Generalized Resolvents of a Class of Symmetric Operators in Krein Spaces

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Abstract. Let A be a closed symmetric operator of defect one in a Krein space $\mathcal{K}$ and assume that A possesses a self-adjoint extension in $\mathcal{K}$ which locally has the same spectral properties as a definitizable operator. We show that the Krein-Naimark formula establishes a bijective correspondence between the compressed resolvents of locally definitizable self-adjoint extensions $\widetilde{A}$ of A acting in Krein spaces $\mathcal{K}\times\mathcal{H}$ and a special subclass of meromorphic functions.

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1. Introduction

Let A be a densely defined closed symmetric operator with defect one in a Hilbert space $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for the adjoint operator A^*. Let A_0 be the self-adjoint extension $A^* \upharpoonright \ker \Gamma_0$ of A in $\mathcal{K}$ and denote the γ-field and Weyl function corresponding to the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ by γ and M, respectively. Here M is a scalar Nevanlinna function, that is, it maps the upper half plane $\mathbb{C}^+$ holomorphically into $\mathbb{C}^+ \cup \mathbb{R}$ and is symmetric with respect to the real axis. It is well known that in this case the Krein-Naimark formula

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^* \tag{1.1}$$

establishes a bijective correspondence between the class of Nevanlinna functions τ (including the constant ∞) and the compressed resolvents of $\mathcal{K}$-minimal self-adjoint extensions $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, cf. [22, 32]. The compressed resolvent on the left hand side of (1.1) is said to be a generalized resolvent of A. We note that if A has equal deficiency indices > 1 the generalized resolvents of A can still be described with formula (1.1), where the parameters τ belong to the class of Nevanlinna families, cf. [11, 23, 30, 31].

Various generalizations of the Krein-Naimark formula in an indefinite setting have been proved in the last decades. The case that A is a symmetric operator in a Pontryagin space $\mathcal{K}$ and $\mathcal{H}$ is a Hilbert space was investigated by M.G. Krein and H. Langer in [24]. Later V. Derkach considered both $\mathcal{K}$ and $\mathcal{H}$ to be Pontryagin or even Krein spaces, cf. [10]. In the general situation of Krein spaces $\mathcal{K}$ and $\mathcal{H}$ one obtains a correspondence between locally holomorphic relation-valued functions τ and self-adjoint extensions $\widetilde{A}$ of A with a non-empty resolvent set. Under additional assumptions other variants of (1.1) were proved in [6, 7, 8, 9, 10, 14, 27]. If e.g. $\mathcal{H}$ is a Pontryagin space, then the parameters τ belong to the class of $\mathcal{N}_\kappa$-families, a class of relation-valued functions which includes the generalized Nevanlinna functions. If $\mathcal{H}$ is a Krein space and the hermitian forms $[A\cdot,\cdot]$ and $[\widetilde{A}\cdot,\cdot]$ both have finitely many negative squares, then τ belongs to a special subclass of the definitizable functions, cf. [7, 19].

It is the aim of this paper to prove a new variant of formula (1.1). Here we allow both $\mathcal{K}$ and $\mathcal{H}$ to be Krein spaces and we assume that A is of defect one and possesses a self-adjoint extension A_0 in $\mathcal{K}$ which locally has the same spectral properties as a definitizable operator or relation, cf. [20, 28]. Under the assumption that $\widetilde{A}$ is also locally definitizable and that its sign types coincide "in essence" (i.e. with the exception of a discrete set, see Definition 2.6) with the sign types of A_0 we prove in Theorem 3.2 that there exists a so-called locally definitizable function τ such that (1.1) holds. The proof is based on a coupling method developed in [11, §5] and a recent perturbation result from [4]. One of the main difficulties here is to show that the symmetric relation $\widetilde{A}\cap\mathcal{H}^2$ possesses a self-adjoint extension in the Krein space $\mathcal{H}$ with a non-empty resolvent set and to choose a boundary value space for the adjoint of $\widetilde{A}\cap\mathcal{H}^2$ in $\mathcal{H}$ such that (1.1) holds with the corresponding Weyl function τ. In connection with a class of abstract λ-dependent boundary value problems the converse direction was already proved in [3], i.e., for a given locally definitizable function τ a self-adjoint extension $\widetilde{A}$ of A in $\mathcal{K}\times\mathcal{H}$ such that (1.1) holds was constructed.

The paper is organized as follows. In Section 2 we recall the definitions and basic properties of locally definitizable self-adjoint operators and relations and the class of locally definitizable functions introduced and studied by P. Jonas, see e.g. [20, 21]. The notion of d-compatibility of sign types of locally definitizable relations and functions is defined in the end of Section 2.3. In the beginning of Section 3 we recall some basics on boundary value spaces and associated Weyl functions. Section 3.2 contains our main result. We prove in Theorem 3.2 that formula (1.1) establishes a bijective correspondence between an appropriate subclass of the locally definitizable functions and the compressed resolvents of locally definitizable $\mathcal{K}$-minimal self-adjoint exit space extensions $\widetilde{A}$ of A in a Krein space $\mathcal{K}\times\mathcal{H}$ with spectral sign types d-compatible to those of A_0. Finally, in the end of Section 3.2, we formulate a variant of the Krein-Naimark formula for self-adjoint extensions A_0 and $\widetilde{A}$ of A in $\mathcal{K}$ and $\mathcal{K}\times\mathcal{H}$, respectively, which locally have the same spectral properties as self-adjoint operators or relations in Pontryagin spaces and functions τ from the local generalized Nevanlinna class.

