AN INDEFINITE LAPLACIAN ON A RECTANGLE

By

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Abstract. In this note, we investigate the nonelliptic differential expression

$$\mathcal{A} = -\mathrm{div}\ \mathrm{sgn}\, \nabla$$

on a rectangular domain Ω in the plane. The seemingly simple problem of associating a self-adjoint operator with the differential expression $\mathcal{A}$ in $L^2(\Omega)$ is solved here. Such indefinite Laplacians arise in mathematical models of metamaterials characterized by negative electric permittivity and/or negative magnetic permeability.

1 Introduction

Consider the domains $\Omega_+ = (0,1) \times (0,1)$ and $\Omega_- = (-1,0) \times (0,1)$, and let $\Omega = (-1,1) \times (0,1)$ and $\mathcal{C} = \{0\} \times (0,1)$. We study the nonelliptic differential expression $\mathcal{A}$ defined on the rectangle Ω by

$$\mathcal{A}f = -\mathrm{div}\,(\mathrm{sgn}\,\nabla f), \tag{1.1}$$

where

$$\mathrm{sgn}(x,y) = \begin{cases} 1, & (x,y) \in \Omega_+, \\ -1, & (x,y) \in \Omega_-. \end{cases}$$

Our aim is to associate a self-adjoint operator in $L^2(\Omega)$ with Dirichlet boundary conditions on $\partial\Omega$ to $\mathcal{A}$. Informally speaking, in this seemingly simple toy problem, this is the partial differential operator

$$\begin{aligned} Af &= \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\ \mathrm{dom}\, A &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{matrix} f_\pm,\ \Delta f_\pm \in L^2(\Omega_\pm),\ f|_{\partial\Omega} = 0, \\ f_+|_{\mathcal{C}} = f_-|_{\mathcal{C}},\ \partial_{\mathbf{n}_+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}} \end{matrix} \right\}, \end{aligned} \tag{1.2}$$

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where $f_\pm$ denote the restrictions of the function $f \in L^2(\Omega)$ onto $\Omega_\pm$, and the normal derivatives $\partial_{\mathbf{n}_+}$ and $\partial_{\mathbf{n}_-}$ point outward of $\Omega_\pm$ (and hence in opposite directions at $\mathcal{C}$). The main peculiarity here is the interface condition

$$\partial_{\mathbf{n}_+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}}, \quad f = (f_+, f_-)^\top \in \operatorname{dom} A,$$

for the normal derivatives, which is due to the sign change and discontinuity of the coefficient sgn at $\mathcal{C}$. Our main result states that (when the Dirichlet and Neumann traces are properly interpreted) the operator A in (1.2) is self-adjoint in $L^2(\Omega)$.

The non-standard interface condition is responsible for unexpected spectral properties of A. Although the domain Ω is bounded, it turns out that the essential spectrum of A is not empty, namely, 0 is an isolated eigenvalue of infinite multiplicity. The remaining part of the spectrum of A consists of discrete eigenvalues which accumulate at $+\infty$ and $-\infty$. We note that the differential equation $\mathcal{A}f = \lambda f$ can, of course, be solved by separation of variables; the main feature of this note is the description of the domain of the corresponding self-adjoint operator A with explicit boundary and interface conditions.

We point out that $\operatorname{dom} A$ contains functions which do not belong to any local Sobolev space H^s, $s > 0$, in a neighbourhood of the interface $\mathcal{C}$. This leads to the difficulties that Green's identity is not valid for functions $f, g \in \operatorname{dom} A$, and the definition of the (local) Dirichlet and Neumann traces is rather subtle and requires a particularly careful analysis. Here we employ recent results on the extension of trace maps onto maximal domains of Laplacians on (quasi-)convex and Lipschitz domains from [2, 12] and rely on the description of the traces of $H^2(\Omega_\pm)$-functions in [13]. Finally, it turns out that the operator A can be viewed as a kind of Krein-von Neumann extension of a non-semibounded symmetric operator with infinite defect and domain contained in $H^2(\Omega_+) \times H^2(\Omega_-)$; thus, only the functions in the infinite dimensional eigenspace $\ker A$ do not possess H^s-regularity near the interface $\mathcal{C}$.

We emphasize that our result complements the results in [4], where the related problem

$$\mathcal{A}_\varepsilon f = -\operatorname{div}(\varepsilon \nabla f), \quad \varepsilon(x, y) = \begin{cases} \varepsilon_+, & (x, y) \in \Omega_+, \\ -\varepsilon_-, & (x, y) \in \Omega_-, \end{cases} \tag{1.3}$$

with $\varepsilon_\pm > 0$ was treated under the assumption $\varepsilon_+ \neq \varepsilon_-$ with the help of boundary integral methods on more general domains $\Omega \subset \mathbb{R}^2$; for related problems, see also [3, 8, 9, 14, 19, 20]. It is shown in [4] that if $\varepsilon_+ \neq \varepsilon_-$, the operator

$$A_\varepsilon f = \mathcal{A}_\varepsilon f, \quad \operatorname{dom} A_\varepsilon = \{ f \in H_0^1(\Omega) : \mathcal{A}_\varepsilon f \in L^2(\Omega) \}, \tag{1.4}$$

is self-adjoint, has a compact resolvent, and has eigenvalues accumulating to $+\infty$ and $-\infty$. The borderline case $\varepsilon_+ = \varepsilon_-$ that we investigate in this note was excluded in [4] and the other works (except for the 1-dimensional situation [20], which is intrinsically different). We also mention that abstract representation theorems for indefinite quadratic forms and related form methods in [18] (see also [11, 19] and [25, 27]) are not directly applicable to the present problem, nor do they lead to a self-adjoint operator in $L^2(\Omega)$. The eigenvalue problem $\mathcal{A}_\varepsilon f = \lambda f$ in our rectangular geometry was previously considered in [19] with the help of separation of variables (cf. Section 5), from which it follows that 0 is an eigenvalue of infinite multiplicity, provided that $\varepsilon_+ = \varepsilon_-$.

The indefinite differential expressions (1.1) and (1.3) arise in mathematical models of metamaterials which are characterized by negative electric permittivity and/or negative magnetic permeability; see [24, 26] for a physical survey and [5, 7, 10] for a rigorous justification of the models via a homogenization of Maxwell's equations in geometrically non-trivial periodic structures. More specifically, our rectangular model can be thought as simulating an interface between a dielectric material in Ω_+ and a metamaterial in Ω_-. It has been known since the seminal work [8] that the problem of the type $\mathcal{A}_\varepsilon f = \rho$ in Ω with a smooth interface is well-posed in $H_0^1(\Omega)$ if and only if the contrast $\kappa := \varepsilon_+/\varepsilon_-$ differs from 1. By proving that (1.2) is self-adjoint, we provide a correct functional setting for the problem on a rectangle in the critical situation $\kappa = 1$. Moreover, we show that the eigenvalues and eigenfunctions of A_ε converge to eigenvalues and eigenfunctions of the operator A as $\kappa \to 1$.

An alternative approach to theoretical studies of metamaterials is to add a small imaginary number to the negative value of sgn, arguing that "real systems are always slightly lossy"; see, e.g., [24]. This leads to a complexified differential expression

$$\mathcal{B}_\eta f = -\operatorname{div}(\varepsilon_\eta \nabla f), \quad \varepsilon_\eta(x,y) = \begin{cases} 1, & (x,y) \in \Omega_+, \\ -1 + i\eta, & (x,y) \in \Omega_-, \end{cases} \tag{1.5}$$

with $\eta > 0$, which immediately provides a well-defined operator

$$B_\eta f = \mathcal{B}_\eta f, \quad \operatorname{dom} B_\eta = \{ f \in H_0^1(\Omega) : \mathcal{B}_\eta f \in L^2(\Omega) \}. \tag{1.6}$$

Indeed, the rotated operator $e^{-i(\pi/2-\eta)} B_\eta$ is an m-sectorial operator with vertex 0 and semi-angle $\pi/2 - \eta$, which is defined via the associated sectorial form defined on $H_0^1(\Omega)$; cf. [21, Sec. VI]. It follows that B_η is an operator with compact resolvent for every $\eta > 0$, albeit non-self-adjoint now. Let us note that considering the complexified problem $B_\eta f = \rho$ in the limit as $\eta \to 0$ is a conventional way of describing the cloaking effects in metamaterials (of different geometric structure) through the "anomalous localized resonance"; see [6, 23]. We show that the eigenvalues and eigenfunctions of B_η converge to eigenvalues and eigenfunctions of our operator A as $\eta \to 0$. Recall that $Af = \rho$ is generally ill-posed since 0 is an eigenvalue of infinite multiplicity.

This note is organized as follows. In Section 2, we establish a modified version of Green's identity and other preliminary results that we frequently use later. In Section 3, we introduce an auxiliary closed symmetric operator R and study its properties. We prove the self-adjointness of A in Section 4, considering a generalized Krein–von Neumann extension of R. In that section, we also discuss qualitative spectral properties of A. More quantitative results about the spectrum of A and the aforementioned convergence results are established in Section 5.

2 A generalized Green's identity on the maximal domain

The Dirichlet realizations $A_{D\pm}$ associated to $\mp\Delta$ in $L^2(\Omega_\pm)$ play an important role in the sequel. Recall that

$$A_{D\pm} = \mp\Delta, \quad \operatorname{dom} A_{D\pm} = H_0^1(\Omega_\pm) \cap H^2(\Omega_\pm) \tag{2.1}$$

are self-adjoint operators on $L^2(\Omega_\pm)$ with compact resolvents, that A_{D+} is uniformly positive, and that A_{D-} is uniformly negative. Here the H^2-regularity is a consequence of $\Omega_\pm$ being convex; cf. [15, 16]. Observe that

$$\operatorname{dom} A_{D\pm} = \{f_\pm \in H^2(\Omega_\pm) : \gamma_D f_\pm = 0\},$$

where γ_D denotes the Dirichlet trace operator defined on $H^2(\Omega_\pm)$.

