Schrödinger operators with δ-potentials supported on unbounded Lipschitz hypersurfaces

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Dedicated to the memory
of our friend and colleague Sergey Naboko

Abstract In this note we consider the self-adjoint Schrödinger operator A_α in $L^2(\mathbb{R}^d)$, $d \geq 2$, with a δ-potential supported on a Lipschitz hypersurface $\Sigma \subseteq \mathbb{R}^d$ of strength $\alpha \in L^p(\Sigma) + L^\infty(\Sigma)$. We show the uniqueness of the ground state and, under some additional conditions on the coefficient α and the hypersurface Σ, we determine the essential spectrum of A_α. In the special case that Σ is a hyperplane we obtain a Birman-Schwinger principle with a relativistic Schrödinger operator as Birman-Schwinger operator. As an application we prove an optimization result for the bottom of the spectrum of A_α.

Key words: Schrödinger operator, singular potential, essential spectrum, ground state, Birman-Schwinger operator, eigenvalue optimization

1 Introduction

In this paper we are interested in spectral properties of a class of self-adjoint operators A_α with singular δ-potentials in the Hilbert space $L^2(\mathbb{R}^d)$, $d \geq 2$, which correspond to the formal differential expression

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$$-\Delta - \alpha\,\delta(x-\Sigma), \tag{1.1}$$

where $\Sigma \subset \mathbb{R}^d$ is the graph of a Lipschitz function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ and the function $\alpha\colon \Sigma \to \mathbb{R}$ is the strength of the δ-potential; cf. [8, 13], the monograph [22] and the references therein. Note that the unbounded Lipschitz surface Σ splits $\mathbb{R}^d$ into two unbounded disjoint parts and that the special choice $\xi = 0$ corresponds to the situation where Σ is the hyperplane in $\mathbb{R}^d$. Assuming $\alpha \in L^p(\Sigma) + L^\infty(\Sigma)$ for some $1 < p < \infty$ in $d = 2$ and for $d-1 \le p < \infty$ in $d \ge 3$ dimensions we define A_α as the semibounded self-adjoint operator in $L^2(\mathbb{R}^d)$ associated with the densely defined, symmetric, semibounded, and closed form

$$\begin{aligned} \mathfrak{a}_\alpha[u,v] &:= (\nabla u, \nabla v)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \int_\Sigma \alpha\, \tau_{\mathrm{D}} u\, \overline{\tau_{\mathrm{D}} v}\, \mathrm{d}x, \\ \operatorname{dom} \mathfrak{a}_\alpha &:= H^1(\mathbb{R}^d), \end{aligned} \tag{1.2}$$

where $\tau_{\mathrm{D}} : H^1(\mathbb{R}^d) \to H^{1/2}(\Sigma)$ is the Dirichlet trace operator. Let us denote the bottom of the spectrum of A_α by

$$\lambda_1(\alpha) := \inf \sigma(\mathsf{A}_\alpha). \tag{1.3}$$

The first issue we discuss in this paper is the essential spectrum of the self-adjoint operator A_α. In the present situation one always has the inclusion $[0,\infty) \subset \sigma_{\mathrm{ess}}(\mathsf{A}_\alpha)$ and in Theorem 2.3 we prove that if Σ is a local deformation of the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ and α is close to a constant $\alpha_0 \in \mathbb{R}$ outside of sets of finite measure (that is, the set $\{\, x \in \Sigma \mid |\alpha(x) - \alpha_0| > \varepsilon \,\}$ is of finite measure for every $\varepsilon > 0$), then

$$\sigma_{\mathrm{ess}}(\mathsf{A}_\alpha) = \begin{cases} [-\frac{\alpha_0^2}{4}, \infty), & \text{if } \alpha_0 \ge 0, \\ [0,\infty), & \text{if } \alpha_0 \le 0; \end{cases}$$

see also [39] for related results. Next we investigate the uniqueness of the ground state of A_α, which is a typical property for Schrödinger operators $-\Delta + V$ with *regular* potentials. More precisely, if $\lambda_1(\alpha)$ in (1.3) is a discrete eigenvalue then it will be shown in Section 2.3 that $\lambda_1(\alpha)$ is simple and the corresponding eigenfunction can be chosen strictly positive on $\mathbb{R}^d \setminus \Sigma$; this observation is based on a standard argument using Harnack's inequality.

In Section 3 we focus on the special case that Σ is the hyperplane and we obtain a Birman-Schwinger principle, where the Birman-Schwinger operator is a relativistic Schrödinger operator in $L^2(\mathbb{R}^{d-1})$. The operators appearing in this context can also be viewed as (extensions of) the γ-field and Weyl function corresponding to a certain quasi boundary triple; cf. [9, Section 8] for more details. Under the additional assumption that α is close to a constant α_0 outside of sets of finite measure, we then provide a more detailed analysis of the spectrum of the Birman-Schwinger operator and link these spectral properties to those of A_α. As an interesting application we prove an optimization result for the bottom of the spectrum of A_α which is formulated in terms of the so-called *symmetric decreasing rearrangement*: Consider

again a real-valued $\alpha \in L^\infty(\mathbb{R}^{d-1}) + L^p(\mathbb{R}^{d-1})$ for some $1 < p < \infty$ in $d = 2$ and for $d - 1 \leq p < \infty$ in $d \geq 3$ dimensions, which is close to a constant α_0 outside of sets of finite measure. Furthermore, let in the following $\alpha_1 := \alpha - \alpha_0$ and $(\alpha_1)_+ = \max\{\alpha_1, 0\}$, and let $(\alpha_1)_+^*$ be the symmetric decreasing rearrangement of $(\alpha_1)_+$ defined in (3.23). Then we have the inequality

$$\lambda_1(\alpha_0 + (\alpha_1)_+^*) \leq \lambda_1(\alpha_0 + \alpha_1). \tag{1.4}$$

Our proof of (1.4) relies on the fact that the symmetric decreasing rearrangement decreases the kinetic energy term corresponding to the relativistic Schrödinger operator. This property of the kinetic energy can be viewed as an analogue of the Pólya-Szegő inequality. We note that a different argument for (1.4) based on Steiner symmetrization was communicated to us; cf. Remark 3.11 for more details. We wish to mention that eigenvalue optimization is a trademark topic in spectral theory; see the monographs [30, 31] and the references therein. In particular, optimization of eigenvalues induced by δ-potentials supported on hypersurfaces is a topic of growing interest [19, 20, 23, 36]. There are also closely related works on eigenvalue optimization for δ-potentials supported on sets of higher co-dimension [7, 21], for the Robin Laplacian [3, 12, 14, 16, 25, 26, 28, 33, 34], for δ'-interactions [37] and for Dirac operators with surface interactions [2, 4].

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2 The Schrödinger operator with δ-potential supported on a Lipschitz graph

In this section let $d \geq 2$ and

$$\Sigma := \left\{ (x, \xi(x)) \,\middle|\, x \in \mathbb{R}^{d-1} \right\} \subset \mathbb{R}^d \tag{2.1}$$

be the graph of a Lipschitz function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$. Furthermore, let

$$\alpha \in L^p(\Sigma) + L^\infty(\Sigma) \tag{2.2}$$

be a real-valued function with $1 < p < \infty$ in $d = 2$ and $d - 1 \leq p < \infty$ in $d \geq 3$ dimensions. In this setting we will define the self-adjoint operator A_α associated to the form (1.2) and study its essential spectrum. In particular, if the support Σ is a local deformation of a hyperplane and if the coefficient α is close to a constant in

the sense of (2.6), we explicitly compute $\sigma_{\mathrm{ess}}(\mathsf{A}_\alpha)$. Furthermore, we verify that the ground state $\lambda_1(\alpha)$ (if it is a discrete eigenvalue) is simple.

2.1 The form $\mathfrak{a}_\alpha$ and the operator A_α

In this subsection we will prove that the form (1.2), which models a δ-potential of strength α supported on Σ, is well defined and gives rise to a self-adjoint operator A_α in $L^2(\mathbb{R}^d)$; cf. [13, 24] and [10, Proposition 3.8]. In the following the Dirichlet trace operator τ_{D} in (1.2) is viewed for $\frac{1}{2} < s < \frac{3}{2}$ as a bounded operator

$$\tau_{\mathrm{D}} \colon H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\Sigma); \tag{2.3}$$

cf. [38, Proof of Theorem 3.38].

Proposition 2.1 *The form $\mathfrak{a}_\alpha$ in* (1.2) *is densely defined, symmetric, semibounded, and closed in $L^2(\mathbb{R}^d)$.*

Proof It is clear, that $\operatorname{dom}\mathfrak{a}_\alpha = H^1(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Furthermore, we split $\mathfrak{a}_\alpha$ into

$$\begin{aligned} \mathfrak{a}_0[u,v] &:= (\nabla u, \nabla v)_{L^2(\mathbb{R}^d;\mathbb{C}^d)}, && \text{with} && \operatorname{dom}\mathfrak{a}_0 := H^1(\mathbb{R}^d), \\ \mathfrak{a}_1[u,v] &:= -\int_\Sigma \alpha\, \tau_{\mathrm{D}} u\, \overline{\tau_{\mathrm{D}} v}\, \mathrm{d}x, && \text{with} && \operatorname{dom}\mathfrak{a}_1 := H^1(\mathbb{R}^d). \end{aligned}$$

Observe that $\mathfrak{a}_0$ is densely defined, nonnegative, and closed in $L^2(\mathbb{R}^d)$. Furthermore, since α is real-valued it is clear that $\mathfrak{a}_1$ is symmetric. The estimate (A.3) shows that for every $\varepsilon > 0$ there exists some $c_\varepsilon \geq 0$, such that

$$\big|\mathfrak{a}_1[u,u]\big| \leq \varepsilon^2 \|\tau_{\mathrm{D}} u\|^2_{H^{\frac{1}{2}}(\Sigma)} + c_\varepsilon^2 \|\tau_{\mathrm{D}} u\|^2_{L^2(\Sigma)}, \qquad u \in H^1(\mathbb{R}^d).$$

Using the boundedness (2.3) of the trace operator, the absolute value of $\mathfrak{a}_1[u,u]$ can further be estimated by

$$\big|\mathfrak{a}_1[u,u]\big| \leq \varepsilon^2 d_1^2 \|u\|^2_{H^1(\mathbb{R}^d)} + c_\varepsilon^2 d_s^2 \|u\|^2_{H^s(\mathbb{R}^d)}, \qquad u \in H^1(\mathbb{R}^d),$$

where d_1 and d_s are the operator norms of (2.3) with $s = 1$ and some fixed $s \in (\frac{1}{2}, 1)$, respectively. Since $s < 1$, we can use [29, Theorem 3.30] to find a constant $\tilde{c}_\varepsilon \geq 0$ with

$$\big|\mathfrak{a}_1[u,u]\big| \leq \varepsilon^2 (d_1^2 + 1) \|u\|^2_{H^1(\mathbb{R}^d)} + \tilde{c}_\varepsilon^2 \|u\|^2_{L^2(\mathbb{R}^d)}, \qquad u \in H^1(\mathbb{R}^d).$$

That is, the form $\mathfrak{a}_1$ is $\mathfrak{a}_0$-bounded with form bound 0. The semiboundedness and closedness of $\mathfrak{a}_\alpha = \mathfrak{a}_0 + \mathfrak{a}_1$ now follow from [32, Chapter VI, Theorem 1.33]. □

Proposition 2.1 combined with the First Representation Theorem [32, Chapter VI, Theorem 2.1] implies that there is a unique self-adjoint operator A_α in $L^2(\mathbb{R}^d)$ representing the form $\mathfrak{a}_\alpha$ in the sense that $\operatorname{dom}\mathsf{A}_\alpha \subset \operatorname{dom}\mathfrak{a}_\alpha$ and

$$(\mathsf{A}_\alpha f, g)_{L^2(\mathbb{R}^d)} = \mathfrak{a}_\alpha[f,g], \quad f \in \operatorname{dom}\mathsf{A}_\alpha,\ g \in \operatorname{dom}\mathfrak{a}_\alpha. \tag{2.4}$$

2.2 Essential spectrum of A_α

In this subsection we investigate the essential spectrum of A_α. The following preparatory lemma shows that in the present situation the essential spectrum of A_α always covers the nonnegative real axis.