2. Locally definitizable self-adjoint relations and locally definitizable functions

2.1. Notations and definitions

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space with a corresponding fundamental symmetry J. The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K})$. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators in $\mathcal{K}$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [15] and [16].

The sum and the direct sum of subspaces in $\mathcal{K}^2$ are denoted by $\boldsymbol{+}$ and $\dot{\boldsymbol{+}}$. We define an indefinite inner product on $\mathcal{K}^2$ by

$$[\![\hat f,\hat g]\!] := i\big([f,g'] - [f',g]\big), \quad \hat f = \begin{pmatrix} f \\ f' \end{pmatrix},\ \hat g = \begin{pmatrix} g \\ g' \end{pmatrix} \in \mathcal{K}^2.$$

Then $(\mathcal{K}^2, [\![\cdot,\cdot]\!])$ is a Krein space and $\mathcal{J} = \begin{pmatrix} 0 & -iJ \\ iJ & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{K}^2)$ is a corresponding fundamental symmetry. If necessary we will indicate the underlying space by subscripts, e.g. $[\![\cdot,\cdot]\!]_{\mathcal{K}^2}$.

Let A be a linear relation in $\mathcal{K}$. The adjoint relation $A^+ \in \widetilde{\mathcal{C}}(\mathcal{K})$ is defined as

$$A^+ := A^{[\![\perp]\!]} = \big\{\hat h \in \mathcal{K}^2 \mid [\![\hat h, \hat f\,]\!] = 0 \text{ for all } \hat f \in A\big\},$$

where $A^{[\![\perp]\!]}$ denotes the orthogonal companion of A with respect to $[\![\cdot,\cdot]\!]$. A is said to be *symmetric* (*self-adjoint*) if $A \subset A^+$ (resp. $A = A^+$).

Let S be a closed linear relation in $\mathcal{K}$. The resolvent set $\rho(S)$ of $S \in \widetilde{\mathcal{C}}(\mathcal{K})$ is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde\sigma(S)$ of S is defined by $\widetilde\sigma(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde\sigma(S) = \sigma(S) \cup \{\infty\}$ otherwise. A point $\lambda \in \mathbb{C}$ is called a *point of regular type* of S, $\lambda \in r(S)$, if $(S-\lambda)^{-1}$ is a bounded operator. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* S, denoted by $\sigma_{ap}(S)$, if there exists a sequence $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in S$, $n = 1,2,\ldots$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$. The *extended approximate point spectrum* $\widetilde\sigma_{ap}(S)$ *of* S is defined by

$$\widetilde\sigma_{ap}(S) := \begin{cases} \sigma_{ap}(S) \cup \{\infty\} & \text{if} \quad 0 \in \sigma_{ap}(S^{-1}) \\ \sigma_{ap}(S) & \text{if} \quad 0 \notin \sigma_{ap}(S^{-1}) \end{cases}.$$

We remark, that the boundary points of $\widetilde\sigma(S)$ in $\overline{\mathbb{C}}$ belong to $\widetilde\sigma_{ap}(S)$.

Next we recall the definitions of the spectra of positive and negative type of self-adjoint relations, cf. [20] (for bounded self-adjoint operators see [29]). For equivalent descriptions of the spectra of positive and negative type we refer to [20, Theorem 3.18].

Definition 2.1. Let A_0 be a self-adjoint relation in $\mathcal{K}$. A point $\lambda \in \sigma_{ap}(A_0)$ is said to be of *positive type* (*negative type*) with respect to A_0, if for every sequence $\binom{x_n}{y_n} \in A_0$, $n = 1, 2 \ldots$, with $\|x_n\| = 1$, $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$ we have

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad \big(\text{resp. } \limsup_{n\to\infty}\, [x_n, x_n] < 0\big).$$

If $\infty \in \widetilde{\sigma}_{ap}(A_0)$, then ∞ is said to be of *positive type* (*negative type*) with respect to A_0 if 0 is of positive type (resp. negative type) with respect to A_0^{-1}. We denote the set of all points of $\widetilde{\sigma}_{ap}(A_0)$ of positive type (negative type) by $\sigma_{++}(A_0)$ (resp. $\sigma_{--}(A_0)$).

We remark that the self-adjointness of the relation A_0 yields that the points of positive and negative type introduced in Definition 2.1 belong to $\overline{\mathbb{R}}$.