The self-adjoint Neumann operators are given by

$$A_{N\pm} = \mp\Delta, \quad \operatorname{dom} A_{N\pm} = \{f_\pm \in H^2(\Omega_\pm) : \gamma_{N_\pm} f_\pm = 0\},$$

where $\gamma_{N_\pm}$ are the Neumann trace operators defined on $H^2(\Omega_\pm)$ with normal pointing outwards $\Omega_\pm$.

We also make use of the spaces

$$\begin{aligned}\mathscr{G}_N(\partial\Omega_\pm) &:= \operatorname{ran}\big(\gamma_{N_\pm}(\operatorname{dom} A_{D\pm})\big) = \{\gamma_{N_\pm} f_\pm : f_\pm \in H^2(\Omega_\pm),\ \gamma_D f_\pm = 0\},\\ \mathscr{G}_D(\partial\Omega_\pm) &:= \operatorname{ran}\big(\gamma_D(\operatorname{dom} A_{N\pm})\big) = \{\gamma_D f_\pm : f_\pm \in H^2(\Omega_\pm),\ \gamma_{N_\pm} f_\pm = 0\},\end{aligned}$$

which were characterized in [12] and denoted by $N^{1/2}(\partial\Omega_\pm)$ and $N^{3/2}(\partial\Omega_\pm)$, respectively, and also appear in [2] in a more general setting. We equip $\mathscr{G}_N(\partial\Omega_\pm)$

and $\mathscr{G}_D(\partial\boldsymbol{\Omega}_\pm)$ with the natural norms [12, (6.6) and (6.42)]. If $\mathbf{n}_\pm$ and $\mathbf{t}_\pm$ denote the unit normal pointing outwards and a corresponding tangential vector, respectively, and $\partial_{\mathbf{t}_\pm}$ is the tangential derivative on $\partial\boldsymbol{\Omega}_\pm$, then according to [13, Theorem 3],

$$\big(\gamma_{N_\pm} f_\pm\big)\mathbf{t}_\pm \in \big(H^{1/2}(\partial\boldsymbol{\Omega}_\pm)\big)^2$$

for all $\gamma_{N_\pm} f_\pm \in \mathscr{G}_N(\partial\boldsymbol{\Omega}_\pm)$, and

$$\big(\partial_{\mathbf{t}_\pm}\gamma_D f_\pm\big)\mathbf{n}_\pm \in \big(H^{1/2}(\partial\boldsymbol{\Omega}_\pm)\big)^2$$

for all $\gamma_D f_\pm \in \mathscr{G}_D(\partial\boldsymbol{\Omega}_\pm)$, where

$$H^{1/2}(\partial\boldsymbol{\Omega}_\pm) = \left\{\varphi \in L^2(\partial\boldsymbol{\Omega}_\pm) : \int_{\partial\boldsymbol{\Omega}_\pm}\int_{\partial\boldsymbol{\Omega}_\pm} \frac{|\varphi(\alpha)-\varphi(\beta)|^2}{|\alpha-\beta|^2}\,d\alpha\,d\beta < \infty\right\}.$$

The following statement on the decomposition of functions in $\mathscr{G}_N(\partial\boldsymbol{\Omega}_\pm)$ and $\mathscr{G}_D(\partial\boldsymbol{\Omega}_\pm)$ into two parts with supports on $\mathcal{C}$ and $\mathcal{C}_\pm := \partial\boldsymbol{\Omega}_\pm\setminus\mathcal{C}$, respectively, is a direct consequence of the abovementioned fact.

Lemma 2.1. *Every function $\varphi \in \mathscr{G}_N(\partial\boldsymbol{\Omega}_\pm)$ (respectively, $\varphi \in \mathscr{G}_D(\partial\boldsymbol{\Omega}_\pm)$) admits a decomposition of the form*

$$\varphi = (\varphi|_{\mathcal{C}})^\sim + (\varphi|_{\mathcal{C}_\pm})^\sim, \tag{2.2}$$

where $(\varphi|_{\mathcal{C}})^\sim \in \mathscr{G}_N(\partial\boldsymbol{\Omega}_\pm)$ (respectively, $(\varphi|_{\mathcal{C}})^\sim \in \mathscr{G}_D(\partial\boldsymbol{\Omega}_\pm)$) is the extension of $\varphi|_{\mathcal{C}}$ to $\partial\boldsymbol{\Omega}_\pm$ by 0, and $(\varphi|_{\mathcal{C}_\pm})^\sim \in \mathscr{G}_N(\partial\boldsymbol{\Omega}_\pm)$ (respectively, $(\varphi|_{\mathcal{C}_\pm})^\sim \in \mathscr{G}_D(\partial\boldsymbol{\Omega}_\pm)$) is the extension of $\varphi|_{\mathcal{C}_\pm}$ to $\partial\boldsymbol{\Omega}_\pm$ by 0.

Consider the symmetric operators $S_\pm = \mp\Delta$, $\operatorname{dom} S_\pm = H_0^2(\boldsymbol{\Omega}_\pm)$, and their adjoints

$$S_\pm^* = \mp\Delta, \quad \operatorname{dom} S_\pm^* = \big\{f_\pm \in L^2(\boldsymbol{\Omega}_\pm) : \Delta f_\pm \in L^2(\boldsymbol{\Omega}_\pm)\big\}. \tag{2.3}$$

Since $0 \notin \sigma(A_{D,\pm})$, one has the direct sum decompositions

$$\operatorname{dom} S_\pm^* = \operatorname{dom} A_{D\pm} \dotplus \ker S_\pm^*. \tag{2.4}$$

In the following, we often decompose functions $f_\pm \in \operatorname{dom} S_\pm^*$ accordingly, that is, we write

$$f_\pm = f_{D\pm} + f_{0\pm}, \quad f_{D\pm} \in \operatorname{dom} A_{D\pm}, \quad f_{0\pm} \in \ker S_\pm^*. \tag{2.5}$$

It is also important to note that the spaces $\ker S_\pm^* \cap H^2(\boldsymbol{\Omega}_\pm)$ are dense in $\ker S_\pm^*$, where the latter spaces are equipped with the L^2-norm (or, equivalently, with the graph norm of $S_\pm^*$). This fact can be shown with the help of the density result [12, (6.30)] for $s = 0$.

Recall from [12, Theorem 6.4] that the Dirichlet traces γ_D admit continuous and surjective extensions

$$\widetilde{\gamma}_D : \operatorname{dom} S_\pm^* \to \big(\mathscr{G}_N(\partial\Omega_\pm)\big)^*,$$

where $\operatorname{dom} S_\pm^*$ is equipped with the graph norm and $(\mathscr{G}_N(\partial\Omega_\pm))^*$ is the conjugate dual space of $\mathscr{G}_N(\partial\Omega_\pm)$ equipped with the corresponding norm. It is important to note that

$$\ker \widetilde{\gamma}_D = \ker \gamma_D = \operatorname{dom} A_{D\pm} = H_0^1(\Omega_\pm) \cap H^2(\Omega_\pm), \tag{2.6}$$

where the first equality has been shown in [2, Section 4.1] and the other identities are clear from the above.

We denote the duality pairing between $\mathscr{G}_N(\partial\Omega_\pm)$ and $(\mathscr{G}_N(\partial\Omega_\pm))^*$ in the form

$$_{\mathscr{G}_N(\partial\Omega_\pm)^*}\langle \psi, \varphi\rangle_{\mathscr{G}_N(\partial\Omega_\pm)}, \quad \psi \in \mathscr{G}_N(\partial\Omega_\pm)^*, \quad \varphi \in \mathscr{G}_N(\partial\Omega_\pm),$$

and occasionally write $\psi(\varphi)$ in this situation.

Note for later use that the Neumann traces $\gamma_{N_\pm}$ admit continuous and surjective extensions

$$\widetilde{\gamma}_{N_\pm} : \operatorname{dom} S_\pm^* \to \big(\mathscr{G}_D(\partial\Omega_\pm)\big)^*;$$

this fact was observed in [12, Theorem 6.10]. Here, again $\operatorname{dom} S_\pm^*$ is equipped with the graph norm and $(\mathscr{G}_D(\partial\Omega_\pm))^*$ is the conjugate dual space of $\mathscr{G}_D(\partial\Omega_\pm)$, equipped with the corresponding norm.

The next proposition shows that a modified Green's identity (with the Neumann trace $\gamma_{N_\pm} f_\pm$ replaced by the regularized Neumann trace $\gamma_{N_\pm} f_{D\pm}$) remains valid on the maximal domains $\operatorname{dom} S_\pm^*$. This fact is essentially a consequence of [12, Theorem 6.4]. We also mention that analogous extensions of Green's identity are well known for elliptic operators on smooth domains; see, e.g., [17].

Proposition 2.2. *The Green's identity*

$$\begin{aligned}\big(S_\pm^* f_\pm, g_\pm\big)_{L^2(\Omega_\pm)} &- \big(f_\pm, S_\pm^* g_\pm\big)_{L^2(\Omega_\pm)} \\ &= \pm\, {}_{\mathscr{G}_N(\partial\Omega_\pm)^*}\langle \widetilde{\gamma}_D f_\pm, \gamma_{N_\pm} g_{D\pm}\rangle_{\mathscr{G}_N(\partial\Omega_\pm)} \mp {}_{\mathscr{G}_N(\partial\Omega_\pm)}\langle \gamma_{N_\pm} f_{D\pm}, \widetilde{\gamma}_D g_\pm\rangle_{\mathscr{G}_N(\partial\Omega_\pm)^*}\end{aligned}$$

holds for all $f_\pm = f_{D\pm} + f_{0\pm}$ *and* $g_\pm = g_{D\pm} + g_{0\pm}$ *in* $\operatorname{dom} S_\pm^*$.