Lemma 2.2 *For any α of the form* (2.2) *we have*

$$[0,\infty) \subseteq \sigma_{\mathrm{ess}}(\mathsf{A}_\alpha). \tag{2.5}$$

Proof In a similar way as in the proof of [17, Chapter 10, Theorem 6.5] one constructs for $\lambda \in (0,\infty)$ an orthonormal sequence $(\Psi_n)_n \in C_0^\infty(\mathbb{R}^d)$ with support in $\mathbb{R}^d \setminus \Sigma$ and

$$\|(-\Delta - \lambda)\Psi_n\|_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} 0.$$

From $\operatorname{supp}\Psi_n \subseteq \mathbb{R}^d \setminus \Sigma$ we have $\tau_{\mathrm{D}}\Psi_n = 0$ and hence it follows from (1.2) that $\mathsf{A}_\alpha\Psi_n = -\Delta\Psi_n$. This implies

$$\|(\mathsf{A}_\alpha - \lambda)\Psi_n\|_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} 0,$$

so that $(\Psi_n)_n$ is a singular sequence and we conclude $\lambda \in \sigma_{\mathrm{ess}}(\mathsf{A}_\alpha)$. This proves that $(0,\infty) \subseteq \sigma_{\mathrm{ess}}(\mathsf{A}_\alpha)$ and since the essential spectrum is closed we obtain (2.5). □

For a subclass of hypersurfaces Σ, which are local deformations of a hyperplane, and interaction strengths that are close to a constant in the sense of (2.6), we are able to determine the essential spectrum explicitly.

Theorem 2.3 *If the function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ in* (2.1) *is compactly supported and if for some $\alpha_0 \in \mathbb{R}$*

$$\{\, x \in \Sigma \mid |\alpha(x) - \alpha_0| > \varepsilon \,\} \text{ has finite measure for every } \varepsilon > 0, \tag{2.6}$$

then the essential spectrum of the corresponding Schrödinger operator A_α is given by

$$\sigma_{\mathrm{ess}}(\mathsf{A}_\alpha) = \begin{cases} [-\frac{\alpha_0^2}{4}, \infty), & \text{if } \alpha_0 \geq 0, \\ [0,\infty), & \text{if } \alpha_0 \leq 0. \end{cases} \tag{2.7}$$

Proof *Step 1.* First, we consider the hyperplane $\Sigma = \mathbb{R}^{d-1} \times \{0\} \cong \mathbb{R}^{d-1}$ and the constant potential $\alpha(x) = \alpha_0$. We introduce two auxiliary closed forms

$$\mathfrak{d}[\phi,\psi] := (\nabla\phi,\nabla\psi)_{L^2(\mathbb{R}^{d-1};\mathbb{C}^{d-1})}, \qquad \text{with} \quad \operatorname{dom}\mathfrak{d} := H^1(\mathbb{R}^{d-1}),$$
$$\mathfrak{t}_{\alpha_0}[f,g] := (f',g')_{L^2(\mathbb{R})} - \alpha_0 f(0)\overline{g(0)}, \qquad \text{with} \quad \operatorname{dom}\mathfrak{t}_{\alpha_0} := H^1(\mathbb{R}),$$

with the corresponding self-adjoint operators $-\Delta$ and T_{α_0} in the Hilbert spaces $L^2(\mathbb{R}^{d-1})$ and $L^2(\mathbb{R})$, respectively. The spectra of these operators are explicitly given by

$$\sigma(-\Delta) = [0,\infty) \quad \text{and} \quad \sigma(\mathsf{T}_{\alpha_0}) = \begin{cases} \{-\frac{\alpha_0^2}{4}\} \cup [0,\infty), & \text{if } \alpha_0 \geq 0, \\ [0,\infty), & \text{if } \alpha_0 \leq 0, \end{cases}$$

where the proof of the latter one can be found in [1, Theorem 3.1.4]. The Schrödinger operator $\widetilde{\mathsf{A}}_{\alpha_0}$ with δ-potential supported on a hyperplane of constant strength α_0 can be decomposed as

$$\widetilde{\mathsf{A}}_{\alpha_0} = (-\Delta) \otimes \mathsf{I}_{\mathbb{R}} + \mathsf{I}_{\mathbb{R}^{d-1}} \otimes \mathsf{T}_{\alpha_0}$$

with respect to $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^{d-1}) \otimes L^2(\mathbb{R})$; here $\mathsf{I}_{\mathbb{R}}$ and $\mathsf{I}_{\mathbb{R}^{d-1}}$ denote the identity operators in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^{d-1})$, respectively. Hence, it follows from [42, Eq. (4.44)] that

$$\sigma(\widetilde{\mathsf{A}}_{\alpha_0}) = \begin{cases} [-\frac{\alpha_0^2}{4},\infty), & \text{if } \alpha_0 \geq 0, \\ [0,\infty), & \text{if } \alpha_0 \leq 0. \end{cases} \tag{2.8}$$

Step 2. Let A_{α_0} be the Schrödinger operator with δ-potential of constant strength α_0 supported on the hypersurface Σ. Since the Lipschitz mapping ξ is compactly supported, the surface Σ is a local deformation of the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ in the sense that $\Sigma \setminus \mathcal{B} = (\mathbb{R}^{d-1} \times \{0\}) \setminus \mathcal{B}$ for a ball $\mathcal{B} \subset \mathbb{R}^d$ of sufficiently large radius. Hence it follows from (2.8) using [6, Theorem 4.7] that

$$\sigma_{\mathrm{ess}}(\mathsf{A}_{\alpha_0}) = \sigma_{\mathrm{ess}}(\widetilde{\mathsf{A}}_{\alpha_0}) = \begin{cases} [-\frac{\alpha_0^2}{4},\infty), & \text{if } \alpha_0 \geq 0, \\ [0,\infty), & \text{if } \alpha_0 \leq 0. \end{cases} \tag{2.9}$$

Step 3. With α_0 from (2.6) we define $\alpha_1 := \alpha - \alpha_0$, such that $\{\, x \in \Sigma \mid |\alpha_1(x)| > \varepsilon \,\}$ has finite measure for every $\varepsilon > 0$. The self-adjoint operators A_{α_0} and A_α are both semibounded since they correspond to semibounded forms. Hence, we can fix $\lambda < \inf(\sigma(\mathsf{A}_{\alpha_0}) \cup \sigma(\mathsf{A}_\alpha))$ and consider the resolvent difference

$$\mathsf{W} := (\mathsf{A}_{\alpha_0} - \lambda)^{-1} - (\mathsf{A}_\alpha - \lambda)^{-1}. \tag{2.10}$$

Our aim is to show that W is a compact operator in $L^2(\mathbb{R}^d)$. For this let $f, g \in L^2(\mathbb{R}^d)$ and set

$$u := (\mathsf{A}_{\alpha_0} - \lambda)^{-1} f \quad \text{and} \quad v := (\mathsf{A}_\alpha - \lambda)^{-1} g. \tag{2.11}$$

Using (2.11) and the definition of the operator W in (2.10) we obtain

$$
\begin{aligned}
(\mathsf{W}f, g)_{L^2(\mathbb{R}^d)} &= \big((\mathsf{A}_{\alpha_0} - \lambda)^{-1} f, g\big)_{L^2(\mathbb{R}^d)} - \big((\mathsf{A}_{\alpha} - \lambda)^{-1} f, g\big)_{L^2(\mathbb{R}^d)} \\
&= (u, g)_{L^2(\mathbb{R}^d)} - (f, v)_{L^2(\mathbb{R}^d)} \\
&= \big(u, (\mathsf{A}_{\alpha} - \lambda) v\big)_{L^2(\mathbb{R}^d)} - \big((\mathsf{A}_{\alpha_0} - \lambda) u, v\big)_{L^2(\mathbb{R}^d)} \\
&= (u, \mathsf{A}_{\alpha} v)_{L^2(\mathbb{R}^d)} - (\mathsf{A}_{\alpha_0} u, v)_{L^2(\mathbb{R}^d)}.
\end{aligned}
\tag{2.12}
$$

We can express the above inner products via the corresponding forms (2.4) and conclude that $(\mathsf{W}f, g)_{L^2(\mathbb{R}^d)}$ reduces to the surface integral

$$
(\mathsf{W}f, g)_{L^2(\mathbb{R}^d)} = -\int_{\Sigma} \alpha_1 \tau_{\mathrm{D}} u \, \overline{\tau_{\mathrm{D}} v} \, \mathrm{d}x = (\mathsf{T}_1 f, \mathsf{T}_2 g)_{L^2(\Sigma)},
$$

where $\mathsf{T}_1, \mathsf{T}_2 : L^2(\mathbb{R}^d) \to L^2(\Sigma)$ are defined by

$$
\mathsf{T}_1 := |\alpha_1|^{\frac{1}{2}} \tau_{\mathrm{D}} (\mathsf{A}_{\alpha_0} - \lambda)^{-1} \quad \text{and} \quad \mathsf{T}_2 := -\operatorname{sgn}(\alpha_1) |\alpha_1|^{\frac{1}{2}} \tau_{\mathrm{D}} (\mathsf{A}_{\alpha} - \lambda)^{-1}.
$$

As $(\mathsf{A}_{\alpha_0} - \lambda)^{-1}$ and $(\mathsf{A}_{\alpha} - \lambda)^{-1}$ are bounded operators from $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$, it follows from (2.3) that $\tau_{\mathrm{D}}(\mathsf{A}_{\alpha_0} - \lambda)^{-1}$ and $\tau_{\mathrm{D}}(\mathsf{A}_{\alpha} - \lambda)^{-1}$ are bounded from $L^2(\mathbb{R}^d)$ into $H^{\frac{1}{2}}(\Sigma)$. Consequently, both T_1 and T_2 are compact as operators from $L^2(\mathbb{R}^d)$ into $L^2(\Sigma)$ by Proposition A.3. Thus the operator $\mathsf{W} = \mathsf{T}_2^* \mathsf{T}_1$ is compact as well and the stability of the essential spectrum under compact perturbations in resolvent sense combined with (2.9) yields the claim. □

2.3 Uniqueness of the ground state

In this subsection we assume that the bottom of the spectrum $\lambda_1(\alpha)$ in (1.3) is a discrete eigenvalue of A_α. The aim is to prove in Theorem 2.7 that this eigenvalue is simple and the corresponding eigenfunction can be chosen strictly positive on $\mathbb{R}^d \setminus \Sigma$.

Lemma 2.4 *Let $u \in H^1(\mathbb{R}^d)$ be a real-valued eigenfunction of A_α corresponding to $\lambda_1(\alpha)$. Then also $|u|$ is an eigenfunction of A_α corresponding to $\lambda_1(\alpha)$.*

Proof From $|\nabla |u|| = |\nabla u|$, cf. [35, Theorem 6.17], and $\tau_{\mathrm{D}}|u| = |\tau_{\mathrm{D}} u|$, we obtain

$$
\frac{\mathfrak{a}_\alpha[|u|]}{\||u|\|^2_{L^2(\mathbb{R}^d)}} = \frac{\mathfrak{a}_\alpha[u]}{\|u\|^2_{L^2(\mathbb{R}^d)}} = \lambda_1(\alpha).
$$

Since $\lambda_1(\alpha)$ is the bottom of the spectrum it can be represented by the min-max principle [40, Theorem XIII.2] as

$$
\lambda_1(\alpha) = \inf_{0 \neq v \in H^1(\mathbb{R}^d)} \frac{\mathfrak{a}_\alpha[v]}{\|v\|^2_{L^2(\mathbb{R}^d)}}.
$$

Since $\lambda_1(\alpha)$ is assumed to be a discrete eigenvalue, it follows from [15, Chapter 10.2, Theorem 1] that $|u|$ is indeed an eigenfunction of A_α corresponding to the eigenvalue $\lambda_1(\alpha)$. □

Lemma 2.5 *Let $\Omega \subseteq \mathbb{R}^d$ be open and connected. Assume that $u \in H^1(\Omega)$ and $\lambda \in \mathbb{R}$ satisfy*

$$(\nabla u, \nabla v)_{L^2(\Omega;\mathbb{C}^d)} = \lambda (u, v)_{L^2(\Omega)}, \quad v \in H^1_0(\Omega).$$

Then $u \in C^\infty(\Omega)$ and if $u \geq 0$ and $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$.

