal{H}^n_{\alpha^n,\beta}$ and $\mathcal{H}_{\alpha^n,\infty}$ can be viewed as self-adjoint extensions of the symmetric operator $\mathcal{H}^n_{\alpha^n,\beta} \cap \mathcal{H}_{\alpha^n,\infty}$, which has finite defect. Hence we have

$$\sigma_{\mathrm{ess}}(\mathcal{H}^n_{\alpha^n,\beta}) = \sigma_{\mathrm{ess}}(\mathcal{H}_{\alpha^n,\infty}) = S_{\mathrm{ess}}$$

by Theorem 3.2 and this shows the first assertion in (6.31).

Now we study the discrete spectrum of the operator $\mathcal{H}^n_{\alpha^n,\beta}$ in $(T_1,T_2)$. It is clear from (6.29) that

$$\sigma(\mathcal{H}^n_{\alpha^n,\beta}) = \sigma\left(\bigoplus_{k\le -n} \mathbf{H}^D_{\mathcal{I}_k,\alpha^n_k}\right) \cup \sigma(\mathbf{H}^n_{\alpha^n,\beta}) \cup \sigma\left(\bigoplus_{k\ge n+1} \mathbf{H}^D_{\mathcal{I}_k,\alpha^n_k}\right).$$

Recall from Proposition 2.2 that $\lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha^n_k})$ coincides with the unique solution of the equation $\alpha^n_k = \mathcal{F}^D_{d_k}(\lambda)$ on the interval $(0,(2\pi/d_k)^2)$. Thus, taking into account (6.30) and the second property in (3.12), we arrive at

$$\lambda_1(\mathbf{H}^D_{\mathcal{I}_k,\alpha^n_k}) = s_k, \qquad k \in \mathbb{Z}\setminus\{1,\dots,n\}. \tag{6.32}$$

Furthermore, we have $\lambda_j(\mathbf{H}^D_{\mathcal{I}_k,\alpha_k}) > T_2$ for $j \ge 2$ by Proposition 2.2 and (3.3). Observe that for $k \in \mathbb{Z}\setminus\mathbb{N}$ the eigenvalues in (6.32) do not contribute to the discrete spectrum of $\mathcal{H}^n_{\alpha^n,\beta}$ in $(T_1,T_2)$ since either $s_k \in \mathcal{O} \subset S_{\mathrm{ess}}$ or $s_k \notin [T_1,T_2]$; cf. (6.1) and (1.4). It follows that

$$\sigma\left(\bigoplus_{k\le -n} \mathbf{H}^D_{\mathcal{I}_k,\alpha^n_k}\right) \cap (T_1,T_2) \subset S_{\mathrm{ess}}.$$

The above considerations also show

$$\sigma\left(\bigoplus_{k\ge n+1} \mathbf{H}^D_{\mathcal{I}_k,\alpha^n_k}\right) \cap (T_1,T_2) = \big\{s_k : k = n+1, n+2, \dots\big\},$$

and all the eigenvalues $s_k$, $k = n+1, n+2, \dots$, are simple by the assumption (1.6). Finally, by Theorem 6.4 the spectrum of $\mathbf{H}^n_{\alpha,\beta}$ in $(T_1,T_2)$ consists of the simple eigenvalues $s_k$, $k = 1,\dots,n$, and the eigenvalues $s^n_{\alpha^n,\beta;\,k} \in B_{\delta_k}(s_k)$ for $k \in \mathcal{K}^n$. However, it follows from (6.1) and (1.4) that $s^n_{\alpha^n,\beta;\,k} \subset S_{\mathrm{ess}}$ for $k \in \mathcal{K}^n$. Summing up we conclude

$$\sigma_{\mathrm{disc}}(\mathcal{H}^n_{\alpha^n,\beta}) \cap (T_1,T_2) = \big\{s_k : k \in \mathbb{N}\big\} = S_{\mathrm{disc}}. \quad \square$$

## 7. Discrete spectrum of the operator $\mathcal{H}_{\alpha,\beta}$

In this section we complete the proof of our main result Theorem 4.4. Recall that the ultimate aim is to show the existence of sequences $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ and $\beta = (\beta_k)_{k\in\mathbb{Z}}$ such that (4.8) and (4.9) hold. We have already shown in Theorem 5.1 that the assertion (4.8) on the essential spectrum of $\mathcal{H}_{\alpha,\beta}$ holds for all sequences $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ and $\beta = (\beta_k)_{k\in\mathbb{Z}}$ that satisfy Hypothesis 3.1 and Hypothesis 4.2, respectively.