Proof. We show the identity only on $L^2(\Omega_+)$; the same argument applies to the identity on Ω_-. Let $f_+ = f_{D+} + f_{0+},\ g_+ = g_{D+} + g_{0+} \in \operatorname{dom} S_+^*$, and recall from [12, Theorem 6.4] that the identity

$$\big(S_+^* f_+, g_{D+}\big) - \big(f_+, A_{D+} g_{D+}\big) = {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \gamma_{N_+} g_{D+}\rangle_{\mathscr{G}_N(\partial\Omega_+)}$$

holds, where $(\cdot,\cdot)$ is the inner product on $L^2(\Omega_+)$. It is clear that since A_{D+} is self-adjoint in $L^2(\Omega_+)$,

$$\big(A_{D+}f_{D+}, g_{D+}\big) - \big(f_{D+}, A_{D+}g_{D+}\big) = 0.$$

Moreover, as $f_{0+}, g_{0+} \in \ker S_+^*$,

$$\big(S_+^* f_{0+}, g_{0+}\big) - \big(f_{0+}, S_+^* g_{0+}\big) = 0.$$

Taking this into account, we compute

$$\begin{aligned}
&(S_+^* f_+, g_+) - (f_+, S_+^* g_+)\\
&\quad= \big(S_+^*(f_{D+}+f_{0+}), g_{D+}+g_{0+}\big) - \big(f_{D+}+f_{0+}, S_+^*(g_{D+}+g_{0+})\big)\\
&\quad= \big(A_{D+}f_{D+}, g_{0+}\big) + \big(S_+^* f_{0+}, g_{D+}\big) - \big(f_{0+}, A_{D+}g_{D+}\big) - \big(f_{D+}, S_+^* g_{0+}\big)\\
&\quad= \big(A_{D+}f_{D+}, g_{0+}\big) - \big(f_{D+}, S_+^* g_{0+}\big) + \big(S_+^* f_{0+}, g_{D+}\big) - \big(f_{0+}, A_{D+}g_{D+}\big)\\
&\quad= -{}_{\mathscr{G}_N(\partial\Omega_+)}\langle \gamma_{N_+} f_{D+}, \widetilde{\gamma}_D g_{0+}\rangle_{\mathscr{G}_N(\partial\Omega_+)^*} + {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_{0+}, \gamma_{N_+} g_{D+}\rangle_{\mathscr{G}_N(\partial\Omega_+)}\\
&\quad= {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \gamma_{N_+} g_{D+}\rangle_{\mathscr{G}_N(\partial\Omega_+)} - {}_{\mathscr{G}_N(\partial\Omega_+)}\langle \gamma_{N_+} f_{D+}, \widetilde{\gamma}_D g_+\rangle_{\mathscr{G}_N(\partial\Omega_+)^*},
\end{aligned}$$

where we have used $\ker \widetilde{\gamma}_D = \ker \gamma_D$ from (2.6) in the last identity. □

Next we consider the subspaces

$$\mathscr{G}_\pm := \big\{\varphi \in \mathscr{G}_N(\partial\Omega_\pm) : \varphi|_{\mathcal{C}} = 0\big\}$$

of $\mathscr{G}_N(\partial\Omega_\pm)$, which consist of functions vanishing on $\mathcal{C}$. Denote by $\mathscr{G}_\pm^\perp \subset (\mathscr{G}_N(\partial\Omega_\pm))^*$ the corresponding annihilators

$$\mathscr{G}_\pm^\perp = \big\{\psi \in (\mathscr{G}_N(\partial\Omega_\pm))^* : \psi(\varphi) = 0 \text{ for all } \varphi \in \mathscr{G}_\pm\big\}.$$

Roughly speaking, $\mathscr{G}_\pm^\perp$ can be viewed as the linear subspaces of functionals from $(\mathscr{G}_N(\partial\Omega_\pm))^*$ that vanish on $\mathcal{C}_\pm = \partial\Omega_\pm \backslash \mathcal{C}$. It is important to note that

$$\mathscr{G}_\pm^\perp \cong \big(\mathscr{G}_N(\partial\Omega_\pm)/\mathscr{G}_\pm\big)^*. \tag{2.7}$$

In particular, if $\psi(\varphi) = 0$ for some $\varphi \in \mathscr{G}_N(\partial\Omega_\pm)$ and all $\psi \in \mathscr{G}_\pm^\perp$, then $\varphi = 0$ when identified with elements in the quotient space $\mathscr{G}_N(\partial\Omega_\pm)/\mathscr{G}_\pm$, and hence $\varphi \in \mathscr{G}_\pm$, that is, $\varphi|_{\mathcal{C}} = 0$.

3 An auxiliary symmetric operator R

In the next proposition, we consider a restriction R of the self-adjoint operator $A_{D+} \oplus A_{D-}$ in $L^2(\Omega)$ and determine the adjoint of R. It turns out that the operator A in (1.2) is a self-adjoint extension of R (and hence a restriction of the adjoint operator R^*).

Proposition 3.1. *The operator*

$$
\begin{aligned}
Rf &= \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\
\operatorname{dom} R &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : f_\pm \in H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm),\ \gamma_{N_+} f_+|_{\mathcal{C}} = \gamma_{N_-} f_-|_{\mathcal{C}} \right\}
\end{aligned}
$$

is a closed symmetric operator with equal infinite deficiency indices in $L^2(\Omega)$, and $R \subset A_{D+} \oplus A_{D-}$. The adjoint operator is given by

$$
\begin{aligned}
R^* f &= \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\
\operatorname{dom} R^* &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : f_\pm,\ \Delta f_\pm \in L^2(\Omega_\pm),\ \widetilde{\gamma}_D f_\pm \in \mathscr{G}_\pm^\perp,\ \widetilde{\gamma}_D f_+|_{\mathcal{C}} = \widetilde{\gamma}_D f_-|_{\mathcal{C}} \right\},
\end{aligned}
$$

where the boundary condition $\widetilde{\gamma}_D f_+|_{\mathcal{C}} = \widetilde{\gamma}_D f_-|_{\mathcal{C}}$ is understood as

$$
{}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \varphi \rangle_{\mathscr{G}_N(\partial\Omega_+)} = {}_{\mathscr{G}_N(\partial\Omega_-)^*}\langle \widetilde{\gamma}_D f_-, \varphi \rangle_{\mathscr{G}_N(\partial\Omega_-)}
$$

for all $\varphi \in \mathscr{G}_N(\partial\Omega_\pm)$ such that $\varphi|_{\mathcal{C}_\pm} = 0$.

Proof. The proof consists of three steps. First, we define the operator

$$
\begin{aligned}
Tf &:= \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\
\operatorname{dom} T &:= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : f_\pm,\ \Delta f_\pm \in L^2(\Omega_\pm),\ \widetilde{\gamma}_D f_\pm \in \mathscr{G}_\pm^\perp,\ \widetilde{\gamma}_D f_+|_{\mathcal{C}} = \widetilde{\gamma}_D f_-|_{\mathcal{C}} \right\},
\end{aligned}
$$

and show in Steps 1 and 2 that $T^* = R$. In Step 3, we verify that T is closed, so that $R^* = T^{**} = \overline{T} = T$.

Step 1: $R \subset T^*$. Fix some $f = (f_+, f_-)^\top \in \operatorname{dom} R$, and note that $f_\pm = f_{D\pm}$ in the decomposition (2.4)–(2.5). As both T and R are restrictions of the orthogonal sum $S_+^* \oplus S_-^*$ of the maximal operators in (2.3), it follows from Proposition 2.2 that for any $g \in \operatorname{dom} T$ decomposed in the form $g_\pm = g_{D\pm} + g_{0\pm}$,

$$
\begin{aligned}
(Rf, g)_{L^2(\Omega)} - (f, Tg)_{L^2(\Omega)} &= \big((S_+^* \oplus S_-^*)f, g\big)_{L^2(\Omega)} - \big(f, (S_+^* \oplus S_-^*)g\big)_{L^2(\Omega)} \\
&= (S_+^* f_+, g_+)_{L^2(\Omega_+)} - (f_+, S_+^* g_+)_{L^2(\Omega_+)} + (S_-^* f_-, g_-)_{L^2(\Omega_-)} - (f_-, S_-^* g_-)_{L^2(\Omega_-)} \\
&= {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \gamma_{N_+} g_{D+} \rangle_{\mathscr{G}_N(\partial\Omega_+)} - {}_{\mathscr{G}_N(\partial\Omega_+)}\langle \gamma_{N_+} f_{D+}, \widetilde{\gamma}_D g_+ \rangle_{\mathscr{G}_N(\partial\Omega_+)^*} \\
&\qquad - {}_{\mathscr{G}_N(\partial\Omega_-)^*}\langle \widetilde{\gamma}_D f_-, \gamma_{N_-} g_{D-} \rangle_{\mathscr{G}_N(\partial\Omega_-)} + {}_{\mathscr{G}_N(\partial\Omega_-)}\langle \gamma_{N_-} f_{D-}, \widetilde{\gamma}_D g_- \rangle_{\mathscr{G}_N(\partial\Omega_-)^*} \\
&= -{}_{\mathscr{G}_N(\partial\Omega_+)}\langle \gamma_{N_+} f_+, \widetilde{\gamma}_D g_+ \rangle_{\mathscr{G}_N(\partial\Omega_+)^*} + {}_{\mathscr{G}_N(\partial\Omega_-)}\langle \gamma_{N_-} f_-, \widetilde{\gamma}_D g_- \rangle_{\mathscr{G}_N(\partial\Omega_-)^*},
\end{aligned}
$$

where in the last step we have used the fact that $f_\pm = f_{D\pm}$, and $f_\pm \in H_0^1(\Omega_\pm)$ for $f = f_+ \oplus f_- \in \operatorname{dom} R$, so that $\widetilde{\gamma}_D f_\pm = 0$; cf. (2.6). Next we decompose $\gamma_{N_\pm} f_\pm$ in the form

$$\gamma_{N_\pm} f_\pm = \big(\gamma_{N_\pm} f_\pm|_{\mathcal{C}}\big)^{\sim} + \big(\gamma_{N_\pm} f_\pm|_{\mathcal{C}_\pm}\big)^{\sim}, \tag{3.1}$$

where both extensions by 0 on the right hand side belong to the space $\mathscr{G}_N(\partial\Omega_\pm)$ (see Lemma 2.1) and, in particular,

$$\big(\gamma_{N_\pm} f_{D\pm}|_{\mathcal{C}_\pm}\big)^{\sim} \in \mathscr{G}_\pm.$$