Proof The interior regularity $u \in C^\infty(\Omega)$ is well known; cf. [18, §6.3. Theorem 3]. Assume now $u \geq 0$ and $u(x_0) = 0$ for some $x_0 \in \Omega$. Since Ω is connected, for every $x \in \Omega$ there exists a path γ connecting x and x_0. Since Ω is also open, there even exists some open and bounded U with $\gamma \subseteq U \subseteq \overline{U} \subseteq \Omega$. Then it follows from Harnack's inequality [27, Corollary 8.21], that

$$\sup_{y \in U} u(y) \leq C \inf_{y \in U} u(y),$$

for some constant $C > 0$. Since $u(x_0) = 0$, the right and hence also the left hand side of this inequality vanishes. Therefore, $u|_U = 0$ and in particular $u(x) = 0$. Since $x \in \Omega$ was arbitrary, we conclude $u \equiv 0$. □

Lemma 2.6 *Let $u \in H^1(\mathbb{R}^d)$ be a real-valued eigenfunction of A_α corresponding to $\lambda_1(\alpha)$. Then $u \in C^\infty(\mathbb{R}^d \setminus \Sigma)$ is either strictly positive or strictly negative on $\mathbb{R}^d \setminus \Sigma$.*

Proof From Lemma 2.5 we conclude $u \in C^\infty(\mathbb{R}^d \setminus \Sigma)$. In order to show that u has no zeros in $\mathbb{R}^d \setminus \Sigma$, we assume the converse, i.e. that $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^d \setminus \Sigma$. It is clear that Σ cuts the whole space $\mathbb{R}^d$ into the two domains

$$\Omega_+ := \left\{ (x, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x_d > \xi(x) \right\},$$
$$\Omega_- := \left\{ (x, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x_d < \xi(x) \right\}.$$

We will assume without loss of generality that $x_0 \in \Omega_+$. Since, by Lemma 2.4, $|u|$ is also an eigenfunction corresponding to $\lambda_1(\alpha)$, we have

$$(\nabla |u|, \nabla v)_{L^2(\Omega_+;\mathbb{C}^d)} = \lambda_1(\alpha)(|u|, v)_{L^2(\Omega_+)}, \quad v \in H^1_0(\Omega_+),$$

and Lemma 2.5 implies $u|_{\Omega_+} \equiv 0$. In particular, we have $\tau_{\mathrm{D}} u = 0$ and the eigenvalue equation for u reduces to

$$(\nabla u, \nabla v)_{L^2(\Omega_-;\mathbb{C}^d)} = \lambda_1(\alpha)(u, v)_{L^2(\Omega_-)}, \quad v \in H^1(\mathbb{R}^d).$$

Since $\lambda_1(\alpha)$ is a discrete eigenvalue, it is negative by Lemma 2.2, and consequently choosing $v = u$, we conclude $u|_{\Omega_-} \equiv 0$. But this is a contradiction to the fact that u is a (nonzero) eigenfunction; hence u has no zeros in $\mathbb{R}^d \setminus \Sigma$.

Since we already know that $u \in C^\infty(\mathbb{R}^d \setminus \Sigma)$ has no zeros in $\mathbb{R}^d \setminus \Sigma$, it has to be either strictly positive or strictly negative on each of the domains $\Omega_\pm$. However, a priori the signs of u may not coincide. If, e.g.

$$u|_{\Omega_+} > 0 \quad \text{and} \quad u|_{\Omega_-} < 0,$$

then $\tau_{\mathrm{D}} u = 0$ and the eigenvalue equation for u reduces to

$$(\nabla u, \nabla v)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} = \lambda_1(\alpha)(u, v)_{L^2(\mathbb{R}^d)}, \quad v \in H^1(\mathbb{R}^d).$$

Choosing $v = u$ we again conclude $u \equiv 0$ by the negativity of $\lambda_1(\alpha)$; a contradiction as u is a (nonzero) eigenfunction. □

Theorem 2.7 *If the bottom* (1.3) *of the spectrum of* A_α *is a discrete eigenvalue, then it is simple and the corresponding eigenfunction can be chosen strictly positive on* $\mathbb{R}^d \setminus \Sigma$.

Proof Note that there exists a real-valued basis of the eigenspace corresponding to $\lambda_1(\alpha)$ since for every eigenfunction the complex conjugate is also an eigenfunction. Now consider two orthogonal real-valued eigenfunctions u_1 and u_2. According to Lemma 2.6 each eigenfunction is either strictly positive or strictly negative on $\mathbb{R}^d \setminus \Sigma$. But this is a contradiction to the orthogonality condition

$$\int_{\mathbb{R}^d} u_1 \, u_2 \, \mathrm{d}x = 0.$$

Hence, the eigenspace is one-dimensional and thus $\lambda_1(\alpha)$ is a simple eigenvalue. □

3 The Birman-Schwinger principle and an optimization result for δ-potentials on a hyperplane

In this section we assume that the support of the δ-potential is a hyperplane and we shall therefore identify $\Sigma = \mathbb{R}^{d-1} \times \{0\} \cong \mathbb{R}^{d-1}$. Moreover, as in (2.2), everywhere in this section we consider a real-valued function

$$\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$$

with $1 < p < \infty$ if $d = 2$ and $d - 1 \leq p < \infty$ if $d \geq 3$. Later we shall also assume that there exists some $\alpha_0 \in \mathbb{R}$ such that

$$\left\{ x \in \mathbb{R}^{d-1} \;\middle|\; |\alpha(x) - \alpha_0| > \varepsilon \right\} \text{ has finite measure for every } \varepsilon > 0. \tag{3.1}$$

We first discuss the Birman-Schwinger principle for the operator A_α in this special situation, by means of which the spectral problem can be reduced to the spectral analysis of a relativistic Schrödinger operator in $L^2(\mathbb{R}^{d-1})$. As an application and illustration we prove an optimization result for the bottom of the spectrum of A_α in Theorem 3.7.

3.1 The Birman-Schwinger principle for δ-potentials supported on a hyperplane

For every $\lambda < 0$ we consider the form

$$\begin{aligned} \mathfrak{d}_{\alpha,\lambda}[\phi,\psi] &:= 2\big((-\Delta-\lambda)^{\frac{1}{4}}\phi, (-\Delta-\lambda)^{\frac{1}{4}}\psi\big)_{L^2(\mathbb{R}^{d-1})} - \int_{\mathbb{R}^{d-1}} \alpha\, \phi\, \overline{\psi}\, \mathrm{d}x, \\ \operatorname{dom} \mathfrak{d}_{\alpha,\lambda} &:= H^{\frac{1}{2}}(\mathbb{R}^{d-1}). \end{aligned} \tag{3.2}$$

It follows from Lemma A.1 that for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that

$$\big\||\alpha|^{\frac{1}{2}}\phi\big\|^2_{L^2(\mathbb{R}^{d-1})} \le \varepsilon^2 \|\phi\|^2_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} + c_\varepsilon^2 \|\phi\|^2_{L^2(\mathbb{R}^{d-1})}, \quad \phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}). \tag{3.3}$$

Using this inequality it follows (see the proof of Proposition 2.1) that $\mathfrak{d}_{\alpha,\lambda}$ is a densely defined, symmetric, semibounded and closed form in $L^2(\mathbb{R}^{d-1})$. We denote the corresponding self-adjoint operator in $L^2(\mathbb{R}^{d-1})$ by $\mathsf{D}_{\alpha,\lambda}$. It turns out in Proposition 3.2 below that the eigenvalue 0 of this *relativistic Schrödinger operator* is directly linked to the eigenvalue λ of the Schrödinger operator A_α.

We first formulate and prove a preparatory lemma; here and in the following we shall denote the extension of the $L^2(\mathbb{R}^{d-1})$ scalar product onto the dual pair $H^{-\frac{1}{2}}(\mathbb{R}^{d-1}) \times H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ by $\langle\,\cdot\,,\,\cdot\,\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})\times H^{\frac{1}{2}}(\mathbb{R}^{d-1})}$.

Lemma 3.1 *For every fixed $\lambda < 0$ there exists a unique bounded linear operator $\gamma(\lambda) : H^{-\frac{1}{2}}(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^d)$ such that the identity*

$$\big(\nabla\gamma(\lambda)\phi, \nabla v\big)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \lambda\,\big(\gamma(\lambda)\phi, v\big)_{L^2(\mathbb{R}^d)} = \langle \phi, \tau_{\mathrm{D}} v\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})\times H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \tag{3.4}$$

holds for all $\phi \in H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$, $v \in H^1(\mathbb{R}^d)$. Moreover, the trace of $\gamma(\lambda)$ is given by

$$\tau_{\mathrm{D}}\gamma(\lambda) = \frac{1}{2}(-\Delta-\lambda)^{-\frac{1}{2}}, \tag{3.5}$$

and acts as a bounded linear operator from $H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$ to $H^{\frac{1}{2}}(\mathbb{R}^{d-1})$.

Proof Let $\mathcal{F}_d$ and $\mathcal{F}_{d-1}$ be the unitary Fourier transforms in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{d-1})$, respectively, and consider Schwartz functions $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$. We first define the operator $\gamma(\lambda)$ in Fourier space as

$$(\mathcal{F}_d\gamma(\lambda)\phi)(\tilde{k}) := \frac{(\mathcal{F}_{d-1}\phi)(k)}{\sqrt{2\pi}(|\tilde{k}|^2-\lambda)}, \quad \tilde{k} = (k, k_d) \in \mathbb{R}^{d-1}\times\mathbb{R}. \tag{3.6}$$

As $\lambda < 0$ and $\mathcal{F}_{d-1}\phi \in \mathcal{S}(\mathbb{R}^{d-1})$, this is a well defined function in $L^2(\mathbb{R}^d)$. The fact, that $\gamma(\lambda)$ is bounded from $H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$ to $H^1(\mathbb{R}^d)$ follows from the estimate

$$
\begin{aligned}
\|\gamma(\lambda)\phi\|^2_{H^1(\mathbb{R}^d)} &= \frac{1}{2\pi}\int_{\mathbb{R}^d}(1+|\tilde{k}|^2)\frac{|(\mathcal{F}_{d-1}\phi)(k)|^2}{(|\tilde{k}|^2-\lambda)^2}\mathsf{d}\tilde{k} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\frac{1+|k|^2+k_d^2}{(|k|^2+k_d^2-\lambda)^2}\mathsf{d}k_d|(\mathcal{F}_{d-1}\phi)(k)|^2\mathsf{d}k \\
&= \frac{1}{4}\int_{\mathbb{R}^{d-1}}\frac{2|k|^2+1-\lambda}{(|k|^2-\lambda)^{\frac{3}{2}}}|(\mathcal{F}_{d-1}\phi)(k)|^2\mathsf{d}k \\
&\leq \frac{c(\lambda)}{4}\|\phi\|^2_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})},
\end{aligned}
$$

where $c(\lambda)$ denotes the maximum of the function

$$
k \mapsto \frac{(2|k|^2+1-\lambda)(|k|^2+1)^{1/2}}{(|k|^2-\lambda)^{3/2}}.
$$

Since $\mathcal{S}(\mathbb{R}^{d-1})$ is dense in $H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$ the operator $\gamma(\lambda)$ can be extended by continuity onto $H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$.