From now on we *fix* a sequence $\beta = (\beta_k)_{k\in\mathbb{Z}}$ that satisfies Hypothesis 6.6 (and hence also Hypothesis 4.2). Now we define a sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ such that Hypothesis 3.1 holds and the statement (4.9) on the discrete spectrum of $\mathcal{H}_{\alpha,\beta}$ is valid: By Theorem 6.7 there exists for each $n \in \mathbb{N}$ a sequence $\alpha^n = (\alpha^n_k)_{k\in\mathbb{Z}}$ such that Hypothesis 3.1, (6.30) and (6.31) hold, and, in particular, we have

$$\alpha^-_k \le \alpha^n_k \le \alpha^+_k, \qquad k \in \mathbb{Z}. \tag{7.1}$$

A usual diagonal process shows that there exist $n_m \in \mathbb{N}$ with $n_m < n_{m+1}$ and $\lim_{m\to\infty} n_m = \infty$, and a sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ such that

$$\alpha^{n_m}_k \to \alpha_k \quad \text{as } m \to \infty, \qquad k \in \mathbb{Z}. \tag{7.2}$$

It also follows that $\alpha_k \in [\alpha^-_k, \alpha^+_k]$ for $k \in \mathbb{Z}$, moreover, by the second property in (3.12) we have $\alpha_k = \alpha^n_k = \mathcal{F}^D_{d_k}(s_k)$ for $k \in \mathbb{Z}\setminus\mathbb{N}$. In other words, the sequence $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ satisfies Hypothesis 3.1. Note also that $\alpha$ depends on the sequence $\beta$ fixed above; cf. Remark 6.5.

**Lemma 7.1.** *For the sequence $\alpha$ defined by (7.2) one has*

$$\|(\mathcal{H}^{n_m}_{\alpha^{n_m},\beta} - \mu \mathrm{I})^{-1} - (\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1}\| \to 0 \quad \text{as } m \to \infty,$$

*where $\mu$ is defined by (5.2).*

Before we prove the above lemma, we observe that Theorem 5.1, Theorem 6.7 together with Lemma 7.1 immediately imply the main result of this section.

**Theorem 7.2.** *For the sequence $\alpha$ defined by (7.2) one has*

$$\sigma_{\mathrm{disc}}(\mathcal{H}_{\alpha,\beta}) \cap (T_1, T_2) = S_{\mathrm{disc}},$$

*moreover, each $s_k$, $k \in \mathbb{N}$, is a simple eigenvalue.*

**Proof of Lemma 7.1.** To simplify the presentation we assume that $n_m = m$ for $m \in \mathbb{N}$. Now let $f, g \in \mathsf{L}^2(\ell_-, \ell_+)$, and consider $u = (\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1} f$ and $v = (\mathcal{H}^m_{\alpha^m,\beta} - \mu \mathrm{I})^{-1} g$. Denote

$$T^m = (\mathcal{H}_{\alpha,\beta} - \mu \mathrm{I})^{-1} - (\mathcal{H}^m_{\alpha^m,\beta} - \mu \mathrm{I})^{-1}.$$

In the same way as in the proof of Theorem 5.1 one computes

$$\begin{aligned}(T^m f, g)_{\mathsf{L}^2(\ell_-,\ell_+)} &= (u, \mathcal{H}^m_{\alpha^m,\beta} v)_{\mathsf{L}^2(\ell_-,\ell_+)} - (\mathcal{H}_{\alpha,\beta} u, v)_{\mathsf{L}^2(\ell_-,\ell_+)} \\ &= \sum_{k\in\mathbb{Z}} u(y_k)\overline{\big(v'(y_k+0) - v'(y_k-0)\big)} - \sum_{k\in\mathbb{Z}} \big(u'(y_k+0) - u'(y_k-0)\big)\overline{v(y_k)} \\ &+ \sum_{k\in\mathbb{Z}} u(x_k)\overline{\big(v'(x_k+0) - v'(x_k-0)\big)} - \sum_{k\in\mathbb{Z}} \big(u'(x_k+0) - u'(x_k-0)\big)\overline{v(x_k)}.\end{aligned} \tag{7.3}$$

Using the boundary conditions for $u \in \mathrm{dom}(\mathcal{H}_{\alpha,\beta})$ and $v \in \mathrm{dom}(\mathcal{H}^m_{\alpha^m,\beta})$ we obtain

$$(T^m f, g)_{\mathsf{L}^2(\ell_-,\ell_+)} = \underbrace{\sum_{k\in\mathbb{N}} (\alpha_k^m - \alpha_k) u(y_k)\overline{v(y_k)}}_{I_1^m :=} + \underbrace{\sum_{|k|\geq m} u(x_k)\overline{\big(v'(x_k+0) - v'(x_k-0)\big)}}_{I_2^m :=}. \tag{7.4}$$