Since $g \in \operatorname{dom} T$, we have $\widetilde{\gamma}_D g_\pm \in \mathscr{G}_\pm^\perp$; therefore,

$${}_{\mathscr{G}_N(\partial\Omega_\pm)}\big\langle \big(\gamma_{N_\pm} f_\pm|_{\mathcal{C}_\pm}\big)^{\sim}, \widetilde{\gamma}_D g_\pm \big\rangle_{\mathscr{G}_N(\partial\Omega_\pm)^*} = 0.$$

We conclude that

$$\begin{aligned}\big(Rf, g\big)_{L^2(\Omega)} - \big(f, Tg\big)_{L^2(\Omega)} = &-{}_{\mathscr{G}_N(\partial\Omega_+)}\big\langle \big(\gamma_{N_+} f_+|_{\mathcal{C}}\big)^{\sim}, \widetilde{\gamma}_D g_+ \big\rangle_{\mathscr{G}_N(\partial\Omega_+)^*} \\ &+ {}_{\mathscr{G}_N(\partial\Omega_-)}\big\langle \big(\gamma_{N_-} f_-|_{\mathcal{C}}\big)^{\sim}, \widetilde{\gamma}_D g_- \big\rangle_{\mathscr{G}_N(\partial\Omega_-)^*}.\end{aligned} \tag{3.2}$$

Since $f \in \operatorname{dom} R$ and $g \in \operatorname{dom} T$, we obtain

$$\gamma_{N_+} f_+|_{\mathcal{C}} = \gamma_{N_-} f_-|_{\mathcal{C}} \quad \text{and} \quad \widetilde{\gamma}_D g_+|_{\mathcal{C}} = \widetilde{\gamma}_D g_-|_{\mathcal{C}}.$$

This and (3.2) imply that $(Rf, g)_{L^2(\Omega)} - (f, Tg)_{L^2(\Omega)} = 0$ holds for all $g \in \operatorname{dom} T$. Therefore, $f \in \operatorname{dom} T^*$ and $T^* f = Rf$. We have shown $R \subset T^*$.

Step 2: $T^* \subset R$. Observe first that the orthogonal sum of the Dirichlet operator $A_{D+} \oplus A_{D-}$ is a self-adjoint restriction of T, and hence

$$T^* \subset A_{D+} \oplus A_{D-} \subset T. \tag{3.3}$$

Let $f = (f_+, f_-)^\top \in \operatorname{dom} T^*$. Then $f_\pm \in H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm)$, and $f_\pm = f_{D\pm}$ in the decomposition (2.4)–(2.5). It remains to show that the boundary condition

$$\gamma_{N_+} f_+|_{\mathcal{C}} = \gamma_{N_-} f_-|_{\mathcal{C}} \tag{3.4}$$

is satisfied. For this, note that by (2.6), we also have $\widetilde{\gamma}_D f_\pm = 0$. For $g \in \operatorname{dom} T$, we obtain in the same way as in Step 1 that

$$\begin{aligned}0 &= \big(T^* f, g\big)_{L^2(\Omega)} - \big(f, Tg\big)_{L^2(\Omega)} \\ &= -{}_{\mathscr{G}_N(\partial\Omega_+)}\langle \gamma_{N_+} f_+, \widetilde{\gamma}_D g_+ \rangle_{\mathscr{G}_N(\partial\Omega_+)^*} + {}_{\mathscr{G}_N(\partial\Omega_-)}\langle \gamma_{N_-} f_-, \widetilde{\gamma}_D g_- \rangle_{\mathscr{G}_N(\partial\Omega_-)^*}.\end{aligned}$$

Next we decompose $\gamma_{N_\pm} f_\pm$ as in (3.1) and use the fact that $\widetilde{\gamma}_D g_\pm \in \mathscr{G}_\pm^\perp$. As in Step 1, this leads to

$$0 = {}_{\mathscr{G}_N(\partial\Omega_+)}\langle (\gamma_{N_+} f_+|_\mathcal{C})^\sim, \widetilde{\gamma}_D g_+\rangle_{\mathscr{G}_N(\partial\Omega_+)^*} - {}_{\mathscr{G}_N(\partial\Omega_-)}\langle (\gamma_{N_-} f_-|_\mathcal{C})^\sim, \widetilde{\gamma}_D g_-\rangle_{\mathscr{G}_N(\partial\Omega_-)^*}$$

for all $g = (g_+, g_-)^\top \in \operatorname{dom} T$. Furthermore, since $\widetilde{\gamma}_D g_+|_\mathcal{C} = \widetilde{\gamma}_D g_-|_\mathcal{C}$, we obtain

$$0 = {}_{\mathscr{G}_N(\partial\Omega_+)}\langle (\gamma_{N_+} f_+|_\mathcal{C})^\sim - (\gamma_{N_-} f_-|_\mathcal{C})^\sim, \widetilde{\gamma}_D g_+\rangle_{\mathscr{G}_N(\partial\Omega_+)^*}$$

for all $g = (g_+, g_-)^\top \in \operatorname{dom} T$ and hence for all $\psi = \widetilde{\gamma}_D g_+ \in \mathscr{G}_+^\perp$. It now follows from (2.7) and the observation below (2.7) that the function

$$(\gamma_{N_+} f_+|_\mathcal{C})^\sim - (\gamma_{N_-} f_-|_\mathcal{C})^\sim$$

vanishes on $\mathcal{C}$. Thus the boundary condition (3.4) is satisfied. We have shown $f \in \operatorname{dom} T$ and hence $R^* \subset T$.

Step 3: T is closed. Let $(f_n) \subset \operatorname{dom} T$ such that $f_n \to f$ and $Tf_n \to h$ for some $f = (f_+, f_-)^\top, h = (h_+, h_-)^\top \in L^2(\Omega)$. Since $T \subset S_+^* \oplus S_-^*$ and $S_+^* \oplus S_-^*$ is closed, $f_\pm \in \operatorname{dom} S_\pm^*$ and $S_\pm^* f_\pm = h_\pm$. It remains to show that the boundary conditions

$$\widetilde{\gamma}_D f_\pm \in \mathscr{G}_\pm^\perp \quad \text{and} \quad \widetilde{\gamma}_D f_+|_\mathcal{C} = \widetilde{\gamma}_D f_-|_\mathcal{C}$$

are satisfied. But this follows immediately, since $f_{n\pm} \to f_\pm$ in the graph norm of $S_\pm^*$ and $\widetilde{\gamma}_D$ is continuous with respect to the graph norm, so that $\widetilde{\gamma}_D f_{n\pm} \to \widetilde{\gamma}_D f_\pm$ in $(\mathscr{G}_N(\partial\Omega_\pm))^*$. □

The following lemma states that the Neumann traces of the functions from $\ker R^*$ coincide on $\mathcal{C}$. This property is essentially a consequence of the symmetry of the domain Ω and the function $\operatorname{sgn}(\cdot)$ with respect to the interface $\mathcal{C}$. For completeness we mention that the functions

$$f_{0,k}(x,y) = \begin{cases} \sinh(k\pi(1-x))\sin(k\pi y), & (x,y) \in \Omega_+, \\ \sinh(k\pi(1+x))\sin(k\pi y), & (x,y) \in \Omega_-, \end{cases} \quad k \in \mathbb{N} = \{1, 2, \dots\}, \tag{3.5}$$

span a dense set in $\ker R^*$; cf. Proposition 5.1(iv).

Lemma 3.2. *Let R and R^* be as in Proposition 3.1. Then*

(i) *the space* $\ker R^*$ *is infinite dimensional, and the functions $f_0 \in \ker R^*$ satisfy*

$$\widetilde{\gamma}_{N_+} f_{0+}|_\mathcal{C} = \widetilde{\gamma}_{N_-} f_{0-}|_\mathcal{C}, \tag{3.6}$$

that is,

$${}_{\mathscr{G}_D(\partial\Omega_+)^*}\langle \widetilde{\gamma}_{N_+} f_{0+}, \varphi\rangle_{\mathscr{G}_D(\partial\Omega_+)} = {}_{\mathscr{G}_D(\partial\Omega_-)^*}\langle \widetilde{\gamma}_{N_-} f_{0-}, \varphi\rangle_{\mathscr{G}_D(\partial\Omega_-)}$$

holds for all $\varphi \in \mathscr{G}_D(\partial\Omega_\pm)$ such that $\varphi|_{\mathcal{C}_\pm} = 0$;