In order to prove the identity (3.4) for Schwartz functions $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$ and $v \in \mathcal{S}(\mathbb{R}^d)$, we use the Fourier representation

$$
(\mathcal{F}_d\nabla v)(\tilde{k}) = i\tilde{k}(\mathcal{F}_d v)(\tilde{k}), \quad \tilde{k} \in \mathbb{R}^d, \tag{3.7}
$$

of the gradient. For $x \in \mathbb{R}^{d-1}$ the trace can be written as

$$
\begin{aligned}
(\tau_{\mathrm{D}}v)(x) &= (\mathcal{F}_d^{-1}\mathcal{F}_d v)(x,0) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i\langle\tilde{k},(x,0)\rangle}(\mathcal{F}_d v)(\tilde{k})\mathsf{d}\tilde{k} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{d-1}} e^{i\langle k,x\rangle}\int_{\mathbb{R}}(\mathcal{F}_d v)(k,k_d)\mathsf{d}k_d\mathsf{d}k \\
&= \frac{1}{\sqrt{2\pi}}\mathcal{F}_{d-1}^{-1}\left[\int_{\mathbb{R}}(\mathcal{F}_d v)(\,\cdot\,,k_d)\mathsf{d}k_d\right](x)
\end{aligned} \tag{3.8}
$$

and hence

$$
(\mathcal{F}_{d-1}\tau_{\mathrm{D}}v)(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathcal{F}_d v)(k,k_d)\mathsf{d}k_d, \quad k \in \mathbb{R}^{d-1}. \tag{3.9}
$$

The definition (3.6) of $\gamma(\lambda)$, together with (3.7) and (3.9) leads to

$$
\begin{aligned}
&\big(\nabla\gamma(\lambda)\phi, \nabla v\big)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \lambda\big(\gamma(\lambda)\phi, v\big)_{L^2(\mathbb{R}^d)} \\
&\qquad = \int_{\mathbb{R}^d} (|\tilde{k}|^2 - \lambda)\, (\mathcal{F}_d\gamma(\lambda)\phi)(\tilde{k})\, \overline{(\mathcal{F}_d v)(\tilde{k})}\, \mathsf{d}\tilde{k} \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} (\mathcal{F}_{d-1}\phi)(k)\, \overline{(\mathcal{F}_d v)(k, k_d)}\, \mathsf{d}k_d \mathsf{d}k \\
&\qquad = \int_{\mathbb{R}^{d-1}} (\mathcal{F}_{d-1}\phi)(k)\, \overline{(\mathcal{F}_{d-1}\tau_{\mathrm{D}} v)(k)}\, \mathsf{d}k \\
&\qquad = (\phi, \tau_{\mathrm{D}} v)_{L^2(\mathbb{R}^{d-1})} \\
&\qquad = \langle \phi, \tau_{\mathrm{D}} v\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})\times H^{\frac{1}{2}}(\mathbb{R}^{d-1})},
\end{aligned}
$$

and hence (3.4) holds for $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$ and $v \in \mathcal{S}(\mathbb{R}^d)$. By density and continuity this identity extends to all $\phi \in H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$ and $v \in H^1(\mathbb{R}^d)$. Also note, that the identity (3.4) uniquely defines the operator $\gamma(\lambda)$.

For the proof of (3.5) note first that the identity (3.9) and its derivation (3.8) remain valid for $v \in H^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ with $\mathcal{F}_d v \in L^1(\mathbb{R}^d)$. In particular, for $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$ it is not difficult to see that $\mathcal{F}_d\gamma(\lambda)\phi \in L^1(\mathbb{R}^d)$ by its definition (3.6) and hence also that $\gamma(\lambda)\phi = \mathcal{F}_d^{-1}\mathcal{F}_d\gamma(\lambda)\phi$ is continuous as the inverse Fourier transform of an L^1-function. This means that from (3.9) we get

$$
\begin{aligned}
(\mathcal{F}_{d-1}\tau_{\mathrm{D}}\gamma(\lambda)\phi)(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}_d\gamma(\lambda)\phi)(k, k_d)\mathsf{d}k_d \\
&= \frac{(\mathcal{F}_{d-1}\phi)(k)}{2\pi} \int_{\mathbb{R}} \frac{\mathsf{d}k_d}{|\tilde{k}|^2 - \lambda} \\
&= \frac{(\mathcal{F}_{d-1}\phi)(k)}{2(|k|^2 - \lambda)^{\frac{1}{2}}},
\end{aligned}
$$

which is exactly equation (3.5) in Fourier space. Again, by continuity this identity also holds for every $\phi \in H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$. □

With this lemma we find a connection between the eigenvalue 0 of the relativistic Schrödinger operator $\mathsf{D}_{\alpha,\lambda}$ and the eigenvalue λ of the Schrödinger operator A_α.

Proposition 3.2 *For every $\lambda < 0$ the restriction of the Dirichlet trace operator*

$$
\tau_{\mathrm{D}} : \ker(\mathsf{A}_\alpha - \lambda) \to \ker \mathsf{D}_{\alpha,\lambda} \tag{3.10}
$$

is bijective and, in particular, $\dim\ker(\mathsf{A}_\alpha - \lambda) = \dim\ker \mathsf{D}_{\alpha,\lambda}$.

Proof In order to see that the restriction of τ_{D} onto $\ker(\mathsf{A}_\alpha - \lambda)$ maps into $\ker \mathsf{D}_{\alpha,\lambda}$ consider some $u \in \ker(\mathsf{A}_\alpha - \lambda)$. By (1.2) we have $u \in H^1(\mathbb{R}^d)$ and

$$
(\nabla u, \nabla v)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \lambda(u, v)_{L^2(\mathbb{R}^d)} = \big(\operatorname{sgn}(\alpha)|\alpha|^{\frac{1}{2}}\tau_{\mathrm{D}} u, |\alpha|^{\frac{1}{2}}\tau_{\mathrm{D}} v\big)_{L^2(\mathbb{R}^{d-1})} \tag{3.11}
$$

for all $v \in H^1(\mathbb{R}^d)$. Since $\tau_{\mathrm{D}} u \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$, we get $|\alpha|^{\frac{1}{2}} \tau_{\mathrm{D}} u \in L^2(\mathbb{R}^{d-1})$ from (3.3) and hence there exist $\psi_n \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ such that

$$\operatorname{sgn}(\alpha)|\alpha|^{\frac{1}{2}} \tau_{\mathrm{D}} u = \lim_{n\to\infty} \psi_n \quad \text{in } L^2(\mathbb{R}^{d-1}). \tag{3.12}$$

Again, by (3.3), we have $|\alpha|^{\frac{1}{2}} \psi_n \in L^2(\mathbb{R}^{d-1})$ and inserting these into (3.4) leads to

$$\big(\nabla\gamma(\lambda)|\alpha|^{\frac{1}{2}}\psi_n, \nabla v\big)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \lambda\big(\gamma(\lambda)|\alpha|^{\frac{1}{2}}\psi_n, v\big)_{L^2(\mathbb{R}^d)} = \big(\psi_n, |\alpha|^{\frac{1}{2}}\tau_{\mathrm{D}} v\big)_{L^2(\mathbb{R}^{d-1})}$$

for all $v \in H^1(\mathbb{R}^d)$. Combining this with (3.11) and (3.12) implies the convergence

$$\gamma(\lambda)|\alpha|^{\frac{1}{2}}\psi_n \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^d).$$

Applying the bounded operator $(-\Delta-\lambda)^{\frac{1}{4}}\tau_{\mathrm{D}} : H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^{d-1})$ on this weak convergence and using (3.5) gives

$$\frac{1}{2}(-\Delta-\lambda)^{-\frac{1}{4}}|\alpha|^{\frac{1}{2}}\psi_n = (-\Delta-\lambda)^{\frac{1}{4}}\tau_{\mathrm{D}}\gamma(\lambda)|\alpha|^{\frac{1}{2}}\psi_n \rightharpoonup (-\Delta-\lambda)^{\frac{1}{4}}\tau_{\mathrm{D}} u$$

weakly in $L^2(\mathbb{R}^{d-1})$. Hence, for every $\psi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ we get

$$\begin{aligned}
\mathfrak{d}_{\alpha,\lambda}[\tau_{\mathrm{D}} u, \psi] &= \lim_{n\to\infty} \big((-\Delta-\lambda)^{-\frac{1}{4}}|\alpha|^{\frac{1}{2}}\psi_n, (-\Delta-\lambda)^{\frac{1}{4}}\psi\big)_{L^2(\mathbb{R}^{d-1})} - \int_{\mathbb{R}^{d-1}} \alpha\, \tau_{\mathrm{D}} u\, \overline{\psi}\, \mathrm{d}x \\
&= \lim_{n\to\infty} \big(\psi_n, |\alpha|^{\frac{1}{2}}\psi\big)_{L^2(\mathbb{R}^{d-1})} - \int_{\mathbb{R}^{d-1}} \alpha\, \tau_{\mathrm{D}} u\, \overline{\psi}\, \mathrm{d}x \\
&= 0,
\end{aligned}$$

where (3.12) was used in the last step. Thus, we conclude $\tau_{\mathrm{D}} u \in \ker \mathsf{D}_{\alpha,\lambda}$.

Next we show that (3.10) is injective. In fact, assume that $\tau_{\mathrm{D}} u = 0$ for some $u \in \ker(\mathsf{A}_\alpha - \lambda)$. Then (1.2) leads to

$$(\nabla u, \nabla v)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} = \lambda(u, v)_{L^2(\mathbb{R}^{d-1})}, \quad v \in H^1(\mathbb{R}^d).$$

Since $\lambda < 0$ we can choose $v = u$ and conclude $u = 0$.

For the surjectivity of (3.10) we choose $\phi \in \ker \mathsf{D}_{\alpha,\lambda}$. By (3.2) we then have $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ and

$$2\big((-\Delta-\lambda)^{\frac{1}{4}}\phi, (-\Delta-\lambda)^{\frac{1}{4}}\psi\big)_{L^2(\mathbb{R}^{d-1})} = \int_{\mathbb{R}^{d-1}} \alpha\, \phi\, \overline{\psi}\, \mathrm{d}x, \quad \psi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}). \tag{3.13}$$

Now define $u_\phi := 2\gamma(\lambda)(-\Delta-\lambda)^{\frac{1}{2}}\phi$. Then $\tau_{\mathrm{D}} u_\phi = \phi$ by (3.5) and using (3.4) with ϕ replaced by $2(-\Delta-\lambda)^{\frac{1}{2}}\phi$, gives for any $v \in H^1(\mathbb{R}^d)$

$$\begin{aligned}\big(\nabla u_\phi, \nabla v\big)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \lambda\,(u_\phi, v)_{L^2(\mathbb{R}^d)} &= 2\langle(-\Delta-\lambda)^{\frac{1}{2}}\phi, \tau_{\mathrm{D}} v\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})\times H^{\frac{1}{2}}(\mathbb{R}^{d-1})}\\ &= \int_{\mathbb{R}^{d-1}} \alpha\,\phi\,\overline{\tau_{\mathrm{D}} v}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^{d-1}} \alpha\,\tau_{\mathrm{D}} u_\phi\,\overline{\tau_{\mathrm{D}} v}\,\mathrm{d}x,\end{aligned}$$

where in the second step we used (3.13) with $\psi = \tau_{\mathrm{D}} v$. Summing up, for $\phi \in \ker \mathsf{D}_{\alpha,\lambda}$ we found $u_\phi \in \ker(\mathsf{A}_\alpha - \lambda)$ such that $\tau_{\mathrm{D}} u_\phi = \phi$, which is the surjectivity of (3.10).□

Next we analyse how the bottom of the spectrum $\sigma(\mathsf{D}_{\alpha,\lambda})$ behaves as a function of $\lambda < 0$.

Lemma 3.3 *For $\lambda < 0$ the mapping*

$$\lambda \mapsto \mu_\alpha(\lambda) := \inf \sigma(\mathsf{D}_{\alpha,\lambda}) = \inf_{0\neq\phi\in H^{1/2}(\mathbb{R}^{d-1})} \frac{\mathfrak{d}_{\alpha,\lambda}[\phi]}{\|\phi\|^2_{L^2(\mathbb{R}^{d-1})}} \tag{3.14}$$

is nonincreasing, continuous and admits the limit

$$\lim_{\lambda\to-\infty} \mu_\alpha(\lambda) = \infty. \tag{3.15}$$

Proof With the help of the Fourier transform in $L^2(\mathbb{R}^{d-1})$ we see that the form $\mathfrak{d}_{\alpha,\lambda}$ admits the representation

$$\mathfrak{d}_{\alpha,\lambda}[\phi] = 2\int_{\mathbb{R}^{d-1}} (|k|^2-\lambda)^{\frac{1}{2}}|(\mathcal{F}_{d-1}\phi)(k)|^2\mathrm{d}k - \int_{\mathbb{R}^{d-1}} \alpha\,|\phi|^2\,\mathrm{d}x, \tag{3.16}$$

for any $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$, which shows that $\mathfrak{d}_{\alpha,\lambda}[\phi]$ is nonincreasing in λ. Hence the same is true for μ_α in (3.14).