Indeed, the first two sums on the right hand side in (7.3) reduce to the first sum in (7.4) since $u$ and $v$ satisfy the $\delta$-jump conditions at $y_k$, $k \in \mathbb{Z}$, of the strength $\alpha_k$ and $\alpha_k^n$, respectively (recall that $\alpha_k^n = \alpha_k = \mathcal{F}^D_{d_k}(s_k)$ as $k \in \mathbb{Z} \setminus \mathbb{N}$). Also, since $u$ and $v$ satisfy the same $\delta$-jump conditions at $x_k$, $k = -m+1, \dots, m-1$, and $v(x_k) = 0$ for all $k \in \mathbb{Z} \setminus \{-m+1, \dots, m-1\}$ the last two sums on the right hand side in (7.3) reduce to the second sum in (7.4).

First we estimate the term $I_1^m$. Fix $\varepsilon > 0$. It is clear that

$$|I_1^m| \leq \left(\sum_{k\in\mathbb{N}} |\alpha_k^m - \alpha_k| \cdot |u(y_k)|^2\right)^{1/2} \left(\sum_{k\in\mathbb{N}} |\alpha_k^m - \alpha_k| \cdot |v(y_k)|^2\right)^{1/2}. \tag{7.5}$$

Let $(c_k)_{k\in\mathbb{N}}$ be the sequence from (3.8). Since $c_k \to 0$ as $k \to \infty$, there exists $K(\varepsilon) \in \mathbb{N}$ such that

$$c_k \leq \varepsilon \text{ as } k > K(\varepsilon). \tag{7.6}$$

Moreover, due to (7.2) there exists $M(\varepsilon)$ such that

$$\text{for } 1 \leq k \leq K(\varepsilon): \ |\alpha_k^m - \alpha_k| < \varepsilon d_k \quad \text{ as } m \geq M(\varepsilon). \tag{7.7}$$

Combining (3.8), (7.1), (7.6)–(7.7) we obtain for $m \geq M(\varepsilon)$:

$$\begin{aligned}&\sum_{k\in\mathbb{N}} |\alpha_k^m - \alpha_k| \cdot |u(y_k)|^2 \\ &\leq \sum_{k=1}^{K(\varepsilon)} |\alpha_k^m - \alpha_k| \cdot |u(y_k)|^2 + \sum_{k=K(\varepsilon)+1}^{\infty} (\alpha_k^+ - \alpha_k^-) \cdot |u(y_k)|^2 \leq \varepsilon \sum_{k\in\mathbb{N}} d_k |u(y_k)|^2.\end{aligned} \tag{7.8}$$

In what follows, we denote by $\mathbf{u}_k$ and $\mathbf{v}_k$, $k \in \mathbb{Z}$, the restrictions of the functions $u$ and $v$ to the interval $\mathcal{I}_k$. Recall that $\widehat{C}$ is a positive constant for which (3.23) holds. Without loss of generality we may assume that $\widehat{C} \geq 1$. We shall also use the following standard Sobolev inequality (see, e.g. [7, Lemma 1.3.8]):

$$\forall w \in \mathsf{W}^{1,2}(a,b): \quad |w(a)| \leq L\|w'\|^2_{\mathsf{L}^2(a,b)} + 2L^{-1}\|w\|^2_{\mathsf{L}^2(a,b)}, \tag{7.9}$$

where $a < b < \infty$, $L \in (0, b-a]$. Applying (7.9) with $(a,b) = (y_k, x_k) \subset \mathcal{I}_k$ and $L = \frac{d_k}{2\widehat{C}}$ we get

$$|u(y_k)|^2 \leq \frac{d_k}{2\widehat{C}}\|\mathbf{u}_k'\|^2_{\mathsf{L}^2(y_k,x_k)} + \frac{4\widehat{C}}{d_k}\|\mathbf{u}_k\|^2_{\mathsf{L}^2(y_k,x_k)} \leq \frac{d_k}{2\widehat{C}}\|\mathbf{u}_k'\|^2_{\mathsf{L}^2(\mathcal{I}_k)} + \frac{4\widehat{C}}{d_k}\|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)}.$$

It is straightforward to check that the above estimate is equivalent to the estimate

$$d_k|u(y_k)|^2 \leq \left(1 + \frac{d_k\alpha_k^m}{2\widehat{C}}\right)^{-1}\left(\frac{d_k^2}{2\widehat{C}}\mathbf{h}_{\mathcal{I}_k,\alpha_k^m}[\mathbf{u}_k,\mathbf{u}_k] + 4\widehat{C}\|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)}\right).$$