(ii) *R is invertible and has closed range.*

Proof. (i) As $A_{D+} \oplus A_{D-} \subset R^*$ and $0 \notin \sigma(A_{D_\pm})$, we have the direct sum decomposition $\operatorname{dom} R^* = \operatorname{dom}\left(A_{D+} \oplus A_{D-}\right) \dot{+} \ker R^*$. Together with (2.6), this yields that the mapping

$$\widetilde{\Gamma}_D : \ker R^* \to \mathscr{G}_N(\partial\Omega_+) \times \mathscr{G}_N(\partial\Omega_-), \quad f_0 = \begin{pmatrix} f_{0+} \\ f_{0-} \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{\gamma}_D f_{0+} \\ \widetilde{\gamma}_D f_{0-} \end{pmatrix} \tag{3.7}$$

is invertible. Suppose now that $f_0 = (f_{0+}, f_{0-})^\top \in \ker R^*$ and assume, in addition, that $f_{0\pm} \in H^2(\Omega_\pm)$. Then $\Delta f_{0\pm} = 0$, and the boundary conditions have the explicit form

$$\gamma_D f_{0\pm}|_{\mathcal{C}_\pm} = 0 \qquad \text{and} \qquad \gamma_D f_{0+}|_{\mathcal{C}} = \gamma_D f_{0-}|_{\mathcal{C}}; \tag{3.8}$$

here, γ_D is the Dirichlet trace operator defined on $H^2(\Omega_\pm)$. It follows that the function $h(x,y) := f_{0+}(-x,y)$, $x \in (-1,0)$, $y \in (0,1)$, belongs to $L^2(\Omega_-)$ and satisfies $\Delta h = 0$, $\gamma_D h|_{\mathcal{C}} = \gamma_D f_{0+}|_{\mathcal{C}}$ and $\gamma_D h|_{\mathcal{C}_-} = 0$. Hence $(f_{0+}, h)^\top \in \ker R^*$; but as the map $\widetilde{\Gamma}_D$ in (3.7) is invertible, we conclude $f_{0-} = h$. In particular, we obtain

$$\gamma_{N_-} f_{0-}|_{\mathcal{C}} = \gamma_{N_-} h|_{\mathcal{C}} = \gamma_{N_+} f_{0+}|_{\mathcal{C}},$$

where $\gamma_{N_\pm}$ denotes the Neumann trace operator on $H^2(\Omega_\pm)$. As $\widetilde{\gamma}_{N_\pm}$ are extensions of $\gamma_{N_\pm}$, this yields ${}_{\mathscr{G}_D(\partial\Omega_+)^*}\langle \widetilde{\gamma}_{N_+} f_{0+}, \varphi\rangle_{\mathscr{G}_D(\partial\Omega_+)} = {}_{\mathscr{G}_D(\partial\Omega_-)^*}\langle \widetilde{\gamma}_{N_-} f_{0-}, \varphi\rangle_{\mathscr{G}_D(\partial\Omega_-)}$ for all $\varphi \in \mathscr{G}_D(\partial\Omega_\pm)$ such that $\varphi|_{\mathcal{C}_\pm} = 0$. We have shown that any function $f_0 \in \ker R^*$ possessing the additional property $f_{0\pm} \in H^2(\Omega_\pm)$ satisfies the condition (3.6). The general case follows from the facts that $R^* \subset S_+^* \oplus S_-^*$ and $\ker S_\pm^* \cap H^2(\Omega_\pm)$ is dense in $\ker S_\pm^*$ and the continuity of the extended Neumann trace maps $\widetilde{\gamma}_{N_\pm}$.

(ii) Since $R \subset A_{D+} \oplus A_{D-} \subset R^*$ and $0 \notin \sigma(A_{D_\pm})$, it follows that $\ker R = \{0\}$. To see that $\operatorname{ran} R$ is closed, assume that $Rf_n \to g$, $n \to \infty$, for some $g \in L^2(\Omega)$. It is clear that also $(A_{D+} \oplus A_{D-})f_n \to g$, $n \to \infty$; and from $0 \notin \sigma(A_{D_\pm})$, we conclude that

$$f_n \to f := \left(A_{D+}^{-1} \oplus A_{D-}^{-1}\right) g, \qquad n \to \infty.$$

Since R is closed, $f \in \operatorname{dom} R$ and $Rf = g$. □

4 The self-adjoint operator A and its qualitative spectral properties

In this section, we present the main result of this note. The operator A (informally written in (1.2)) is now defined rigorously with explicit boundary conditions as a restriction of the maximal operator $S_+^* \oplus S_-^*$. It is shown that A is self-adjoint in $L^2(\Omega)$, and it turns out that A can be viewed as a generalized Krein-von Neumann extension of the non-semibounded symmetric operator R; see also Proposition 4.2 below.

Theorem 4.1. *The operator*

$$\begin{aligned} Af &= \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\ \operatorname{dom} A &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{array}{c} f_\pm,\ \Delta f_\pm \in L^2(\Omega_\pm),\ \widetilde{\gamma}_D f_\pm \in \mathscr{G}_\pm^\perp, \\ \widetilde{\gamma}_D f_+|_{\mathcal{C}} = \widetilde{\gamma}_D f_-|_{\mathcal{C}},\ \widetilde{\gamma}_{N_+} f_+|_{\mathcal{C}} = \widetilde{\gamma}_{N_-} f_-|_{\mathcal{C}} \end{array} \right\}, \end{aligned} \tag{4.1}$$

is self-adjoint in $L^2(\Omega)$ and coincides with the operator $R^* \upharpoonright \operatorname{dom} R \dotplus \ker R^*$. The boundary conditions $\widetilde{\gamma}_D f_+|_{\mathcal{C}} = \widetilde{\gamma}_D f_-|_{\mathcal{C}}$ and $\widetilde{\gamma}_{N_+} f_+|_{\mathcal{C}} = \widetilde{\gamma}_{N_-} f_-|_{\mathcal{C}}$ are understood as

$$_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \varphi \rangle_{\mathscr{G}_N(\partial\Omega_+)} = {}_{\mathscr{G}_N(\partial\Omega_-)^*}\langle \widetilde{\gamma}_D f_-, \varphi \rangle_{\mathscr{G}_N(\partial\Omega_-)}$$

for all $\varphi \in \mathscr{G}_N(\partial\Omega_\pm)$ such that $\varphi|_{\mathcal{C}_\pm} = 0$, and

$$_{\mathscr{G}_D(\partial\Omega_+)^*}\langle \widetilde{\gamma}_N f_+, \psi \rangle_{\mathscr{G}_D(\partial\Omega_+)} = {}_{\mathscr{G}_D(\partial\Omega_-)^*}\langle \widetilde{\gamma}_N f_-, \psi \rangle_{\mathscr{G}_D(\partial\Omega_-)}$$

for all $\psi \in \mathscr{G}_D(\partial\Omega_\pm)$ such that $\psi|_{\mathcal{C}_\pm} = 0$, respectively.

Proof. We first show that $A \subset A^*$. Since $A \subset R^* \subset S_+^* \oplus S_-^*$, for $f, g \in \operatorname{dom} A$ decomposed in the usual form $f_\pm = f_{D\pm} + f_{0\pm}$, $g_\pm = g_{D\pm} + g_{0\pm}$ (see (2.4)–(2.5)),

$$\begin{aligned} \big(Af, g\big)_{L^2(\Omega)} - \big(f, Ag\big)_{L^2(\Omega)} &= \big((S_+^* \oplus S_-^*)f, g\big)_{L^2(\Omega)} - \big(f, (S_+^* \oplus S_-^*)g\big)_{L^2(\Omega)} \\ &= {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \gamma_{N_+} g_{D+} \rangle_{\mathscr{G}_N(\partial\Omega_+)} - {}_{\mathscr{G}_N(\partial\Omega_+)}\langle \gamma_{N_+} f_{D+}, \widetilde{\gamma}_D g_+ \rangle_{\mathscr{G}_N(\partial\Omega_+)^*} \\ &\quad - {}_{\mathscr{G}_N(\partial\Omega_-)^*}\langle \widetilde{\gamma}_D f_-, \gamma_{N_-} g_{D-} \rangle_{\mathscr{G}_N(\partial\Omega_-)} + {}_{\mathscr{G}_N(\partial\Omega_-)}\langle \gamma_{N_-} f_{D-}, \widetilde{\gamma}_D g_- \rangle_{\mathscr{G}_N(\partial\Omega_-)^*}; \end{aligned}$$

cf. Proposition 2.2 and Step 1 of the proof of Proposition 3.1. Taking into account that $\widetilde{\gamma}_D f_\pm, \widetilde{\gamma}_D g_\pm \in \mathscr{G}_\pm^\perp$ and decomposing $\gamma_{N_\pm} f_{D\pm}$ and $\gamma_{N_\pm} g_{D\pm}$ as

$$\begin{aligned} \gamma_{N_\pm} f_{D\pm} &= \big(\gamma_{N_\pm} f_{D\pm}|_{\mathcal{C}}\big)^\sim + \big(\gamma_{N_\pm} f_{D\pm}|_{\mathcal{C}_\pm}\big)^\sim, \\ \gamma_{N_\pm} g_{D\pm} &= \big(\gamma_{N_\pm} g_{D\pm}|_{\mathcal{C}}\big)^\sim + \big(\gamma_{N_\pm} g_{D\pm}|_{\mathcal{C}_\pm}\big)^\sim, \end{aligned}$$

where the extensions by 0 on the right-hand side belong to the spaces $\mathscr{G}_N(\partial\Omega_\pm)$ by Lemma 2.1, we find that

$$\begin{aligned} \big(Af, g\big)_{L^2(\Omega)} - \big(f, Ag\big)_{L^2(\Omega)} &= {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, \big(\gamma_{N_+} g_{D+}|_{\mathcal{C}}\big)^\sim \rangle_{\mathscr{G}_N(\partial\Omega_+)} \\ &\quad - {}_{\mathscr{G}_N(\partial\Omega_+)}\langle \big(\gamma_{N_+} f_{D+}|_{\mathcal{C}}\big)^\sim, \widetilde{\gamma}_D g_+ \rangle_{\mathscr{G}_N(\partial\Omega_+)^*} - {}_{\mathscr{G}_N(\partial\Omega_-)^*}\langle \widetilde{\gamma}_D f_-, \big(\gamma_{N_-} g_{D-}|_{\mathcal{C}}\big)^\sim \rangle_{\mathscr{G}_N(\partial\Omega_-)} \\ &\quad + {}_{\mathscr{G}_N(\partial\Omega_-)}\langle \big(\gamma_{N_-} f_{D-}|_{\mathcal{C}}\big)^\sim, \widetilde{\gamma}_D g_- \rangle_{\mathscr{G}_N(\partial\Omega_-)^*}. \end{aligned}$$