For the continuity of the function μ_α consider $\lambda_1 \le \lambda_2 < 0$. Then for every $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ we can estimate the difference

$$\begin{aligned}\mathfrak{d}_{\alpha,\lambda_1}[\phi] - \mathfrak{d}_{\alpha,\lambda_2}[\phi] &= 2\int_{\mathbb{R}^{d-1}} \big((|k|^2-\lambda_1)^{\frac{1}{2}} - (|k|^2-\lambda_2)^{\frac{1}{2}}\big)|(\mathcal{F}_{d-1}\phi)(k)|^2\mathrm{d}k\\ &\le 2\big(\sqrt{-\lambda_1}-\sqrt{-\lambda_2}\big)\,\|\phi\|^2_{L^2(\mathbb{R}^{d-1})},\end{aligned}$$

and via (3.14) we also conclude

$$\mu_\alpha(\lambda_1) - \mu_\alpha(\lambda_2) \le 2\big(\sqrt{-\lambda_1}-\sqrt{-\lambda_2}\big),$$

which proves the continuity of $\lambda \mapsto \mu_\alpha(\lambda)$.

It remains to verify (3.15). For this we use the estimate

$$\Big|\int_{\mathbb{R}^{d-1}} \alpha\,|\phi|^2\mathrm{d}x\Big| \le \|\phi\|^2_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} + c_1^2\|\phi\|^2_{L^2(\mathbb{R}^{d-1})}, \quad \phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}),$$

from (3.3). Plugging this in (3.16) gives

$$\begin{aligned}\mathfrak{d}_{\alpha,\lambda}[\phi] &\geq \int_{\mathbb{R}^{d-1}} \big(2(|k|^2-\lambda)^{\frac{1}{2}} - (1+|k|^2)^{\frac{1}{2}}\big)|(\mathcal{F}_{d-1}\phi)(k)|^2 \mathsf{d}k - c_1^2\|\phi\|^2_{L^2(\mathbb{R}^{d-1})} \\ &\geq (c(\lambda)-c_1^2)\|\phi\|^2_{L^2(\mathbb{R}^{d-1})},\end{aligned}$$

where $c(\lambda) \in \mathbb{R}$ is the minimum of $k \mapsto 2(|k|^2-\lambda)^{\frac{1}{2}} - (1+|k|^2)^{\frac{1}{2}}$. From (3.14) we then conclude

$$\mu_\alpha(\lambda) \geq c(\lambda) - c_1^2 \overset{\lambda\to-\infty}{\longrightarrow} \infty. \qquad \square$$

Next, we compute the essential spectrum of $\mathsf{D}_{\alpha,\lambda}$ under the additional assumption that α is close to a constant in the sense of (3.1).

Proposition 3.4 *Assume that α satisfies* (3.1) *with some $\alpha_0 \in \mathbb{R}$. Then for every $\lambda < 0$ the essential spectrum of $\mathsf{D}_{\alpha,\lambda}$ is given by*

$$\sigma_{\mathrm{ess}}(\mathsf{D}_{\alpha,\lambda}) = \big[2\sqrt{-\lambda} - \alpha_0, \infty\big). \tag{3.17}$$

Furthermore, the mapping $\lambda \mapsto \mu_\alpha(\lambda)$ from (3.14) *is strictly decreasing on* $(-\infty, 0)$.

Proof It is clear that for constant $\alpha(x) = \alpha_0 \in \mathbb{R}$ the relativistic Schrödinger operator is given by $\mathsf{D}_{\alpha_0,\lambda} = 2(-\Delta-\lambda)^{\frac{1}{2}} - \alpha_0$ with $\operatorname{dom} \mathsf{D}_{\alpha_0,\lambda} = H^1(\mathbb{R}^{d-1})$. Hence we have

$$\sigma(\mathsf{D}_{\alpha_0,\lambda}) = \sigma_{\mathrm{ess}}(\mathsf{D}_{\alpha_0,\lambda}) = \big[2\sqrt{-\lambda} - \alpha_0, \infty\big). \tag{3.18}$$

For a nonconstant function α we define the function $\alpha_1(x) := \alpha(x) - \alpha_0$. Then the set $\{x \in \mathbb{R}^{d-1} \mid |\alpha_1(x)| > \varepsilon\}$ has finite measure for every $\varepsilon > 0$ by the property (3.1). To prove (3.17) we proceed in the same way as in Step 3 of the proof of Theorem 3.4 and check that for some $\mu < \inf(\sigma(\mathsf{D}_{\alpha_0,\lambda}) \cup \sigma(\mathsf{D}_{\alpha,\lambda}))$ the resolvent difference

$$\mathsf{W} := (\mathsf{D}_{\alpha_0,\lambda} - \mu)^{-1} - (\mathsf{D}_{\alpha,\lambda} - \mu)^{-1}$$

is a compact operator in $L^2(\mathbb{R}^{d-1})$. For this let $\phi, \psi \in L^2(\mathbb{R}^{d-1})$ and set

$$\phi_\mu := (\mathsf{D}_{\alpha_0,\lambda} - \mu)^{-1}\phi \quad \text{and} \quad \psi_\mu := (\mathsf{D}_{\alpha,\lambda} - \mu)^{-1}\psi.$$

In the same way as in (2.12) one verifies

$$\begin{aligned}(\mathsf{W}\phi, \psi)_{L^2(\mathbb{R}^{d-1})} &= \big(\phi_\mu, \mathsf{D}_{\alpha,\lambda}\psi_\mu\big)_{L^2(\mathbb{R}^{d-1})} - \big(\mathsf{D}_{\alpha_0,\lambda}\phi_\mu, \psi_\mu\big)_{L^2(\mathbb{R}^{d-1})} \\ &= -\int_{\mathbb{R}^{d-1}} \alpha_1\, \phi_\mu \overline{\psi_\mu} \mathsf{d}x \\ &= (\mathsf{T}_1\phi, \mathsf{T}_2\psi)_{L^2(\mathbb{R}^{d-1})},\end{aligned}$$

where

$$\mathsf{T}_1 := |\alpha_1|^{\frac{1}{2}}(\mathsf{D}_{\alpha_0,\lambda} - \mu)^{-1} \quad \text{and} \quad \mathsf{T}_2 := -\operatorname{sgn}(\alpha_1)|\alpha_1|^{\frac{1}{2}}(\mathsf{D}_{\alpha,\lambda} - \mu)^{-1}.$$

As $(\mathsf{D}_{\alpha_0,\lambda} - \mu)^{-1}$ and $(\mathsf{D}_{\alpha,\lambda} - \mu)^{-1}$ are bounded operators from $L^2(\mathbb{R}^{d-1})$ into $H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ it follows from Proposition A.3 that both T_1 and T_2 are compact operators in $L^2(\mathbb{R}^{d-1})$. Thus the resolvent difference $\mathsf{W} = \mathsf{T}_2^*\mathsf{T}_1$ is compact as well, which implies $\sigma_{\mathrm{ess}}(\mathsf{D}_{\alpha_0,\lambda}) = \sigma_{\mathrm{ess}}(\mathsf{D}_{\alpha,\lambda})$ and (3.17) follows from (3.18).

For the proof of the strict monotonicity of $\lambda \mapsto \mu_\alpha(\lambda)$, let $\lambda_1 < \lambda_2 < 0$. Then

$$\mu_\alpha(\lambda_j) \leq 2\sqrt{-\lambda_j} - \alpha_0, \qquad j = 1, 2, \tag{3.19}$$

by (3.17). If $\mu_\alpha(\lambda_1) = 2\sqrt{-\lambda_1} - \alpha_0$ we conclude from $\mu_\alpha(\lambda_2) \leq 2\sqrt{-\lambda_2} - \alpha_0$ that $\mu_\alpha(\lambda_2) < \mu_\alpha(\lambda_1)$. If $\mu_\alpha(\lambda_1) < 2\sqrt{-\lambda_1} - \alpha_0$ we know from (3.17) that $\mu_\alpha(\lambda_1)$ is a discrete eigenvalue of $\mathsf{D}_{\alpha,\lambda_1}$ and hence there is a corresponding eigenfunction $\phi \in \operatorname{dom} \mathsf{D}_{\alpha,\lambda_1} \subset H^{\frac{1}{2}}(\mathbb{R}^{d-1})$. Since, in particular, $\phi \neq 0$ we conclude from (3.16) that $\lambda \mapsto \mathfrak{d}_{\alpha,\lambda}[\phi]$ is strictly decreasing, and hence

$$\mu_\alpha(\lambda_1) = \frac{\mathfrak{d}_{\alpha,\lambda_1}[\phi]}{\|\phi\|^2_{L^2(\mathbb{R}^{d-1})}} > \frac{\mathfrak{d}_{\alpha,\lambda_2}[\phi]}{\|\phi\|^2_{L^2(\mathbb{R}^{d-1})}} \geq \mu_\alpha(\lambda_2). \qquad \square$$

Lemma 3.5 *Assume that α satisfies* (3.1) *with some $\alpha_0 \in \mathbb{R}$. For the lowest spectral point $\lambda_1(\alpha)$ of A_α in* (1.3) *and the lowest spectral point $\mu_\alpha(\lambda)$ of $\mathsf{D}_{\alpha,\lambda}$ in* (3.14) *the following are equivalent:*

(i) $\lambda_1(\alpha) \in \sigma_{\mathrm{d}}(\mathsf{A}_\alpha)$

(ii) μ_α *admits a zero strictly below* $\begin{cases} -\frac{\alpha_0^2}{4}, & \text{if } \alpha_0 \geq 0 \\ 0, & \text{if } \alpha_0 \leq 0. \end{cases}$

In this situation the zero of μ_α coincides with $\lambda_1(\alpha)$.

Proof For an easier notation we write $\lambda_1 := \lambda_1(\alpha)$. For the implication (i) $\Rightarrow$ (ii) let $\lambda_1 \in \sigma_{\mathrm{d}}(\mathsf{A}_\alpha)$ and note that due to the explicit form of the essential spectrum (2.7) we have

$$\lambda_1 < \begin{cases} -\frac{\alpha_0^2}{4}, & \text{if } \alpha_0 \geq 0, \\ 0, & \text{if } \alpha_0 \leq 0. \end{cases} \tag{3.20}$$

It follows from Proposition 3.2 that zero is an eigenvalue of $\mathsf{D}_{\alpha,\lambda_1}$. Assume now $\mu_\alpha(\lambda_1) \neq 0$.

- The case $\mu_\alpha(\lambda_1) = \inf \sigma(\mathsf{D}_{\alpha,\lambda_1}) > 0$ is a contradiction to the fact that zero is an eigenvalue of $\mathsf{D}_{\alpha,\lambda_1}$.
- If $\mu_\alpha(\lambda_1) < 0$, then $\mu_\alpha(\tilde{\lambda}) = 0$ for some $\tilde{\lambda} < \lambda_1$ by Lemma 3.3. Also note, that

$$\inf \sigma_{\mathrm{ess}}(\mathsf{D}_{\alpha,\tilde{\lambda}}) = 2\sqrt{-\tilde{\lambda}} - \alpha_0 \geq 2\sqrt{-\lambda_1} - \alpha_0 > 0$$

by Proposition 3.4 and the estimate (3.20). But then the bottom of the spectrum

$$0 = \mu_\alpha(\tilde{\lambda}) = \inf \sigma(\mathsf{D}_{\alpha,\tilde{\lambda}})$$

is a point in the discrete spectrum and hence an eigenvalue of $\mathsf{D}_{\alpha,\tilde{\lambda}}$. Consequently, Proposition 3.2 implies that $\tilde{\lambda} < \lambda_1$ is an eigenvalue of A_α; a contradiction as λ_1 is the smallest spectral point of A_α.

Hence our assumption is wrong and we conclude $\mu_\alpha(\lambda_1) = 0$. Due to the strict monotonicity in Proposition 3.4, this is also the only zero of μ_α.