Therefore, since $\alpha_k^m \in [-\widehat{C}d_k^{-1}, 0)$ by (3.23), $d_k < \ell_+ - \ell_-$, and

$$\mathbf{h}_{\mathcal{I}_k,\alpha_k^m}[\mathbf{u}_k,\mathbf{u}_k] \leq \mathbf{h}_{\mathcal{I}_k,\alpha_k^m,\beta_{k-1},\beta_k}[\mathbf{u}_k,\mathbf{u}_k]$$

(this inequality holds since $\beta_k \geq 0$), we obtain

$$d_k|u(y_k)|^2 \leq C_1\mathbf{h}_{\mathcal{I}_k,\alpha_k^m,\beta_{k-1},\beta_k}[\mathbf{u}_k,\mathbf{u}_k] + C_2\|\mathbf{u}_k\|^2_{\mathsf{L}^2(\mathcal{I}_k)},$$

where $C_1 = \frac{(\ell_+-\ell_-)^2}{\widehat{C}}$, $C_2 = 8\widehat{C}$. From the above estimate and (7.8) we conclude

$$\begin{aligned}\sum_{k\in\mathbb{N}} |\alpha_k^m - \alpha_k| \cdot |u(y_k)|^2 &\leq C_1\varepsilon\mathfrak{h}_{\alpha,\beta}[u,u] + C_2\varepsilon\|u\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} \\ &\leq C\varepsilon\|f\|^2_{\mathsf{L}^2(\ell_-,\ell_+)},\ m \geq M(\varepsilon).\end{aligned} \tag{7.10}$$

Similarly,

$$\sum_{k\in\mathbb{N}} |\alpha_k^m - \alpha_k| \cdot |v(y_k)|^2 \leq C\varepsilon\|g\|^2_{\mathsf{L}^2(\ell_-,\ell_+)},\ m \geq M(\varepsilon). \tag{7.11}$$

Combining (7.5), (7.10), (7.11) we conclude that

$$\forall\varepsilon > 0 \quad \exists M(\varepsilon) \in \mathbb{N}: \quad |I_1^m| \leq C\varepsilon\|f\|_{\mathsf{L}^2(\ell_-,\ell_+)}\|g\|_{\mathsf{L}^2(\ell_-,\ell_+)} \text{ as } m \geq M(\varepsilon). \tag{7.12}$$

It remains to estimate the term $I_2^m$. Recall that $D_k = \min\{d_k, d_{k+1}\}$, $k \in \mathbb{Z}$. One has

$$|I_2^m| \le \left(\sum_{|k|\ge m} D_k^{-3}|u(x_k)|^2\right)^{1/2} \left(\sum_{|k|\ge m} D_k^3|v'(x_k+0) - v'(x_k-0)|^2\right)^{1/2}. \qquad (7.13)$$

Repeating verbatim the arguments of the proofs of (5.13) and (5.17) we obtain

$$\sum_{|k|\ge m} D_k^{-3}|u(x_k)|^2 \le C_m\|f\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}, \text{ where } C_m \to 0 \text{ for } m \to \infty, \qquad (7.14)$$

and

$$\sum_{k\in\mathbb{Z}} D_k^3|v'(x_k+0) - v'(x_k-0)|^2 \le C\|g\|^2_{\mathsf{L}^2(\ell_-,\ell_+)} \qquad (7.15)$$

(note that the function $v$ in the estimate (5.17) vanishes at $x_k$ for *all* $k \in \mathbb{Z}$, however this property is not utilized in the proof of (5.17)). Combining (7.14) and (7.15) we arrive at

$$|I_2^m|^2 \le C\,C_m\,\|f\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}\|g\|^2_{\mathsf{L}^2(\ell_-,\ell_+)}, \qquad \text{where } C_m \to 0 \text{ for } m \to \infty. \qquad (7.16)$$

The statement of the lemma follows immediately from (7.4), (7.12), (7.16). □

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## Appendix A

For the convenience of the reader we briefly discuss in this appendix infinite orthogonal sums of densely defined, closed, uniformly semibounded forms in Hilbert spaces and the associated self-adjoint operators.