As $f, g \in \operatorname{dom} A$, we also have $\widetilde{\gamma}_D f_+|_{\mathcal{C}} = \widetilde{\gamma}_D f_-|_{\mathcal{C}}$ and $\widetilde{\gamma}_D g_+|_{\mathcal{C}} = \widetilde{\gamma}_D g_-|_{\mathcal{C}}$, and hence the terms on the right-hand side simplify to

$$\begin{aligned}(4.2)\qquad & {}_{\mathscr{G}_N(\partial\Omega_+)^*}\langle \widetilde{\gamma}_D f_+, (\gamma_{N_+} g_{D+}|_{\mathcal{C}})^{\sim} - (\gamma_{N_-} g_{D-}|_{\mathcal{C}})^{\sim}\rangle_{\mathscr{G}_N(\partial\Omega_+)} \\ & - {}_{\mathscr{G}_N(\partial\Omega_+)}\langle (\gamma_{N_+} f_{D+}|_{\mathcal{C}})^{\sim} - (\gamma_{N_-} f_{D-}|_{\mathcal{C}})^{\sim}, \widetilde{\gamma}_D g_+\rangle_{\mathscr{G}_N(\partial\Omega_+)^*}.\end{aligned}$$

According to Lemma 3.2(ii), the functions $f_{0\pm}, g_{0\pm} \in \ker R^*$ satisfy $\widetilde{\gamma}_{N_+} f_{0+}|_{\mathcal{C}} = \widetilde{\gamma}_{N_-} f_{0-}|_{\mathcal{C}}$ and $\widetilde{\gamma}_{N_+} g_{0+}|_{\mathcal{C}} = \widetilde{\gamma}_{N_-} g_{0-}|_{\mathcal{C}}$. Thus

$$\begin{aligned} 0 &= \widetilde{\gamma}_{N_+} f_+|_{\mathcal{C}} - \widetilde{\gamma}_{N_-} f_-|_{\mathcal{C}} = \widetilde{\gamma}_{N_+}(f_{D+} + f_{0+})|_{\mathcal{C}} - \widetilde{\gamma}_{N_-}(f_{D-} + f_{0-})|_{\mathcal{C}} \\ &= \gamma_{N_+} f_{D+}|_{\mathcal{C}} - \gamma_{N_-} f_{D-}|_{\mathcal{C}} \end{aligned}$$

and

$$\begin{aligned} 0 &= \widetilde{\gamma}_{N_+} g_+|_{\mathcal{C}} - \widetilde{\gamma}_{N_-} g_-|_{\mathcal{C}} = \widetilde{\gamma}_{N_+}(g_{D+} + g_{0+})|_{\mathcal{C}} - \widetilde{\gamma}_{N_-}(g_{D-} + g_{0-})|_{\mathcal{C}} \\ &= \gamma_{N_+} g_{D+}|_{\mathcal{C}} - \gamma_{N_-} g_{D-}|_{\mathcal{C}}, \end{aligned}$$

and hence the corresponding entries in (4.2) vanish, that is,

$$(Af, g)_{L^2(\Omega)} - (f, Ag)_{L^2(\Omega)} = 0, \quad f, g \in \operatorname{dom} A.$$

We have shown that $A \subset A^*$.

Next we verify that the operator $R_0 := R^* \restriction \operatorname{dom} R \dotplus \ker R^*$ is contained in A. The inclusion $\operatorname{dom} R \subset \operatorname{dom} A$ is obvious; hence it remains to show that $\ker R^* \subset \operatorname{dom} A$. It is clear from the definition of $\operatorname{dom} R^*$ that any function $f_0 = (f_{0+}, f_{0-}) \in \ker R^*$ satisfies the boundary conditions for functions in $\operatorname{dom} A$, with the exception of the condition $\widetilde{\gamma}_{N_+} f_{0+}|_{\mathcal{C}} = \widetilde{\gamma}_{N_-} f_{0-}|_{\mathcal{C}}$. But this last condition holds by Lemma 3.2(i). Therefore, $R_0 \subset A$. We claim that R_0 is self-adjoint. In fact, R_0 is symmetric since for $f = f_R + f_0 \in \operatorname{dom} R \dotplus \ker R^*$,

$$(R_0 f, f)_{L^2(\Omega)} = (R_0(f_R + f_0), f_R + f_0)_{L^2(\Omega)} = (R f_R, f_R)_{L^2(\Omega)}.$$

Moreover, by Lemma 3.2(ii), 0 is a point of regular type of R, that is, $\ker R = \{0\}$ and $\operatorname{ran} R$ is closed. This leads to the direct sum decomposition

$$\operatorname{ran}(R_0 - \mu) = \operatorname{ran}(R - \mu) \dotplus \ker R^* = L^2(\Omega), \quad \mu \in \mathbb{C} \setminus \mathbb{R},$$

from which we then conclude that R_0 is a self-adjoint operator in $L^2(\Omega)$. In summary, we have shown that A is a symmetric operator which contains the self-adjoint operator R_0, so $A = R_0$ is self-adjoint. □

Finally we state a result on the spectral properties of the operator A. Our proof is a variant of that of [1, Lemma 2.3]; see also [22].

Proposition 4.2. *Let A be the self-adjoint operator from Theorem 4.1. Then* 0 *is an isolated eigenvalue of infinite multiplicity, and the corresponding eigenspace is given by* $\ker R^*$. *The spectrum in* $\mathbb{R}\backslash\{0\}$ *is discrete (i.e., composed of isolated eigenvalues of finite multiplicities) and accumulates at* $+\infty$ *and* $-\infty$.

Proof. It is clear that the eigenspace $\ker A = \ker R^*$ is an infinite dimensional closed subspace of $L^2(\Omega)$. Moreover,

$$\mathcal{H} := \operatorname{ran} A = (\ker A)^{\perp} = (\ker R^*)^{\perp} = \operatorname{ran} R \tag{4.3}$$

is closed, according to Lemma 3.2(ii). In the following, we denote the orthogonal projection onto the subspace $\mathcal{H}$ by P and the embedding of $\mathcal{H}$ into $L^2(\Omega)$ by ι. We denote the restriction of A to $\mathcal{H}$ by A'. Note that A' is a bijective self-adjoint operator on the Hilbert space $\mathcal{H}$, so that $0 \notin \sigma(A')$. With respect to the decomposition $L^2(\Omega) = \mathcal{H} \oplus \mathcal{H}^{\perp}$, we have $A = A' \oplus 0$, and hence

$$Af = \iota A' P f, \qquad f \in \operatorname{dom} A. \tag{4.4}$$

We also use below the facts that the orthogonal sum $A_D = A_{D+} \oplus A_{D-}$ of the Dirichlet operators $A_{D\pm}$ is a self-adjoint operator on $L^2(\Omega)$ and that $0 \notin \sigma(A_D)$.

Now let $f = f_R + f_0 \in \operatorname{dom} A$, where $f_R \in \operatorname{dom} R$ and $f_0 \in \ker A$. As $R \subset A_D$ and $R \subset A$, we have

$$f = f_R + f_0 = A_D^{-1} R f_R + f_0 = A_D^{-1} A f_R + f_0 = A_D^{-1} A f + f_0,$$

and hence

$$Pf = P\big(A_D^{-1} A f + f_0\big) = P A_D^{-1} A f = P A_D^{-1} \iota A' P f,$$

where we have used (4.4) in the last equality. This leads to

$$A'^{-1}(A' P f) = P f = P A_D^{-1} \iota (A' P f);$$

and as $0 \notin \sigma(A')$, we conclude that $A'^{-1} = P A_D^{-1} \iota$. Since A_D^{-1} is a compact operator on $L^2(\Omega)$, A'^{-1} is a compact operator on $\mathcal{H}$. Moreover, for $g \in \mathcal{H}$,

$$(A'^{-1} g, g)_{\mathcal{H}} = (P A_D^{-1} \iota g, g)_{\mathcal{H}} = (A_D^{-1} \iota g, \iota g)_{L^2(\Omega)}. \tag{4.5}$$

Since $S_+ \oplus S_- \subset R$, we conclude that for all $f_\pm \in \operatorname{dom} S_\pm = H_0^2(\Omega_\pm)$,

$$(S_+ f_+, 0)^\top \in \operatorname{ran} R = \mathcal{H} \quad \text{and} \quad (0, S_- f_-)^\top \in \operatorname{ran} R = \mathcal{H}.$$

It follows that both the spaces $\mathcal{H} \cap L^2(\Omega_\pm)$ are infinite dimensional. It is clear that the form on the right-hand side of (4.5) is positive (negative) for functions in

$\mathcal{H} \cap L^2(\Omega_+)$ (respectively, $\mathcal{H} \cap L^2(\Omega_-)$). This implies that both the positive and negative spectra of A'^{-1} are infinite. It now follows from the compactness that the spectrum of A' (and hence of A) in $\mathbb{R}\backslash\{0\}$ is discrete and accumulates at $+\infty$ and $-\infty$. □

5 Quantitive spectral properties of the self-adjoint operator *A*

According to Proposition 4.2, the spectrum of the self-adjoint operator A consists of eigenvalues which accumulate at $+\infty$ and $-\infty$. The eigenvalue 0 is of infinite multiplicity and the multiplicities of the nonzero eigenvalues are finite. In the next proposition, we identify the eigenvalues of A with the roots of an elementary algebraic equation and specify the eigenfunctions of A.