For the implication (ii) $\Rightarrow$ (i) assume that μ_α admits a zero

$$\tilde{\lambda} < \begin{cases} -\frac{\alpha_0^2}{4}, & \text{if } \alpha_0 \geq 0, \\ 0, & \text{if } \alpha_0 \leq 0, \end{cases} \tag{3.21}$$

that is, $0 = \mu_\alpha(\tilde{\lambda}) = \inf \sigma(\mathsf{D}_{\alpha,\tilde{\lambda}})$. Since $2\sqrt{-\tilde{\lambda}} - \alpha_0 > 0$ by (3.21) we conclude from (3.17) that zero belongs to the discrete spectrum of $\mathsf{D}_{\alpha,\tilde{\lambda}}$, and hence Proposition 3.2 implies that $\tilde{\lambda}$ is an eigenvalue of A_α. Hence, also the bottom of the spectrum

$$\lambda_1 = \inf \sigma(\mathsf{A}_\alpha) \leq \tilde{\lambda} < \begin{cases} -\frac{\alpha_0^2}{4}, & \text{if } \alpha_0 \geq 0, \\ 0, & \text{if } \alpha_0 \leq 0, \end{cases}$$

belongs to the discrete spectrum of A_α by (2.7). □

3.2 Optimization of $\lambda_1(\alpha)$ and the symmetric decreasing rearrangement

In this subsection we prove an optimization result for the bottom of the spectrum of A_α, which will be formulated in terms of the so-called symmetric decreasing rearrangement of the positive part of the function $\alpha_1(x) := \alpha(x) - \alpha_0$, with $\alpha_0 \in \mathbb{R}$ from (3.1). We first briefly recall the definition and some basic properties of the symmetric decreasing rearrangement and formulate our main result in Theorem 3.7 below. Further details on symmetric decreasing rearrangements can be found in the monographs [5, 35].

Let $\mathcal{A} \subseteq \mathbb{R}^{d-1}$, $d \geq 2$, be a measurable set of finite volume. Then its *symmetric rearrangement* $\mathcal{A}^*$ is defined as the open ball centered at the origin and having the same volume. Let $u\colon \mathbb{R}^{d-1} \to \mathbb{R}$ be a nonnegative measurable function, that vanishes at infinity in the sense that

$$\left\{ x \in \mathbb{R}^{d-1} \;\middle|\; u(x) > t \right\} \text{ has finite measure for every } t > 0. \tag{3.22}$$

We define the *symmetric decreasing rearrangement* u^* of u by symmetrizing its level sets as

$$u^*(x) := \int_0^\infty \chi_{\{u>t\}^*}(x)\, \mathrm{d}t. \tag{3.23}$$

Here $\chi_{\mathcal{A}}\colon \mathbb{R}^{d-1} \to \mathbb{R}$ denotes the characteristic function. The rearrangement u^* has a number of straightforward properties, which will be needed below in the proofs of Theorem 3.7 and Lemma 3.9; cf. [35, Section 3.3 (iv) and Theorem 3.4].

Lemma 3.6 *Let $u, v\colon \mathbb{R}^{d-1} \to \mathbb{R}$ be nonnegative measurable functions satisfying* (3.22)*. Then the following holds:*

(i) *u^* is nonnegative;*

(ii) *u^* is radially symmetric and nonincreasing;*

(iii) *u and u^* are equi-measurable, i.e.,*

$$\left|\left\{ x \in \mathbb{R}^{d-1} \;\middle|\; u(x) > t \right\}\right| = \left|\left\{ x \in \mathbb{R}^{d-1} \;\middle|\; u^*(x) > t \right\}\right|, \quad t > 0;$$

(iv) $(u^*)^2 = (u^2)^*$.

(v) $\|u\|_{L^p(\mathbb{R}^{d-1})} = \|u^*\|_{L^p(\mathbb{R}^{d-1})}, \quad p \geq 1$ *(Conservation of L^p-norm);*

(vi) $\int_{\mathbb{R}^{d-1}} u\, v \,\mathrm{d}x \leq \int_{\mathbb{R}^{d-1}} u^* v^* \mathrm{d}x$ *(Hardy-Littlewood inequality).*

Next we formulate our optimization result for the bottom of the spectrum of A_α.

Theorem 3.7 *Assume that α satisfies* (3.1) *with some $\alpha_0 \in \mathbb{R}$. For $\alpha_1(x) := \alpha(x) - \alpha_0$ we then have the inequality*

$$\lambda_1(\alpha_0 + (\alpha_1)_+^*) \leq \lambda_1(\alpha_0 + \alpha_1),$$

where $(\alpha_1)_+ := \max\{\alpha_1, 0\}$ is the positive part and $(\alpha_1)_+^*$ its symmetric decreasing rearrangement defined in (3.23)*.*

Corollary 3.8 *Let $\omega \subset \mathbb{R}^{d-1}$ be a set of finite measure and $\omega^* \subset \mathbb{R}^{d-1}$ be a ball with the same volume as ω, and let χ_ω and χ_{ω^*} be the characteristic functions of ω and ω^*, respectively. Then for $\beta \geq 0$ we have the inequality*

$$\lambda_1(\beta\chi_{\omega^*}) \leq \lambda_1(\beta\chi_\omega).$$

The proof of Theorem 3.7 relies on the Birman-Schwinger principle for the operator A_α, by means of which the problem is reduced to an eigenvalue inequality for the relativistic Schrödinger operator in $L^2(\mathbb{R}^{d-1})$. The latter is proven with the help of the fact that the symmetric decreasing rearrangement decreases the kinetic energy term corresponding to the relativistic Schrödinger operator; cf. Lemma 3.9. This property of the kinetic energy can be viewed as an analogue of the Pólya-Szegő inequality.

Lemma 3.9 *For every $\lambda < 0$ and nonnegative $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ the rearrangements $(\alpha_1)_+^*$, ϕ^* in* (3.23) *and the form* (3.2) *satisfy*

$$\mathfrak{d}_{\alpha_0+(\alpha_1)_+^*,\lambda}[\phi^*] \leq \mathfrak{d}_{\alpha_0+\alpha_1,\lambda}[\phi]. \tag{3.24}$$

Proof First, in view of Lemma 3.6 (iv), (v) and (vi) we have

$$\int_{\mathbb{R}^{d-1}} (\alpha_0+\alpha_1)\phi^2 \mathrm{d}x \leq \int_{\mathbb{R}^{d-1}} (\alpha_0+(\alpha_1)_+)\phi^2 \mathrm{d}x \leq \int_{\mathbb{R}^{d-1}} (\alpha_0+(\alpha_1)_+^*)(\phi^*)^2 \mathrm{d}x. \quad (3.25)$$

Moreover, it is proven in [35, Section 7.11 (5), Section 7.17 (2) and the remark afterwards] that

$$\left\|(-\Delta-\lambda)^{\frac{1}{4}}\phi^*\right\|^2_{L^2(\mathbb{R}^{d-1})} \leq \left\|(-\Delta-\lambda)^{\frac{1}{4}}\phi\right\|_{L^2(\mathbb{R}^{d-1})}. \quad (3.26)$$

Combining (3.25) and (3.26) then proves the stated inequality (3.24). □

Proof of Theorem 3.7 Observe that by Theorem 2.3 and Lemma 3.6 (v) the essential spectra of the Schrödinger operators $\mathsf{A}_{\alpha_0+\alpha_1}$ and $\mathsf{A}_{\alpha_0+(\alpha_1)_+^*}$ are given by

$$\sigma_{\mathrm{ess}}(\mathsf{A}_{\alpha_0+\alpha_1}) = \sigma_{\mathrm{ess}}(\mathsf{A}_{\alpha_0+(\alpha_1)_+^*}) = \begin{cases} [-\frac{\alpha_0^2}{4},\infty), & \text{if } \alpha_0 \geq 0, \\ [0,\infty), & \text{if } \alpha_0 \leq 0. \end{cases}$$

We assume that α_1 is such that

$$\lambda_1 := \lambda_1(\alpha_0+\alpha_1) < \begin{cases} -\frac{\alpha_0^2}{4}, & \text{if } \alpha_0 \geq 0, \\ 0, & \text{if } \alpha_0 \leq 0, \end{cases}$$

as otherwise the statement of the theorem is clear. Then $\lambda_1 \in \sigma_{\mathrm{d}}(\mathsf{A}_{\alpha_0+\alpha_1})$ and by Theorem 2.7 the eigenfunction $u_1 \in \ker(\mathsf{A}_{\alpha_0+\alpha_1}-\lambda_1)$ can be chosen nonnegative. By Proposition 3.2, we have $\phi_1 := \tau_D u_1 \in \ker \mathsf{D}_{\alpha_0+\alpha_1,\lambda_1}$ for the trace of the eigenfunction, and also $\phi_1 \geq 0$ follows from $u_1 \geq 0$. Lemma 3.6 (v) and Lemma 3.9 give the estimate

$$0 = \frac{\mathfrak{d}_{\alpha_0+\alpha_1,\lambda_1}[\phi_1]}{\|\phi_1\|^2_{L^2(\mathbb{R}^{d-1})}} \geq \frac{\mathfrak{d}_{\alpha_0+(\alpha_1)_+^*,\lambda_1}[\phi_1^*]}{\|\phi_1^*\|^2_{L^2(\mathbb{R}^{d-1})}} \geq \mu_{\alpha_0+(\alpha_1)_+^*}(\lambda_1).$$

Since $\mu_{\alpha_0+(\alpha_1)_+^*}$ is nonincreasing by Lemma 3.3 it admits a zero

$$\tilde{\lambda}_1 \leq \lambda_1 < \begin{cases} -\frac{\alpha_0^2}{4}, & \text{if } \alpha_0 \geq 0, \\ 0, & \text{if } \alpha_0 \leq 0. \end{cases}$$

Hence $\tilde{\lambda}_1 \in \sigma_{\mathrm{d}}(\mathsf{A}_{\alpha_0+(\alpha_1)_+^*})$ and we have $\lambda_1(\alpha_0+(\alpha_1)_+^*) \leq \tilde{\lambda}_1 \leq \lambda_1$, which proves the theorem. □

Remark 3.10 The above results remain valid for Robin Laplacians on the upper half-space $\mathbb{R}^d_+$. More precisely, if B_α denotes the self-adjoint operator in $L^2(\mathbb{R}^d_+)$ associated with the densely defined, symmetric, semibounded, and closed form

$$\mathfrak{b}_\alpha[u,v] := (\nabla u, \nabla v)_{L^2(\mathbb{R}^d_+,\mathbb{C}^d)} - \int_{\mathbb{R}^{d-1}} \alpha\, \tau_{\mathrm{D}} u\, \overline{\tau_{\mathrm{D}} v}\, \mathrm{d}x,$$
$$\operatorname{dom} \mathfrak{b}_\alpha := H^1(\mathbb{R}^d_+),$$

and if we replace $\lambda_1(\alpha) = \inf \sigma(\mathsf{A}_\alpha)$ by $\lambda_1(\alpha) := \inf \sigma(\mathsf{B}_\alpha)$, then Theorem 3.7 and Corollary 3.8 hold.