### *A.1. Direct sums of Hilbert spaces*

Let $(\mathbf{V}_k)_{k\in\mathbb{Z}}$ be a family of Hilbert spaces and let $\prod_{k\in\mathbb{Z}} \mathbf{V}_k$ be the Cartesian product of $\mathbf{V}_k, k \in \mathbb{Z}$. The elements of $\prod_{k\in\mathbb{Z}} \mathbf{V}_k$ will be denoted by roman letters, while bold letters are used for their components, e.g., $u = (\mathbf{u}_k)_{k\in\mathbb{Z}}$, $\mathbf{u}_k \in \mathbf{V}_k$. The direct sum of $\mathbf{V}_k$,

$$V = \bigoplus_{k\in\mathbb{Z}} \mathbf{V}_k,$$

consists of all $u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in \prod_{k\in\mathbb{Z}} \mathbf{V}_k$ such that

$$\|u\|_V^2 = \sum_{k\in\mathbb{Z}} \|\mathbf{u}_k\|_{\mathbf{V}_k}^2 < \infty. \tag{A.1}$$

Due to (A.1) one can introduce a scalar product on $V$ by

$$(u,v)_V = \sum_{k\in\mathbb{Z}} (u_k, v_k)_{\mathbf{V}_k}. \tag{A.2}$$

It then turns out that $V$ is a Hilbert space; cf. [16, Chapter 1.6, Theorem 6.2].

**Proposition A.1.** *The space $V$ equipped with the scalar product (A.2) is a Hilbert space.*

*A.2. Direct sums of non-negative forms and associated operators*

Let $(\mathbf{V}_k)_{k\in\mathbb{Z}}$ be a family of Hilbert spaces and let $(\mathbf{h}_k)_{k\in\mathbb{Z}}$ be a family of closed, non-negative, densely defined sesquilinear forms (for each $k \in \mathbb{Z}$ the form $\mathbf{h}_k$ acts in the space $\mathbf{V}_k$). By the first representation theorem [20, §-VI. Theorem 2.1] there exists a unique self-adjoint operator $\mathbf{H}_k$ associated with the form $\mathbf{h}_k$, i.e. $\mathrm{dom}(\mathbf{H}_k) \subset \mathrm{dom}(\mathbf{h}_k)$ and

$$\mathbf{h}_k[\mathbf{u},\mathbf{v}] = (\mathbf{H}_k\mathbf{u},\mathbf{v})_{\mathbf{V}_k}, \qquad \mathbf{u} \in \mathrm{dom}(\mathbf{H}_k),\ \mathbf{v} \in \mathrm{dom}(\mathbf{h}_k).$$

In the space $V$ we define the form $\mathfrak{h}$ by

$$\mathfrak{h}[u,v] = \sum_{k\in\mathbb{Z}} \mathbf{h}_k[\mathbf{u}_k,\mathbf{v}_k],$$
$$\mathrm{dom}(\mathfrak{h}) = \left\{ u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in V : \mathbf{u}_k \in \mathrm{dom}(\mathbf{h}_k),\ \sum_{k\in\mathbb{Z}} \mathbf{h}_k[\mathbf{u}_k,\mathbf{u}_k] < \infty \right\}.$$

The form $\mathfrak{h}$ is referred to as the direct sum of the forms $\mathbf{h}_k$; we also use the notation

$$\mathfrak{h} = \bigoplus_{k\in\mathbb{Z}} \mathbf{h}_k.$$

**Proposition A.2.** *The form $\mathfrak{h}$ is non-negative, densely defined, and closed in $V$.*

**Proof.** It is clear that the form $\mathfrak{h}$ is non-negative. In order to prove that $\mathfrak{h}$ is densely defined in $V$ fix $v = (\mathbf{v}_k)_{k\in\mathbb{Z}} \in V$ and assume that

$$(u, v)_V = 0, \qquad u \in \mathrm{dom}(\mathfrak{h}). \tag{A.3}$$

For arbitrary $l \in \mathbb{Z}$ and $\mathbf{w} \in \mathrm{dom}(\mathbf{h}_l)$ we consider

$$w^l = (\mathbf{w}^l_k)_{k\in\mathbb{Z}} = \begin{cases} \mathbf{w} & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

Then $w^l \in \mathrm{dom}(\mathfrak{h})$ and (A.3) holds with $u = w^l$, which implies $(\mathbf{w}, \mathbf{v}_l)_{\mathbf{V}_l} = 0$. As the form $\mathbf{h}_l$ is densely defined in $\mathbf{V}_l$ it follows that $\mathbf{v}_l = 0$. Since $l \in \mathbb{Z}$ is arbitrary we conclude $v = (\mathbf{v}_l)_{l\in\mathbb{Z}} = 0$, which implies that $\mathfrak{h}$ is densely defined in $V$.

