Selfadjoint Schrödinger operators on the half-space with compactly supported Robin boundary conditions

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Schrödinger differential operators on a half-space with compactly supported Robin boundary conditions are studied.

1 Introduction

We investigate realizations of the differential expression $-\Delta + V$ on the half-space $\mathbb{R}^n_+ = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, $n \geq 2$, with a real-valued, bounded potential V. More precisely, we study the differential operator

$$A_g u = -\Delta u + Vu, \quad \operatorname{dom} A_g = \left\{ u \in H^{3/2}_\Delta(\mathbb{R}^n_+) : \partial_\nu u|_{\mathbb{R}^{n-1}} = g \cdot (u|_{\mathbb{R}^{n-1}}) \right\}, \tag{1}$$

in $L^2(\mathbb{R}^n_+)$, where $H^{3/2}_\Delta(\mathbb{R}^n_+) = \{u \in H^{3/2}(\mathbb{R}^n_+) : \Delta u \in L^2(\mathbb{R}^n_+)\}$ and $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is a bounded, real function with compact support. The aim of the present note is to show that A_g is a selfadjoint, compact perturbation in the resolvent sense of the selfadjoint realization A_N of $-\Delta + V$ with Neumann boundary conditions. In particular this guarantees that A_g and A_N have the same essential spectrum. We point out that the latter can still be proved under slightly weaker assumptions on g, see [10] for a more general approach and [8] for a result with a more regular g in dimension $n = 2$. Our proofs make use of techniques which were originally developed in [2, 3] for the treatment of elliptic differential operators on domains with a compact boundary. For further recent developments in this area we refer the reader to [1, 4, 6, 11, 12].

2 Preliminaries

In this section we fix some notation and recall some known facts on Sobolev spaces and Schrödinger operators; proofs and further details can be found in [9] and, e.g., [7, Chapter 9]. Let $K \subset \mathbb{R}^{n-1}$ be a compact set and let $H^s(\mathbb{R}^n_+)$ and $H^s(K) = \{f|_K : f \in H^s(\mathbb{R}^{n-1})\}$ be the Sobolev spaces of order $s > 0$ on $\mathbb{R}^n_+$ and K, respectively. For $u \in H^{3/2}_\Delta(\mathbb{R}^n_+)$ we denote by $u|_{\mathbb{R}^{n-1}}$ the trace of u on the boundary $\mathbb{R}^{n-1}$ of $\mathbb{R}^n_+$, by $\partial_\nu u|_{\mathbb{R}^{n-1}} = -\frac{\partial u}{\partial x_n}|_{\mathbb{R}^{n-1}}$ the derivative of u along the outer normal vector field on $\mathbb{R}^{n-1}$, and by $u|_K$ and $\partial_\nu u|_K$, $\partial_\nu u|_{\mathbb{R}^{n-1}\setminus K}$ their restrictions to K and $\mathbb{R}^{n-1} \setminus K$, respectively. The mappings Γ_0 and Γ_1 given by

$$\Gamma_0 : H^{3/2}_\Delta(\mathbb{R}^n_+) \to L^2(K), \quad \Gamma_0 u = \partial_\nu u|_K \quad \text{and} \quad \Gamma_1 : H^{3/2}_\Delta(\mathbb{R}^n_+) \to H^1(K), \quad \Gamma_1 u = u|_K \tag{2}$$

are surjective.

Here and in the following let $V \in L^\infty(\mathbb{R}^n_+)$ be real-valued. It is well known that the *Neumann operator*

$$A_N u = -\Delta u + Vu, \qquad \operatorname{dom} A_N = \left\{ u \in H^{3/2}_\Delta(\mathbb{R}^n_+) : \partial_\nu u|_{\mathbb{R}^{n-1}} = 0 \right\}$$