Remark 3.11 Using Steiner symmetrization the Theorem 3.7 can also be proven in a different way; the following elegant argument was communicated to us recently. Consider a nonnegative function $u\colon \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{R}^{d-1} \ni x' \mapsto u(x', x_d)$ is vanishing at infinity for all $x_d \in \mathbb{R}$. Following the lines of [5, Chapter 6] we recall that the $(d-1, d)$-Steiner symmetrization $u^\sharp$ of the function u is defined as

$$u^\sharp(x', x_d) := (u^*(\cdot, x_d))(x', x_d),$$

where the symmetric decreasing rearrangement in the right hand side is taken for each $x_d \in \mathbb{R}$ with respect to first $d-1$ variables. Let the nonnegative function $u_1 \in H^1(\mathbb{R}^d)$ be the normalized ground state of the operator $\mathsf{A}_{\alpha_0+\alpha_1}$. It is not difficult to check that u_1 is vanishing at infinity slice-wise in the above sense; cf. [5, §6.8]. According to [5, Theorem 6.8] we have

$$\|u_1^\sharp\|_{L^2(\mathbb{R}^d)} = \|u_1\|_{L^2(\mathbb{R}^d)} = 1. \tag{3.27}$$

In view of [5, Theorem 6.19] we get $u_1^\sharp \in H^1(\mathbb{R}^d)$ and

$$\|\nabla u_1^\sharp\|_{L^2(\mathbb{R}^d;\mathbb{C}^d)} \le \|\nabla u_1\|_{L^2(\mathbb{R}^d;\mathbb{C}^d)}. \tag{3.28}$$

Lemma 3.6 (iv), (v) and (vi) yield

$$\int_{\mathbb{R}^{d-1}} (\alpha_0 + (\alpha_1)_+^*)|\tau_{\mathrm{D}} u_1^\sharp|^2 \mathrm{d}x \ge \int_{\mathbb{R}^{d-1}} (\alpha_0 + \alpha_1)|\tau_{\mathrm{D}} u_1|^2 \mathrm{d}x. \tag{3.29}$$

Finally, combining (3.27), (3.28), and (3.29) we obtain by the min-max principle

$$\begin{aligned}
\lambda_1(\alpha_0 + (\alpha_1)_+^*) &\le \mathfrak{a}_{\alpha_0+(\alpha_1)_+^*}[u_1^\sharp] \\
&= \|\nabla u_1^\sharp\|^2_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \int_{\mathbb{R}^{d-1}} (\alpha_0 + (\alpha_1)_+^*)|\tau_{\mathrm{D}} u_1^\sharp|^2 \mathrm{d}x \\
&\le \|\nabla u_1\|^2_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \int_{\mathbb{R}^{d-1}} (\alpha_0 + \alpha_1)|\tau_{\mathrm{D}} u_1|^2 \mathrm{d}x \\
&= \mathfrak{a}_{\alpha_0+\alpha_1}[u_1] = \lambda_1(\alpha_0 + \alpha_1).
\end{aligned}$$

Appendix A

In this appendix let again Σ be a Lipschitz hypersurface as in (2.1) and assume that $\alpha \in L^p(\Sigma) + L^\infty(\Sigma)$ for some $1 < p < \infty$ in $d = 2$ and for $d - 1 \leq p < \infty$ in $d \geq 3$ dimensions, as in (2.2). In this setting we consider the multiplication operator

$$M_\alpha : H^{\frac{1}{2}}(\Sigma) \to L^2(\Sigma) \quad \text{with} \quad M_\alpha \phi := |\alpha|^{\frac{1}{2}} \phi, \quad \phi \in H^{\frac{1}{2}}(\Sigma), \tag{A.1}$$

which plays a crucial role in the well definedness of the form $\mathfrak{a}_\alpha$ in Proposition 2.1 and in the derivation of the essential spectrum in Theorem 2.3. If, in addition, (2.6) (or (A.14)) holds, then it turns out that the operator M_α is compact; for the convenience of the reader we will provide a complete proof below. The preparatory estimate in Lemma A.1 is also used to conclude the semiboundedness of the form $\mathfrak{a}_\alpha$ in Proposition 2.1.

We also want to mention that we consider Sobolev and Lebesgue spaces on the surface Σ in the sense that for every $s > 0$ and $q \in [1, \infty]$

$$\begin{aligned} &\phi \in H^s(\Sigma) \Leftrightarrow \phi \circ \Xi \in H^s(\mathbb{R}^{d-1}) \text{ and } \|\phi\|_{H^s(\Sigma)} := \|\phi \circ \Xi\|_{H^s(\mathbb{R}^{d-1})}, \\ &\phi \in L^q(\Sigma) \Leftrightarrow \phi \circ \Xi \in L^q(\mathbb{R}^{d-1}) \text{ and } \|\phi\|_{L^q(\Sigma)} := \|\phi \circ \Xi\|_{L^q(\mathbb{R}^{d-1})}, \end{aligned} \tag{A.2}$$

where $\Xi(x) := (x, \xi(x))$ is a bijective map from $\mathbb{R}^{d-1}$ onto Σ.

Lemma A.1 *For every $\varepsilon > 0$ there exists some $c_\varepsilon \geq 0$, depending on α, such that*

$$\|M_\alpha \phi\|^2_{L^2(\Sigma)} \leq \varepsilon^2 \|\phi\|^2_{H^{\frac{1}{2}}(\Sigma)} + c_\varepsilon^2 \|\phi\|^2_{L^2(\Sigma)}, \quad \phi \in H^{\frac{1}{2}}(\Sigma). \tag{A.3}$$

Proof We decompose $\alpha \in L^p(\Sigma) + L^\infty(\Sigma)$ into

$$\alpha = \beta + \gamma, \quad \beta \in L^p(\Sigma),\ \gamma \in L^\infty(\Sigma).$$

Fix $\varepsilon > 0$. Then the integrability condition $\beta \in L^p(\Sigma)$ ensures the existence of some $C_\varepsilon \geq 0$ such that $\beta = \beta_1 + \beta_2$, where

$$\beta_1(x) := \begin{cases} 0, & |\beta(x)| \leq C_\varepsilon, \\ \beta(x), & |\beta(x)| > C_\varepsilon, \end{cases} \quad \text{and} \quad \beta_2(x) := \begin{cases} \beta(x), & |\beta(x)| \leq C_\varepsilon, \\ 0, & |\beta(x)| > C_\varepsilon, \end{cases}$$

and

$$\|\beta_1\|_{L^p(\Sigma)} \leq \varepsilon^2. \tag{A.4}$$

We now split $\alpha = \beta_1 + (\beta_2 + \gamma)$ into a bounded part $\beta_2 + \gamma$ and an unbounded remainder β_1 and estimate both parts separately. For β_1 we use Hölder's inequality and the estimate (A.4) to get

$$\left\| |\beta_1|^{\frac{1}{2}} \phi \right\|^2_{L^2(\Sigma)} \leq \|\beta_1\|_{L^p(\Sigma)} \|\phi\|^2_{L^{\frac{2p}{p-1}}(\Sigma)} \leq \varepsilon^2 \|\phi\|^2_{L^{\frac{2p}{p-1}}(\Sigma)} \leq \varepsilon^2 c_E^2 \|\phi\|^2_{H^{\frac{1}{2}}(\Sigma)}, \tag{A.5}$$

where in the last inequality we additionally used the Sobolev embedding

$$\|\cdot\|_{L^{\frac{2p}{p-1}}(\Sigma)} \leq c_E \|\cdot\|_{H^{\frac{1}{2}}(\Sigma)}$$

on the surface, which follows from the classical Sobolev embedding [11, Theorem 8.12.6] on $\mathbb{R}^{d-1}$ and the definition of the Sobolev and Lebesgue norms in (A.2).

On the other hand, $\beta_2 + \gamma$ can simply be estimated by

$$\big\||\beta_2 + \gamma|^{\frac{1}{2}}\phi\big\|^2_{L^2(\Sigma)} \leq \big(C_\varepsilon + \|\gamma\|_{L^\infty(\Sigma)}\big)\|\phi\|^2_{L^2(\Sigma)}. \tag{A.6}$$

Now the estimate (A.3) follows from (A.5) and (A.6). □

The next lemma treats the transition from weak $H^{\frac{1}{2}}$-convergence on Σ to strong L^2-convergence on subsets of finite measure of Σ; this observation is preparatory for the compactness result in Proposition A.3.

Lemma A.2 *For every $\phi_0, (\phi_n)_n \in H^{\frac{1}{2}}(\Sigma)$, the convergence*

$$\phi_n \rightharpoonup \phi_0 \quad \textit{weakly in } H^{\frac{1}{2}}(\Sigma), \tag{A.7}$$

implies for any Borel set $A \subseteq \Sigma$ with finite measure, the convergence

$$\phi_n \to \phi_0 \quad \textit{strongly in } L^2(A). \tag{A.8}$$

Proof In *Step 1* we consider the hyperplane case $\Sigma = \mathbb{R}^{d-1} \times \{0\} \cong \mathbb{R}^{d-1}$. For every $t > 0$, we define the mollifier

$$\varphi_t(x) := \frac{1}{(4\pi t)^{\frac{d-1}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^{d-1}. \tag{A.9}$$

Then by the weak convergence (A.7), we conclude the pointwise convergence of the convolution

$$\begin{aligned} \lim_{n\to\infty} (\varphi_t * \phi_n)(x) &= \lim_{n\to\infty} \big\langle \varphi_t(x - \cdot), \phi_n \big\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1}) \times H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \\ &= \big\langle \varphi_t(x - \cdot), \phi_0 \big\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1}) \times H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \\ &= (\varphi_t * \phi_0)(x). \end{aligned} \tag{A.10}$$

As the weakly convergent sequence $(\phi_n)_n$ is bounded, i.e. $\|\phi_n\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \leq M$ for some $M \geq 0$, we also conclude the uniform boundedness of the convolution

$$|(\varphi_t * \phi_n)(x)| \leq \|\varphi_t\|_{L^2(\mathbb{R}^{d-1})} \|\phi_n\|_{L^2(\mathbb{R}^{d-1})} \leq M \|\varphi_t\|_{L^2(\mathbb{R}^{d-1})} \tag{A.11}$$

for every $x \in \mathbb{R}^{d-1}$, $n \in \mathbb{N}$. Since A is a set of finite measure, (A.10) & (A.11) are sufficient to apply the dominated convergence theorem, which leads to the norm convergence

$$\lim_{n\to\infty} \|\varphi_t * (\phi_n - \phi_0)\|_{L^2(A)} = 0. \tag{A.12}$$

For the Fourier transform of the mollifier (A.9) we have

$$
\begin{aligned}
(\mathcal{F}\varphi_t)(k) &= \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int_{\mathbb{R}^{d-1}} e^{-ikx}\varphi_t(x)\mathrm{d}x = \frac{1}{(8\pi^2 t)^{\frac{d-1}{2}}}\int_{\mathbb{R}^{d-1}} e^{-ikx}e^{-\frac{|x|^2}{4t}}\mathrm{d}x \\
&= \frac{1}{(8\pi^2 t)^{\frac{d-1}{2}}}e^{-t|k|^2}\int_{\mathbb{R}^{d-1}} e^{-\frac{(x+2itk)^2}{4t}}\mathrm{d}x = \frac{1}{(2\pi)^{\frac{d-1}{2}}}e^{-t|k|^2}
\end{aligned}
$$

for $k \in \mathbb{R}^{d-1}$ and we use the estimate

$$
\left|1-(2\pi)^{\frac{d-1}{2}}(\mathcal{F}\varphi_t)(k)\right| = 1-e^{-t|k|^2} \le c(t|k|^2)^{\frac{1}{4}} \le ct^{\frac{1}{4}}(1+|k|^2)^{\frac{1}{4}}, \quad k \in \mathbb{R}^{d-1},
$$

where $c := \sup_{y>0}(1-e^{-y})y^{-\frac{1}{4}}$. Since the Fourier transform of the convolution can be written as the product $\mathcal{F}(\varphi_t * \phi_n) = (2\pi)^{\frac{d-1}{2}}(\mathcal{F}\varphi_t)(\mathcal{F}\phi_n)$, we can estimate

$$
\begin{aligned}
\|\phi_n - \varphi_t * \phi_n\|_{L^2(\mathbb{R}^{d-1})} &= \left\|\left(1-(2\pi)^{\frac{d-1}{2}}\mathcal{F}\varphi_t\right)\mathcal{F}\phi_n\right\|_{L^2(\mathbb{R}^{d-1})} \\
&\le ct^{\frac{1}{4}}\left\|(1+|\cdot|^2)^{\frac{1}{4}}\mathcal{F}\phi_n\right\|_{L^2(\mathbb{R}^{d-1})} \\
&= ct^{\frac{1}{4}}\|\phi_n\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})}.
\end{aligned} \tag{A.13}
$$

The inequality (A.13) of course also holds with ϕ_n replaced by ϕ_0, which leads to the estimate

$$
\|\phi_n - \phi_0\|_{L^2(A)} \le ct^{\frac{1}{4}}M + \|\varphi_t * (\phi_n - \phi_0)\|_{L^2(A)} + ct^{\frac{1}{4}}\|\phi_0\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})},
$$

for every $n \in \mathbb{N}$ and $t > 0$. The first and third term can be made arbitrary small by the choice of $t > 0$ and the second term converges by (A.12). This proves the statement of the lemma for $\Sigma \cong \mathbb{R}^{d-1} \times \{0\}$.