Finally, we verify that $\mathfrak{h}$ is closed. Let us equip $\mathrm{dom}(\mathbf{h}_k)$ with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathrm{dom}(\mathbf{h}_k)} = \mathbf{h}_k[\mathbf{u}, \mathbf{v}] + (u, v)_{\mathbf{V}_k}, \qquad \mathbf{u}, \mathbf{v} \in \mathrm{dom}(\mathbf{h}_k). \tag{A.4}$$

Since $\mathbf{h}_k$ is closed by assumption $\mathrm{dom}(\mathbf{h}_k)$ equipped with the scalar product (A.4) is a Hilbert space. On $\mathrm{dom}(\mathfrak{h})$ we consider the scalar product

$$(u, v)_{\mathrm{dom}(\mathfrak{h})} := \mathfrak{h}[u, v] + (u, v)_V = \sum_{k\in\mathbb{Z}} (\mathbf{u}_k, \mathbf{v}_k)_{\mathrm{dom}(\mathbf{h}_k)} \tag{A.5}$$

for $u = (\mathbf{u}_k)_{k\in\mathbb{Z}}, v = (\mathbf{v}_k)_{k\in\mathbb{Z}} \in \mathrm{dom}(\mathfrak{h})$. By Proposition A.1 $\mathrm{dom}(\mathfrak{h})$ together with the scalar product (A.5) is also a Hilbert space, that is, the form $\mathfrak{h}$ is closed. □

The proposition above implies that there exists a unique self-adjoint and non-negative operator $\mathcal{H}$ associated to the form $\mathfrak{h}$. We refer to $\mathcal{H}$ as the direct sum of $\mathbf{H}_k$ and use the notation

$$\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} \mathbf{H}_k.$$

As a consequence one obtains the following statement.

**Proposition A.3.** *The non-negative self-adjoint operator $\mathcal{H}$ associated to $\mathfrak{h}$ in $V$ is given by*

$$\mathcal{H}u = (\mathbf{H}_k \mathbf{u}_k)_{k\in\mathbb{Z}},$$
$$\mathrm{dom}(\mathcal{H}) = \left\{ u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in V : \mathbf{u}_k \in \mathrm{dom}(\mathbf{H}_k),\ \sum_{k\in\mathbb{Z}} \|\mathbf{H}_k \mathbf{u}_k\|^2_{\mathbf{V}_k} < \infty \right\}.$$

### A.3. Direct sums of uniformly semibounded forms and associated operators

Let again $(\mathbf{V}_k)_{k\in\mathbb{Z}}$ be a family of Hilbert spaces and let $(\mathbf{h}_k)_{k\in\mathbb{Z}}$ be a family of densely defined, semibounded, closed forms. We assume, in addition, that there is a uniform lower bound

$$C_{\inf} = \inf_{k\in\mathbb{Z}} \inf_{\mathbf{u}\in\mathrm{dom}(\mathbf{h}_k):\ \|\mathbf{u}\|_{\mathbf{V}_k}=1} \mathbf{h}_k[\mathbf{u},\mathbf{u}] > -\infty,$$

and consider the family of densely defined, non-negative, closed forms

$$\widetilde{\mathbf{h}}_k[\mathbf{u},\mathbf{v}] = \mathbf{h}_k[\mathbf{u},\mathbf{v}] - C_{\inf}(\mathbf{u},\mathbf{v})_{\mathbf{V}_k}, \quad \mathrm{dom}(\widetilde{\mathbf{h}}_k) = \mathrm{dom}(\mathbf{h}_k).$$

By Proposition A.2 the form

$$\widetilde{\mathfrak{h}} = \bigoplus_{k\in\mathbb{Z}} \widetilde{\mathbf{h}}_k$$

is non-negative, densely defined, and closed in $V = (\mathbf{V}_k)_{k\in\mathbb{Z}}$.

Now, we define the direct sum $\mathfrak{h} = \bigoplus_{k\in\mathbb{Z}} \mathbf{h}_k$ in $V$ by

$$\mathfrak{h}[u,v] = \widetilde{\mathfrak{h}}[u,v] + C_{\inf}(u,v)_V, \quad \mathrm{dom}(\mathfrak{h}) = \mathrm{dom}(\widetilde{\mathfrak{h}}).$$

It is clear that the form $\mathfrak{h}$ is densely defined, semibounded, and closed in $V$; moreover $\mathrm{dom}(\mathfrak{h})$ consists of all $u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in V$ such that

$$\mathbf{u}_k \in \mathrm{dom}(\mathbf{h}_k) \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |\mathbf{h}_k[\mathbf{u}_k,\mathbf{u}_k]| < \infty. \tag{A.6}$$

As in the non-negative case the self-adjoint operator $\mathcal{H}$ associated to $\mathfrak{h}$ is referred to as the direct sum of the operators $\mathbf{H}_k$. Then one obtains the following variant of Proposition A.3.