is a selfadjoint realization of $-\Delta + V$ in $L^2(\mathbb{R}^n_+)$, and by elliptic regularity $\operatorname{dom} A_N \subset H^2(\mathbb{R}^n_+)$ holds. Note that this yields the decomposition $\{u \in H^{3/2}_\Delta(\mathbb{R}^n_+) : \partial_\nu u|_{\mathbb{R}^{n-1}\setminus K} = 0\} = \operatorname{dom} A_N \dotplus \mathcal{N}_\lambda$ for each λ in the resolvent set $\rho(A_N)$ of A_N, where $\mathcal{N}_\lambda := \{u \in H^{3/2}(\mathbb{R}^n_+) : -\Delta u + Vu = \lambda u, \partial_\nu u|_{\mathbb{R}^{n-1}\setminus K} = 0\}$. This, together with (2), ensures that the *Poisson operator*

$$\gamma(\lambda) : L^2(K) \to L^2(\mathbb{R}^n_+), \quad \partial_\nu u_\lambda|_K \mapsto u_\lambda, \quad u_\lambda \in \mathcal{N}_\lambda, \tag{3}$$

and the *Neumann-to-Dirichlet operator*

$$M(\lambda) : L^2(K) \to L^2(K), \quad \partial_\nu u_\lambda|_K \mapsto u_\lambda|_K, \quad u_\lambda \in \mathcal{N}_\lambda, \tag{4}$$

are well-defined for each $\lambda \in \rho(A_N)$. Moreover, $\gamma(\lambda)$ and $M(\lambda)$ are bounded and $\operatorname{ran} M(\lambda) = H^1(K)$ holds.

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3 Selfadjoint Schrödinger operators on the half-space

The following theorem is the main result of this note. For $g \in L^\infty(\mathbb{R}^{n-1})$, $\operatorname{supp} g = K$, we denote by G the operator of multiplication with the function $g|_K$ in $L^2(K)$.

Theorem 3.1 *Let $K \subset \mathbb{R}^{n-1}$ be a compact set and let $g \in L^\infty(\mathbb{R}^{n-1})$ be a real-valued function with supp $g = K$. Then the operator A_g in* (1) *is selfadjoint in $L^2(\mathbb{R}^n_+)$ and $\lambda \in \rho(A_N)$ is an eigenvalue of A_g if and only if* 1 *is an eigenvalue of $GM(\lambda)$. The resolvent difference*

$$(A_g - \lambda)^{-1} - (A_N - \lambda)^{-1} = \gamma(\lambda)\big(I - GM(\lambda)\big)^{-1} G\gamma(\overline{\lambda})^*, \quad \lambda \in \rho(A_g) \cap \rho(A_N), \tag{5}$$

is compact and, in particular, the essential spectra of A_g and A_N coincide.

P r o o f. Let us first show that $\lambda \in \rho(A_N)$ is an eigenvalue of A_g if and only if 1 is an eigenvalue of $GM(\lambda)$. For $u \in \ker(A_g - \lambda)$, $u \neq 0$, we have $\Gamma_0 u \neq 0$ and $GM(\lambda)\Gamma_0 u = G\Gamma_1 u = \Gamma_0 u$. Thus $I - GM(\lambda)$ is not injective. Conversely, $f \in \ker(I - GM(\lambda))$, $f \neq 0$, implies $\gamma(\lambda) f \in \operatorname{dom} A_g$, $(A_g - \lambda)\gamma(\lambda) f = 0$, and $\gamma(\lambda) f \neq 0$. Thus $\gamma(\lambda) f$ is an eigenfunction of A_g corresponding to the eigenvalue λ.