In *Step 2* we consider the general case of a Lipschitz graph Σ. By the definition of the boundary spaces (A.2), it follows immediately from the weak convergence (A.7), that also

$$
\phi_n \circ \Xi \rightharpoonup \phi_0 \circ \Xi \quad \text{weakly in } H^{\frac{1}{2}}(\mathbb{R}^{d-1}).
$$

Since A has finite measure, the preimage $\Xi^{-1}(A) = \{x \in \mathbb{R}^{d-1} \mid \Xi(x) \in A\}$ has finite measure as well, and we conclude from the first step

$$
\phi_n \circ \Xi \to \phi_0 \circ \Xi \quad \text{strongly in } L^2(\Xi^{-1}(A)).
$$

By the definition of the boundary spaces (A.2) this implies (A.8). □

Next we prove the compactness of the multiplication operator M_α for functions α which are close to a constant in the sense of (A.14). Note that, although stated for α, this property only affects the L^∞-part of α. Any function in $L^p(\mathbb{R}^{d-1})$ satisfies (A.14) automatically.

Proposition A.3 *Assume that the function α satisfies*

$$
\{\, x \in \Sigma \mid |\alpha(x)| > \varepsilon \,\} \text{ has finite measure for every } \varepsilon > 0. \tag{A.14}
$$

Then the multiplication operator M_α in (A.1) *is compact.*

Proof From Lemma A.1 we conclude that M_α in (A.1) is an everywhere defined and bounded operator. In order to prove that M_α is compact, we verify that for any sequence $\phi_n \rightharpoonup \phi_0$ weakly in $H^{\frac{1}{2}}(\Sigma)$, the sequence $M_\alpha\phi_n \to M_\alpha\phi_0$ converges strongly in $L^2(\Sigma)$. As in the proof of Lemma A.1, let $\varepsilon > 0$ and decompose the potential into

$$\alpha = \beta_1 + \beta_2 + \gamma.$$

Next, we define the set

$$A_\varepsilon := \left\{ x \in \Sigma \,\middle|\, |\beta_2(x)| > \varepsilon^2 \right\} \cup \left\{ x \in \Sigma \,\middle|\, |\gamma(x)| > \varepsilon^2 \right\}. \tag{A.15}$$

The integrability condition $\beta_2 \in L^p(\Sigma)$ implies that the set $\{|\beta_2| > \varepsilon^2\}$ has finite measure. Furthermore, since $\{|\gamma| > \varepsilon^2\} \subseteq \{|\beta| > \frac{\varepsilon^2}{2}\} \cup \{|\alpha| > \frac{\varepsilon^2}{2}\}$ it follows from the integrability condition $\beta \in L^p(\Sigma)$ and from (A.14) that $\{|\gamma| > \varepsilon^2\}$ also has finite measure. Then Lemma A.2 shows

$$\lim_{n\to\infty} \|\phi_n - \phi_0\|_{L^2(A_\varepsilon)} = 0.$$

This convergence in particular gives an index $N_\varepsilon \in \mathbb{N}$, such that

$$\|\phi_n - \phi_0\|^2_{L^2(A_\varepsilon)} \leq \frac{\varepsilon^2}{C_\varepsilon + \|\gamma\|_{L^\infty(\Sigma)}}, \quad n \geq N_\varepsilon, \tag{A.16}$$

with C_ε the cut-off from (A.4). Then the equations (A.5) & (A.16), as well as the fact that $|\beta_2+\gamma| \leq C_\varepsilon + \|\gamma\|_{L^\infty(\Sigma)}$ on Σ and $|\beta_2+\gamma| \leq 2\varepsilon^2$ on $\Sigma \setminus A_\varepsilon$, we can estimate

$$\begin{aligned}
\left\| |\alpha|^{\frac{1}{2}}(\phi_n - \phi_0) \right\|^2_{L^2(\Sigma)} &\leq \left\| |\beta_1|^{\frac{1}{2}}(\phi_n - \phi_0) \right\|^2_{L^2(\Sigma)} + \left\| |\beta_2 + \gamma|^{\frac{1}{2}}(\phi_n - \phi_0) \right\|^2_{L^2(A_\varepsilon)} \\
&\quad + \left\| |\beta_2 + \gamma|^{\frac{1}{2}}(\phi_n - \phi_0) \right\|^2_{L^2(\Sigma\setminus A_\varepsilon)} \\
&\leq \varepsilon^2 c_E^2 \|\phi_n - \phi_0\|^2_{H^{\frac{1}{2}}(\Sigma)} + \varepsilon^2 + 2\varepsilon^2 \|\phi_n - \phi_0\|^2_{L^2(\Sigma\setminus A_\varepsilon)} \\
&\leq \varepsilon^2 \Big((c_E^2 + 2)\|\phi_n - \phi_0\|^2_{H^{\frac{1}{2}}(\Sigma)} + 1 \Big), \quad n \geq N_\varepsilon.
\end{aligned}$$

Since $\|\phi_n - \phi_0\|_{H^{1/2}(\Sigma)}$ on the right hand side is bounded as a consequence of the weak $H^{1/2}$-convergence, this inequality implies the norm convergence

$$\lim_{n\to\infty} \left\| |\alpha|^{\frac{1}{2}}(\phi_n - \phi_0) \right\|^2_{L^2(\Sigma)} = 0,$$

and hence the compactness of the operator M_α. □

References

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics*, AMS Chelsea Publishing, Providence (2005).
2. P. R. S. Antunes, R. D. Benguria, V. Lotoreichik, T. Ourmières-Bonafos, *A variational formulation for Dirac operators in bounded domains. Applications to spectral geometric inequalities*, Commun. Math. Phys. **386** (2021), 781–818.
3. P. R. S. Antunes, P. Freitas, D. Krejčiřík, *Bounds and extremal domains for Robin eigenvalues with negative boundary parameter*, Adv. Calc. Var. **10** (2017), 357–380.
4. N. Arrizabalaga, A. Mas, L. Vega, *An isoperimetric-type inequality for electrostatic shell interactions for Dirac operators*, Commun. Math. Phys. **344** (2016), 483–505.
5. I. A. Baernstein, *Symmetrization in analysis*, Cambridge University Press, Cambridge, 2019. With David Drasin and Richard S. Laugesen, With a foreword by Walter Hayman.
6. J. Behrndt, P. Exner, and V. Lotoreichik, Schrödinger operators with δ- and δ'-interactions on Lipschitz surfaces and chromatic numbers of associated partitions, *Rev. Math. Phys.* **26** (2014), 1450015, 43 pp.
7. J. Behrndt, R. L. Frank, C. Kühn, V. Lotoreichik, J. Rohleder, *Spectral theory for Schrödinger operators with δ-interactions supported on curves in $\mathbb{R}^3$*, Ann. Henri Poincaré **18** (2017), 1305–1347.
8. J. Behrndt, M. Langer, V. Lotoreichik, *Schrödinger operators with δ and δ'-potentials supported on hypersurfaces*, Ann. Henri Poincaré **14** (2013), 385–423.
9. J. Behrndt, M. Langer, V. Lotoreichik, J. Rohleder, *Spectral enclosures for non-self-adjoint extensions of symmetric operators*, J. Funct. Anal. **275** (2018), 1808–1888.
10. J. Behrndt, P. Schlosser, *Quasi boundary triples, self-adjoint extensions, and Robin Laplacians on the half-space*, Operator Theory Advances Applications **275** (2019), 49–66.
11. P. M. Bhattacharyya, *Distributions. Generalized Functions with Applications in Sobolev Spaces*, De Gruyter, Berlin (2012).
12. M. H. Bossel, *Membranes élastiquement liées: Extension du théoréme de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger*, C. R. Acad. Sci. Paris Sér. Math. **302** (1986), 47–50.
13. J. F. Brasche, P. Exner, Y. A. Kuperin, P. Šeba, *Schrödinger operators with singular interactions*, J. Math. Anal. Appl. **184** (1994), 112–139.
14. D. Bucur, V. Ferone, C. Nitsch, C. Trombetti, *A sharp estimate for the first Robin-Laplacian eigenvalue with negative boundary parameter*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. **30** (2019), 665–676.
15. M. S. Birman, M. Z. Solomjak, *Spectral Theory of Self-adjoint Operators in Hilbert Space*, Kluwer, Dordrecht (1987).
16. D. Daners, *A Faber-Krahn inequality for Robin problems in any space dimension*, Math. Ann. **335** (2006), 767–785.
17. D. E. Edmunds, W. D. Evans, *Spectral Theory and Differential Operators*, Oxford University Press (2018).
18. L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence (2010).
19. P. Exner, *An isoperimetric problem for leaky loops and related mean-chord inequalities*, J. Math. Phys. **46** (2005), 062105.
20. P. Exner, E. M. Harrell, M. Loss, *Inequalities for means of chords, with application to isoperimetric problems*, Lett. Math. Phys. **75** (2006), 225–233.
21. P. Exner, S. Kondej, *Spectral optimization for strongly singular Schrödinger operators with a star-shaped interaction*, Lett. Math. Phys. **110** (2020), 735–751.
22. P. Exner, H. Kovařík, *Quantum waveguides*, Theoretical and Mathematical Physics, Springer, Cham (2015).
23. P. Exner, V. Lotoreichik, *A spectral isoperimetric inequality for cones*, Lett. Math. Phys. **107** (2017), 717–732.
24. R. L. Frank, A. Laptev, *Spectral inequalities for Schrödinger operators with surface potentials*, Spectral Theory of Differential Operators , Amer. Math. Soc. Transl. Ser. 2, **225** (2008), pp. 91–102.

25. P. Freitas, D. Krejčiřík, *The first Robin eigenvalue with negative boundary parameter*, Adv. Math. **280** (2015), 322–339.
26. P. Freitas, R. S. Laugesen, *From Steklov to Neumann and beyond, via Robin: the Szegő way*, Can. J. Math. **72(4)** (2020), 1024-1043.
27. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin (2001).
28. A. Girouard, R. S. Laugesen, *Robin spectrum: two disks maximize the third eigenvalue*, `arXiv:1907.13173`.
29. D. Haroske, H. Triebel, *Distributions, Sobolev Spaces, Elliptic Equations*, Eur. Math. Soc., Zürich (2008).
30. A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Birkhäuser, Basel (2006).
31. A. Henrot, *Shape Optimization and Spectral Theory*, De Gruyter, Warsaw (2017).
32. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin (1976).
33. M. Khalile, V. Lotoreichik, *Spectral isoperimetric inequalities for Robin Laplacians on 2-manifolds and unbounded cones*,to appear in J. Spectral Theory, `arXiv:1909.10842`.
34. D. Krejčiřík, V. Lotoreichik, *Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, II: nonconvex domains and higher dimensions*, Potential Anal. **52** (2020), 601–614.
35. E. H. Lieb, M. Loss, *Analysis. Second Edition*, Amer. Math. Soc., Providence (2001).
36. V. Lotoreichik, *Spectral isoperimetric inequalities for singular interactions on open arcs*, Appl. Anal. **98** (2019), 1451–1460.
37. V. Lotoreichik, *Spectral isoperimetric inequality for the δ'-interaction on a contour*, Michelangeli A. (eds) Mathematical Challenges of Zero-Range Physics, Springer INdAM Series **42**, Springer, Cham (2021)
38. W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge (2000).
39. V. Rabinovich, *Essential spectrum of Schrödinger operators with δ and δ'-interactions on systems of unbounded smooth hypersurfaces in $\mathbb{R}^n$*, Differential Equations, Mathematical Physics, and Applications: Selim Grigorievich Krein centennial, 293–310, Contemp. Math. 734, Amer. Math. Soc., Providence, (2019).
40. M. Reed, B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press Inc. (1978)
41. B. Simon, *Trace Ideals and their Applications*, Amer. Math. Soc., Providence (2005).
42. G. Teschl, *Mathematical Methods in Quantum Mechanics. With applications to Schrödinger operators*, Amer. Math. Soc., Providence (2009)