Proposition 5.1. *Let A be the self-adjoint operator from Theorem 4.1. Then*

(i) *the spectrum of A is symmetric with respect to* 0*;*

(ii) $\sigma(A) = \bigcup_{n=1}^{\infty} \bigcup_{m=-\infty}^{\infty} \{\lambda_{n,m}\}$, *where* $\{\lambda_{n,m}\}_{m\in\mathbb{Z}}$ *for each fixed* $n \in \mathbb{N}$ *is an increasing sequence of simple roots of the algebraic equation*

$$\frac{\tanh\sqrt{\lambda + (n\pi)^2}}{\sqrt{\lambda + (n\pi)^2}} = \frac{\tan\sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}} \tag{5.1}$$

for $\lambda \neq \pm(n\pi)^2$*; we arrange the sequence in such a way that* $\lambda_{n,0} = 0$ (0 *is a solution of* (5.1) *for any* $n \in \mathbb{N}$).

(iii) *for each* $n \in \mathbb{N}$, (5.1) *has no root in* $(-(n\pi)^2, 0) \cup (0, (n\pi)^2)$*; in particular,* $\big([-\pi^2, 0) \cup (0, \pi^2]\big) \cap \sigma(A) = \varnothing$*;*

(iv) *the eigenfunction of A corresponding to* $\lambda_{n,m}$ *is given by*

$$f_{n,m}(x, y) = \psi_{n,m}(x)\chi_n(y),$$

where $\chi_n(y) = \sqrt{2}\sin(n\pi y)$ *and*

$$\psi_{n,m}(x) = \begin{cases} N_{n,m} \sinh\sqrt{\lambda_{n,m} + (n\pi)^2}\, \sin\left(\sqrt{\lambda_{n,m} - (n\pi)^2}\,(1 - x)\right), & x > 0, \\ N_{n,m} \sin\sqrt{\lambda_{n,m} - (n\pi)^2}\, \sinh\left(\sqrt{\lambda_{n,m} + (n\pi)^2}\,(1 + x)\right), & x < 0, \end{cases} \tag{5.2}$$

with any $N_{n,m} \in \mathbb{C} \setminus \{0\}$*; with the normalization constants* $N_{n,m}$ *satisfying*

$$|N_{n,m}|^{-2} = \sinh^2\sqrt{\lambda_{n,m} + (n\pi)^2}\left[\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_{n,m} - (n\pi)^2}\right)}{4\sqrt{\lambda_{n,m} - (n\pi)^2}}\right] + \sin^2\sqrt{\lambda_{n,m} - (n\pi)^2}\left[-\frac{1}{2} + \frac{\sinh\left(2\sqrt{\lambda_{n,m} + (n\pi)^2}\right)}{4\sqrt{\lambda_{n,m} + (n\pi)^2}}\right],$$

the functions $f_{n,m}$ $(n \in \mathbb{N},\ m \in \mathbb{Z})$ *form a complete orthonormal set in* $L^2(\Omega)$.

Proof. The eigenvalues λ and the corresponding eigenfunctions f of A can be obtained as nontrivial solutions of the differential equations $\mp\Delta f_\pm = \lambda f_\pm$ in $\Omega_\pm$, subject to the boundary and interface conditions determined in (4.1). From this boundary transmission problem, it is immediately seen that if λ is an eigenvalue of A (with eigenfunction $f(x, y)$), then $-\lambda$ is also an eigenvalue of A (with eigenfunction $f(-x, y)$). This establishes (i).

The other properties (ii)–(iv) are obtained by a separation of variables. Decomposing any eigenfunction $f \in L^2(\Omega)$ of A into the transverse orthonormal Dirichlet basis $\{\chi_n\}_{n=1}^{\infty}$, i.e., writing

$$f(x, y) = \sum_{n=1}^{\infty} \psi_n(x)\chi_n(y), \qquad \chi_n(y) = \sqrt{2}\sin(n\pi y),$$

we easily obtain from the boundary transmission problem in Ω that for each $n \in \mathbb{N}$, the function $\psi_n = (\psi_{n+}, \psi_{n-})^\top \in L^2((0, 1)) \times L^2((-1, 0))$ is a nontrivial solution of the problem

$$\begin{aligned} -\psi''_{n+} &= (\lambda - (n\pi)^2)\psi_{n+} \quad \text{in} \quad (0, 1), \\ \psi''_{n-} &= (\lambda + (n\pi)^2)\psi_{n-} \quad \text{in} \quad (-1, 0), \end{aligned} \tag{5.3}$$

subject to the boundary and interface conditions

$$\psi_{n+}(1) = \psi_{n-}(-1) = 0, \qquad \psi_{n+}(0) = \psi_{n-}(0), \qquad \text{and} \qquad \psi'_{n+}(0) = -\psi'_{n-}(0). \tag{5.4}$$

Solving the differential equations in (5.3) in terms of exponentials and subjecting the latter to the boundary and interface conditions (5.4), we find that any nontrivial solution ψ_n is of the form (5.2) with the constraint that the eigenvalue λ solves (5.1). There are infinitely many such solutions because (5.1) always contains an oscillatory tangent function for large values of λ. For each fixed $n \in \mathbb{N}$, we arrange the roots of (5.1) in an increasing sequence $\{\lambda_{n,m}\}_{m\in\mathbb{Z}}$ such that $\lambda_{n,0} = 0$. Notice that $\lambda = \pm(n\pi)^2$ are not admissible solutions of (5.3) for any $n \in \mathbb{N}$. This is consistent with (5.1), because the limit $\lambda \to \pm(n\pi)^2$ casts (5.1) into $\tanh\sqrt{2(n\pi^2)} = \sqrt{2(n\pi^2)}$, which is never satisfied for nonzero n. We have thus proved (ii), except for the simplicity of the roots of (5.1), which will be established at the end of this proof.

As for (iv), it remains only to recall that eigenfunctions of a self-adjoint operator with pure point spectrum form a complete orthonormal set when normalized properly ($N_{n,m}$ is chosen so that all $\psi_{n,m}$ have norm 1 in $L^2((-1, 1))$ and χ_n are already normalized to 1 in $L^2((0, 1))$).

Now we turn to (iii). Recall that we already know that no eigenvalue can equal $\pm(n\pi)^2$ with $n \in \mathbb{N}$. To show that (5.1) has no root in $(0, (n\pi)^2)$, it suffices to show that the function

$$G(\lambda) = \frac{\sqrt{\lambda + (n\pi)^2}}{\tanh\sqrt{\lambda + (n\pi)^2}} - \frac{\sqrt{(n\pi)^2 - \lambda}}{\tanh\sqrt{(n\pi)^2 - \lambda}}$$

does not vanish in $(0, (n\pi)^2)$. This follows since $G(0) = 0$ and

$$G'(\lambda) = \frac{1}{4}\left[\frac{\sinh\left(2\sqrt{\lambda + (n\pi)^2}\right) - 2\sqrt{\lambda + (n\pi)^2}}{\sqrt{\lambda + (n\pi)^2}\,\sinh^2\sqrt{\lambda + (n\pi)^2}} + \frac{\sinh\left(2\sqrt{(n\pi)^2 - \lambda}\right) - 2\sqrt{(n\pi)^2 - \lambda}}{\sqrt{(n\pi)^2 - \lambda}\,\sinh^2\sqrt{(n\pi)^2 - \lambda}}\right] > 0,$$

for $\lambda \in (0, (n\pi)^2)$, where the crucial inequality is due to the elementary bound $\sinh(x) > x$, valid for all $x > 0$. Since (5.1) is symmetric with respect to the transformation $\lambda \mapsto -\lambda$, the claim on the absence of roots extends to the symmetric set $(-(n\pi)^2, 0) \cup (0, (n\pi)^2)$.

It remains to prove the simplicity of roots stated in (ii). By symmetry of (5.1), it again suffices to show it for non-negative roots $\lambda_{n,m}$ only. Define

$$F(\lambda) = \frac{\tanh\sqrt{\lambda + (n\pi)^2}}{\sqrt{\lambda + (n\pi)^2}} - \frac{\tan\sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}}. \tag{5.5}$$

Then $\lambda_{n,m}$ is a root of (5.1) if and only if $F(\lambda_{n,m}) = 0$. Using this identity, it is straightforward to cast the derivative of F at $\lambda_{n,m}$ into the form

$$F'(\lambda_{n,m}) = -\frac{\tanh^2\sqrt{\lambda_{n,m} + (n\pi)^2}}{\lambda_{n,m} + (n\pi)^2} + \frac{(n\pi)^2}{\lambda_{n,m}^2 - (n\pi)^4}\left[\frac{\tanh\sqrt{\lambda_{n,m} + (n\pi)^2}}{\sqrt{\lambda_{n,m} + (n\pi)^2}} - 1\right].$$

We know by (iii) that if $\lambda_{n,m} > 0$, then $\lambda_{n,m} > (n\pi)^2$. Using the elementary bound $\tanh(x) < x$ for all $x > 0$, we thus obtain

$$F'(\lambda_{n,m}) < -\frac{\tanh^2\sqrt{\lambda_{n,m} + (n\pi)^2}}{\lambda_{n,m} + (n\pi)^2} < 0.$$

On the other hand, employing standard algebraic expressions for hyperbolic functions, it is easy to check that the formula for $F'(\lambda_{n,m})$ above reduces for $\lambda_{n,0} = 0$ to

$$F'(0) = \frac{2n\pi - \sinh(2n\pi)}{2(n\pi)^3\cosh^2(n\pi)} < 0,$$

where the inequality follows by the elementary bound used above in the proof of (iii). In summary, $F'(\lambda) \neq 0$ whenever $F(\lambda) = 0$, which proves the simplicity of the roots of (5.1) and completes the proof of the proposition. □

The simplicity of roots of (5.1) stated in point (ii) of the above proposition does not mean that all the eigenvalues of A are simple. Indeed, we already know from Proposition 4.2 that 0 is an eigenvalue of infinite multiplicity.