**Proposition A.4.** *The semibounded self-adjoint operator $\mathcal{H}$ associated to $\mathfrak{h}$ in $V$ is given by*

$$\mathcal{H}u = (\mathbf{H}_k\mathbf{u}_k)_{k\in\mathbb{Z}},$$
$$\mathrm{dom}(\mathcal{H}) = \left\{ u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in V : \mathbf{u}_k \in \mathrm{dom}(\mathbf{H}_k),\ \sum_{k\in\mathbb{Z}} \|\mathbf{H}_k\mathbf{u}_k\|^2_{\mathbf{V}_k} < \infty \right\}.$$

Let us now assume that the spectrum of each semibounded self-adjoint operator $\mathbf{H}_k$ is discrete; we denote the corresponding eigenvalues (in nondecreasing order with multiplicities taken into account) by $s_{jk}$, $j \in \mathbb{N}$. Furthermore, we introduce the sequence $S = (s_{jk})_{j\in\mathbb{N},k\in\mathbb{Z}}$. The next goal is to describe the spectrum of the operator $\mathcal{H}$.

**Theorem A.5.** *Assume that the spectra of all $\mathbf{H}_k$ are discrete and let $S = (s_{jk})_{j\in\mathbb{N},k\in\mathbb{Z}}$ be the set of all eigenvalues. Then the following assertions hold for the spectrum of the semibounded self-adjoint operator $\mathcal{H}$ in Proposition A.4.*

(i) *$\lambda$ is an eigenvalue of $\mathcal{H}$ if and only if $\lambda \in \sigma(\mathbf{H}_k)$ for some $k \in \mathbb{Z}$. More precisely, one has*

$$\ker(\mathcal{H} - \lambda \mathrm{I}) = \bigoplus_{k\in\mathbb{Z}} \ker(\mathcal{H}_k - \lambda \mathrm{I}) \tag{A.7}$$

*and, in particular,*

$$\dim\big(\ker(\mathcal{H} - \lambda \mathrm{I})\big) = \#\left\{(j,k) \in \mathbb{N}\times\mathbb{Z} :\ s_{jk} = \lambda\right\}; \tag{A.8}$$

(ii) *$\sigma(\mathcal{H}) = \overline{S}$;*
(iii) *$\sigma_{\mathrm{ess}}(\mathcal{H}) = \{$accumulation points of $S\}$.*

**Proof.** (i) Let $\lambda$ be an eigenvalue of $\mathcal{H}$ and let $u = (\mathbf{u}_k)_{k\in\mathbb{Z}} \in V$ be a corresponding eigenfunction. Then one has $\mathbf{H}_k\mathbf{u}_k = \lambda\mathbf{u}_k$ for all $k \in \mathbb{Z}$ by Proposition A.4. Moreover, since $u \neq 0$ there exists $k \in \mathbb{Z}$ such that $\mathbf{u}_k \neq 0$. Therefore, $\lambda$ is an eigenvalue of $\mathbf{H}_k$. Conversely, if $\lambda \in \sigma(\mathbf{H}_k)$ for some $k \in \mathbb{Z}$ then $\lambda$ is an eigenvalue of $\mathbf{H}_k$. If $\mathbf{w}$ is a corresponding eigenvector then $\lambda$ is an eigenvalue of $\mathcal{H}$

$$u = (\mathbf{u}_k)_{k\in\mathbb{Z}} = \begin{cases} \mathbf{w} & \text{if } l = k, \\ 0 & \text{if } l \neq k, \end{cases}$$

is a corresponding eigenvector. This also shows the equality (A.7) and the last statement (A.8) is obvious.

(ii) The inclusion $\sigma(\mathcal{H}) \supset S$ follows from (i). Since $\sigma(\mathcal{H})$ is closed we conclude $\sigma(\mathcal{H}) \supset \overline{S}$. To prove the reverse inclusion assume that $\lambda \in \mathbb{R} \setminus \overline{S}$. Then there exists $\delta > 0$ such that

$$\operatorname{dist}(\lambda, \sigma(\mathbf{H}_k)) > \delta, \qquad k \in \mathbb{Z},$$

and, in particular, $\lambda$ belongs to the resolvent set of each operator $\mathbf{H}_k$. Now pick some $f = (\mathbf{f}_k)_{k\in\mathbb{N}} \in V$ and consider $\mathbf{u}_k = (\mathbf{H}_k - \lambda \mathrm{I})^{-1}\mathbf{f}_k \in \operatorname{dom}(\mathbf{H}_k) \subset \operatorname{dom}(\mathbf{h}_k)$. Then $\|\mathbf{u}_k\|_{\mathbf{V}_k} \leq \delta^{-1}\|\mathbf{f}_k\|_{\mathbf{V}_k}$ and for $u = (\mathbf{u}_k)_{k\in\mathbb{Z}}$ one has $\mathbf{H}_k\mathbf{u}_k = \mathbf{f}_k + \lambda\mathbf{u}_k$ and