Next we show that A_g is a selfadjoint operator in $L^2(\mathbb{R}^n_+)$. For this note first that for $u \in \operatorname{dom} A_g$ we have

$$(A_g u, u) = \int_{\mathbb{R}^n_+} (-\Delta + V) u\, \overline{u}\, dx = \int_{\mathbb{R}^n_+} |\nabla u|^2 + V|u|^2\, dx - \int_K g\, |u|^2\, d\sigma \ \in \mathbb{R},$$

so that A_g is a symmetric in operator in $L^2(\mathbb{R}^n_+)$. Hence it is sufficient to verify that $A_g - \lambda$ is surjective for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Fix some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, choose an arbitrary $u \in L^2(\mathbb{R}^n_+)$, and define

$$v := (A_N - \lambda)^{-1} u + \gamma(\lambda)\big(I - GM(\lambda)\big)^{-1} G\gamma(\overline{\lambda})^* u. \tag{6}$$

In the following we will show that v is well-defined and belongs to $\operatorname{dom} A_g$ with $(A_g - \lambda)v = u$. The operator $\gamma(\lambda)$ and hence also $\gamma(\overline{\lambda})^*$ and $G\gamma(\overline{\lambda})^*$ are bounded and everywhere defined. Furthermore, since $\operatorname{ran} M(\lambda) = H^1(K)$ and the embedding from $H^1(K)$ into $L^2(K)$ is compact, $M(\lambda)$ and $GM(\lambda)$ are also compact operators in $L^2(K)$. Together with the fact that 1 is not an eigenvalue of $GM(\lambda)$ we conclude that the operator $I - GM(\lambda)$ has an everywhere defined, bounded inverse, i.e., v in (6) is well-defined. From the definition of v it is easy to see that $v \in H^{3/2}_\Delta(\mathbb{R}^n_+)$ and $\partial_\nu v|_{\mathbb{R}^{n-1}\setminus K} = 0$ holds.

It remains to show $G\Gamma_1 v = \Gamma_0 v$ and $(A_g - \lambda)v = u$. In fact, as a consequence of the second Green identity we find $\Gamma_1(A_N - \lambda)^{-1} u = \gamma(\overline{\lambda})^* u$ and therefore we conclude from (6)

$$G\Gamma_1 v = G\gamma(\overline{\lambda})^* u + GM(\lambda)\big(I - GM(\lambda)\big)^{-1} G\gamma(\overline{\lambda})^* u = \big(I - GM(\lambda)\big)^{-1} G\gamma(\overline{\lambda})^* u = \Gamma_0 v.$$

Thus we have shown $v \in \operatorname{dom} A_g$ and from $(A_g - \lambda)v = (-\Delta + V - \lambda)v = u$ we obtain that $A_g - \lambda$ is surjective and, hence, A_g is selfadjoint. Moreover, we have shown the formula (5) for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the same reasoning applies for real $\lambda \in \rho(A_N) \cap \rho(A_g)$. As mentioned above, $\gamma(\overline{\lambda})^* = \Gamma_1(A_N - \lambda)^{-1}$, in particular, $\operatorname{ran} \gamma(\overline{\lambda})^* \subset H^{3/2}(K)$, which is compactly embedded in $L^2(K)$. This shows that the right hand side in (5) is compact for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence $(A_g - \lambda)^{-1} - (A_N - \lambda)^{-1}$ is compact for each $\lambda \in \rho(A_g) \cap \rho(A_N)$, and, in particular, A_g and A_N have the same essential spectrum. ☐

We obtain the following corollary in the case $V = 0$.

Corollary 3.2 *Let $V = 0$. Then the essential spectrum of the operator A_g in* (1) *is given by* $[0, +\infty)$. *Moreover, $\lambda < 0$ is an eigenvalue of A_g if and only if* 1 *is an eigenvalue of $G\iota^*(-\Delta_{\mathbb{R}^{n-1}} - \lambda)^{-1/2}\iota$, where ι denotes the embedding from $L^2(K)$ into $L^2(\mathbb{R}^{n-1})$ and $\Delta_{\mathbb{R}^{n-1}}$ is the Laplacian on $\mathbb{R}^{n-1}$.*

P r o o f. In the case $V = 0$ it is well-known that the spectrum and essential spectrum of A_N is given by $[0, +\infty)$. Moreover, one computes similarly as in [7, Chapter 9] that $M(\lambda) = \iota^*(-\Delta_{\mathbb{R}^{n-1}} - \lambda)^{-1/2}\iota$ holds. ☐

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