In order to establish the convergence results announced in the Introduction in a unified way, we consider now the more general differential expression

$$\mathcal{T}_\delta f = -\mathrm{div}\,(a_\delta \nabla f), \quad a_\delta(x,y) = \begin{cases} 1, & (x,y) \in \Omega_+, \\ -\frac{1}{1+\delta}, & (x,y) \in \Omega_-, \end{cases} \tag{5.6}$$

where δ is an arbitrary complex number with $|\delta| < 1$. We also introduce the associated operator

$$T_\delta f = \mathcal{T}_\delta f, \quad \mathrm{dom}\, T_\delta = \left\{ f \in H_0^1(\Omega) : \mathcal{T}_\delta f \in L^2(\Omega) \right\}. \tag{5.7}$$

Clearly, by choosing δ appropriately, we can cast the eigenvalue problems for the self-adjoint operator A_ε from (1.4) and the (up to a rotation) m-sectorial operator B_η from (1.6) into the form of the eigenvalue problem for T_δ. The latter reads

$$\begin{aligned} -\Delta f_+ &= \lambda f_+ && \text{in} \quad \Omega_+\,, \\ \Delta f_- &= (1+\delta)\lambda f_- && \text{in} \quad \Omega_-\,, \end{aligned} \tag{5.8}$$

where, in addition, $f = (f_+, f_-)^\top \in \mathrm{dom}\, T_\delta \subset H_0^1(\Omega)$ satisfies the interface condition

$$(1+\delta)\partial_{\mathbf{n}_+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}}\,. \tag{5.9}$$

Proposition 5.2. *Let T_δ be the operator introduced in* (5.7)*. There exists an absolute constant $c > 0$ such that for $|\delta| \le c$,*

(i) $\sigma_{\mathrm{p}}(T_\delta) = \bigcup_{n=1}^{\infty} \bigcup_{m=-\infty}^{\infty} \{\lambda_{n,m}^\delta\}$, *where $\{\lambda_{n,m}^\delta\}_{m\in\mathbb{Z}}$ for each fixed $n \in \mathbb{N}$ is a sequence of roots of the algebraic equation*

$$(1+\delta)\,\frac{\tanh\sqrt{(1+\delta)\lambda + (n\pi)^2}}{\sqrt{(1+\delta)\lambda + (n\pi)^2}} = \frac{\tan\sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}} \tag{5.10}$$

for $\lambda \ne (n\pi)^2$ and $\lambda \ne -(n\pi)^2/(1+\delta)$;

(ii) *the eigenfunction of T_δ corresponding to $\lambda_{n,m}^\delta$ is given by*

$$f_{n,m}^\delta(x,y) = \psi_{n,m}^\delta(x)\chi_n(y),$$

where $\chi_n(y) = \sqrt{2}\sin(n\pi y)$ and

(5.11)
$$\psi^{\delta}_{n,m}(x) = \begin{cases} N^{\delta}_{n,m} \sinh\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}\, \sin\left(\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}\,(1-x)\right), & x > 0, \\ N^{\delta}_{n,m} \sin\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}\, \sinh\left(\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}\,(1+x)\right), & x < 0, \end{cases}$$

with any $N^{\delta}_{n,m} \in \mathbb{C} \setminus \{0\}$*; with the normalization constants* $N^{\delta}_{n,m}$ *satisfying*

$$\begin{aligned} |N^{\delta}_{n,m}|^{-2} = {} & \left|\sinh\left(\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}\right)\right|^2 \\ & \times \left[\frac{\sinh\left(2\,\mathrm{Im}\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}\right)}{4\,\mathrm{Im}\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}} - \frac{\sin\left(2\,\mathrm{Re}\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}\right)}{4\,\mathrm{Re}\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}}\right] \\ & + \left|\sin\left(\sqrt{\lambda^{\delta}_{n,m} - (n\pi)^2}\right)\right|^2 \\ & \times \left[-\frac{\sin\left(2\,\mathrm{Im}\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}\right)}{4\,\mathrm{Im}\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}} \right. \\ & \left. + \frac{\sinh\left(2\,\mathrm{Re}\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}\right)}{4\,\mathrm{Re}\sqrt{(1+\delta)\lambda^{\delta}_{n,m} + (n\pi)^2}}\right], \end{aligned}$$

the functions $f^{\delta}_{n,m}$ ($n \in \mathbb{N}$*,* $m \in \mathbb{Z}$) *are normalized to* 1 *in* $L^2(\Omega)$*.*

Proof. The results follow by the separation of variables, as in the proof of Proposition 5.1. Contrary to the symmetric situation $\delta = 0$, however, (5.10) can have solutions $\lambda = (n\pi)^2$ and $\lambda = -(n\pi)^2/(1+\delta)$. Compatibility conditions for the existence of such solutions are

$$\frac{\tanh\sqrt{(2+\delta)(n\pi)^2}}{\sqrt{(2+\delta)(n\pi)^2}} = \frac{1}{1+\delta}, \qquad \frac{\tanh\sqrt{\frac{2+\delta}{1+\delta}\,(n\pi)^2}}{\sqrt{\frac{2+\delta}{1+\delta}\,(n\pi)^2}} = 1+\delta, \tag{5.12}$$

respectively, (they can be obtained from (5.10) after the limit $\lambda \to (n\pi)^2$ and $\lambda \to -(n\pi)^2/(1+\delta)$, respectively). We claim that these "exceptional" solutions do not exist for all δ small in the absolute value, uniformly in $n \in \mathbb{N}$. This can be proved straightforwardly by comparing the real parts of the left- and right-hand sides of (5.12). More specifically, we have

$$\left|\mathrm{Re}\left(\frac{\tanh z}{z}\right)\right| = \frac{1}{|z|^2}\left|\frac{z_1\sinh(2z_1) + z_2\sin(2z_2)}{\cosh(2z_1) + \cos(2z_2)}\right| \le \frac{1}{|z_1|}\frac{\sinh(2|z_1|) + 1}{\cosh(2|z_1|) - 1}$$

for all $z = z_1 + iz_2 \in \mathbb{C}$ with $z_1 = \mathrm{Re}\, z \neq 0$, where the right hand side is decreasing as a function of $|z_1|$. Employing the elementary inequality $|\mathrm{Re}\sqrt{\xi}| \ge |\sqrt{\mathrm{Re}\,\xi}|$

(valid for every $\xi \in \mathbb{C}$ with $\operatorname{Re}\xi \geq 0$) and $|\delta| < 1$, we estimate

$$\left|\operatorname{Re}\sqrt{(2+\delta)(n\pi)^2}\right| \geq \left|\sqrt{(2+\operatorname{Re}\delta)(n\pi)^2}\right| \geq \pi,$$

$$\left|\operatorname{Re}\sqrt{\frac{2+\delta}{1+\delta}(n\pi)^2}\right| \geq \left|\sqrt{\left(1+\frac{1+\operatorname{Re}\delta}{1+2\operatorname{Re}\delta+|\delta|^2}\right)(n\pi)^2}\right| \geq \pi .$$

Consequently, a necessary condition for equality to hold in (5.12) is

$$0.32 \approx \frac{1}{\pi}\frac{\sinh(2\pi)+1}{\cosh(2\pi)-1} \geq \min\left\{\operatorname{Re}\left(\frac{1}{1+\delta}\right), \operatorname{Re}(1+\delta)\right\} \geq \frac{1-|\delta|}{(1+|\delta|)^2},$$

which is clearly impossible if $|\delta|$ is small enough (the present estimates yield $c \geq 0.38$). □

We are now in a position to establish the convergence of the eigenvalues and eigenfunctions of T_δ to the eigenvalues and eigenfunctions of A as $\delta \to 0$. In the next theorem, we show, in particular, that the operators A_ε and B_η in the Introduction represent an "approximation" of the self-adjoint operator A, at least on the spectral level. However, the resolvents of A_ε and B_η are compact for all $\kappa \neq 1$ and $\eta > 0$, while the resolvent of A is not compact (0 is an eigenvalue of infinite multiplicity).

Theorem 5.3. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, let $\lambda_{n,m}$ and $\psi_{n,m}$ be the eigenvalues and eigenfunctions, respectively, of A specified in Proposition 5.1, and let $\lambda_{n,m}^\delta$ and $\psi_{n,m}^\delta$ be the eigenvalues and eigenfunctions, respectively, of T_δ specified in Proposition 5.2. For any $n \in \mathbb{N}$, the sequence $\{\lambda_{n,m}^\delta\}_{m\in\mathbb{Z}}$ can be arranged so that*

$$\lim_{\delta\to 0}\left|\lambda_{n,m}^\delta - \lambda_{n,m}\right| = 0 \quad \text{and} \quad \lim_{\delta\to 0}\left\|\psi_{n,m}^\delta - \psi_{n,m}\right\|_{L^\infty(\Omega)} = 0.$$

Proof. The convergence of the eigenvalues follows by the Implicit Function Theorem applied to

$$H(\lambda,\delta) = (1+\delta)\frac{\tanh\sqrt{(1+\delta)\lambda+(n\pi)^2}}{\sqrt{(1+\delta)\lambda+(n\pi)^2}} - \frac{\tan\sqrt{\lambda-(n\pi)^2}}{\sqrt{\lambda-(n\pi)^2}}.$$

Clearly, $H(\lambda,0) = F(\lambda)$, where F is introduced in (5.5) based on (5.1). Hence $H(\lambda_{n,m},0) = 0$. We need only to check that the derivative $\partial_1 H(\lambda_{n,m},0)$ does not vanish. However, $\partial_1 H(\lambda_{n,m},0) = F'(\lambda_{n,m}) \neq 0$, due to the proof of simplicity of the roots of (5.1) established in the proof of Proposition 5.1. The convergence of the eigenfunctions is then clear from the expressions (5.2) and (5.11). □

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