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\mathbf{h}_k[\mathbf{u}_k,\mathbf{u}_k]| &= \sum_{k\in\mathbb{Z}} |(\mathbf{H}_k\mathbf{u}_k,\mathbf{u}_k)_{\mathbf{V}_k}| \\ &= \sum_{k\in\mathbb{Z}} \left|(\mathbf{f}_k,\mathbf{u}_k)_{\mathbf{V}_k} + \lambda\|\mathbf{u}_k\|^2_{\mathbf{V}_k}\right| \\ &\le \sum_{k\in\mathbb{Z}} \left(\frac{1}{\delta}\|\mathbf{f}_k\|^2_{\mathbf{V}_k} + \frac{|\lambda|}{\delta^2}\|\mathbf{f}_k\|^2_{\mathbf{V}_k}\right) \\ &\le \left(\frac{1}{\delta} + \frac{|\lambda|}{\delta^2}\right)\|f\|^2_V.\end{aligned}$$

Thus, $u \in \mathrm{dom}(\mathfrak{h})$; cf. (A.6). Furthermore, for $v = (\mathbf{v}_k)_{k\in\mathbb{Z}} \in \mathrm{dom}(\mathfrak{h})$ a similar argument shows

$$\mathfrak{h}[u,v] = \sum_{k\in\mathbb{Z}} \mathbf{h}_k[\mathbf{u}_k,\mathbf{v}_k] = \sum_{k\in\mathbb{Z}} [(\mathbf{f}_k,\mathbf{v}_k)_{\mathbf{V}_k} + \lambda(\mathbf{u}_k,\mathbf{v}_k)_{\mathbf{V}_k}] = (f+\lambda u, v)_V,$$

and we conclude $u \in \mathrm{dom}(\mathcal{H})$ and $\mathcal{H}u = f + \lambda u$ from the first representation theorem. Consequently, $\mathrm{ran}(\mathcal{H} - \lambda\mathrm{I}) = V$ and as $\mathcal{H}$ is self-adjoint this shows that $\lambda$ is in the resolvent set of $\mathcal{H}$. Therefore, we conclude $\sigma(\mathcal{H}) \subset \overline{S}$.

(iii) Let $\lambda$ be an accumulation point of $S$. Then any open neighborhood of $\lambda$ contains infinitely many elements of $S$. Therefore, either

(a) there is a sequence $(\lambda_l)_{l\in\mathbb{N}}$ such that $\lambda_l \in \sigma(\mathbf{H}_{k_l})_{l\in\mathbb{N}}$ with $\lambda_l \neq \lambda$ and $\lambda_l \to \lambda$ as $l \to \infty$, or
(b) there exists an infinite set $K \subset \mathbb{Z}$ such $\lambda \in \sigma(\mathbf{H}_k)$ for $k \in K$.

Using (i) we conclude in the case (a) that each punctured neighborhood of $\lambda$ contains an eigenvalue of $\mathcal{H}$, or in the case (b) $\lambda$ is an eigenvalue of $\sigma(\mathcal{H})$ with $\dim(\ker(\mathcal{H}-\lambda\mathrm{I})) = \infty$. In both situations we have $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{H})$.

Conversely, we have $\sigma_{\mathrm{ess}}(H) = \sigma(\mathcal{H}) \setminus \sigma_{\mathrm{disc}}(\mathcal{H}) = \overline{S} \setminus \sigma_{\mathrm{disc}}(\mathcal{H})$ by (ii). One concludes from (i) that the set $\sigma_{\mathrm{disc}}(\mathcal{H})$ consists of those $\lambda \in S$ which are isolated and satisfy

$$\#\{(j,k) \in \mathbb{N}\times\mathbb{Z} :\ s_{jk} = \lambda\} < \infty.$$

Now, if $\lambda \in \sigma_{\mathrm{ess}}(H)$ then it follows that $\lambda \in \overline{S}$ but $\lambda$ is not isolated or

$$\#\{(j,k) \in \mathbb{N}\times\mathbb{Z} :\ s_{jk} = \lambda\} = \infty.$$

In both cases we conclude that $\lambda$ is an accumulation point of $S$. □

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