An open subset Δ of $\overline{\mathbb{R}}$ is said to be of *positive type* (*negative type*) with respect to A_0 if each point $\lambda \in \Delta \cap \widetilde{\sigma}(A_0)$ is of positive type (resp. negative type) with respect to A_0. An open set Δ is called of *definite type* with respect to A_0 if it is either of positive or of negative type with respect to A_0.

For each $\lambda \in \sigma_{++}(A_0)$ ($\sigma_{--}(A_0)$) there exists an open neighbourhood $\mathcal{U}_\lambda$ in $\overline{\mathbb{C}}$ such that $(\mathcal{U}_\lambda \cap \widetilde{\sigma}(A_0) \cap \overline{\mathbb{R}}) \subset \sigma_{++}(A_0)$ (resp. $(\mathcal{U}_\lambda \cap \widetilde{\sigma}(A_0) \cap \overline{\mathbb{R}}) \subset \sigma_{--}(A_0)$), $\mathcal{U}_\lambda \backslash \overline{\mathbb{R}} \subset \rho(A_0)$ and

$$\|(A_0 - \lambda)^{-1}\| \leq M |\text{Im}\ \lambda|^{-1}$$

holds for some $M > 0$ and all $\lambda \in \mathcal{U}_\lambda \backslash \overline{\mathbb{R}}$, cf. [1], [20] (and [29] for bounded operators).

2.2. Locally definitizable self-adjoint relations

In this section we briefly recall the notion of locally definitizable self-adjoint relations and intervals of type π_+ and type π_- from [20].

Let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

Definition 2.2. Let A_0 be a self-adjoint relation in the Krein space $\mathcal{K}$ such that $\sigma(A_0) \cap (\Omega \backslash \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A_0, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A_0 in Ω. The relation A_0 is said to be *definitizable over* Ω, if the following holds.

(i) Every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to A_0.

(ii) For every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, there exists $m \geq 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that

$$\|(A_0 - \lambda)^{-1}\| \leq M(1 + |\lambda|)^{2m-2}\, |\text{Im}\ \lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U} \backslash \overline{\mathbb{R}}$.

By [20, Theorem 4.7] a self-adjoint relation A_0 in $\mathcal{K}$ is definitizable over $\overline{\mathbb{C}}$ if and only if A_0 is *definitizable*, that is, the resolvent set of A_0 is non-empty and there exists a rational function $r \neq 0$ with poles only in $\rho(A_0)$ such that $r(A_0) \in \mathcal{L}(\mathcal{K})$ is a nonnegative operator in $\mathcal{K}$, that is

$$[r(A_0)x, x] \geq 0$$

holds for all $x \in \mathcal{K}$ (see [28] and [16, §4 and §5]).

Let $A_0 = A_0^+$ be definitizable over Ω and let $\delta \mapsto E(\delta)$ be the local spectral function of A_0 on $\Omega \cap \overline{\mathbb{R}}$. Recall that $E(\delta)$ is defined for all finite unions δ of connected subsets of $\Omega \cap \overline{\mathbb{R}}$, $\overline{\delta} \subset \Omega \cap \overline{\mathbb{R}}$, the endpoints of which belong to $\Omega \cap \overline{\mathbb{R}}$ and are of definite type with respect to A_0 (see [20, Section 3.4 and Remark 4.9]). With the help of the local spectral function $E(\cdot)$ the open subsets of definite type in $\Omega \cap \overline{\mathbb{R}}$ can be characterized in the following way. An open subset Δ, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, is of positive type (negative type) with respect to A_0 if and only if for every finite union δ of open connected subsets of Δ, $\overline{\delta} \subset \Delta$, such that the boundary points of δ in $\overline{\mathbb{R}}$ are of definite type with respect to A_0, the spectral subspace $(E(\delta)\mathcal{K}, [\cdot,\cdot])$ (resp. $(E(\delta)\mathcal{K}, -[\cdot,\cdot])$) is a Hilbert space (cf. [20, Theorem 3.18]).

We say that an open subset Δ, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, is of *type* π_+ (*type* π_-) with respect to A_0 if for every finite union δ of open connected subsets of Δ, $\overline{\delta} \subset \Delta$, such that the boundary points of δ in $\overline{\mathbb{R}}$ are of definite type with respect to A_0 the spectral subspace $(E(\delta)\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (resp. positivity). We shall say that A_0 is of *type* π_+ *over* Ω (*type* π_- *over* Ω) if $\Omega \cap \overline{\mathbb{R}}$ is of type π_+ (resp. type π_-) with respect to A_0 and $\sigma(A_0) \cap \Omega \backslash \overline{\mathbb{R}}$ consists of eigenvalues with finite algebraic multiplicity.

We remark, that spectral points in sets of type π_+ and type π_- can also be characterized with the help of approximative eigensequences (see [1, 2]).

2.3. Matrix-valued locally definitizable functions

In this section we recall the definition of matrix-valued locally definitizable functions from [21]. Although in the formulation of the main theorem in Section 3.2 below only scalar locally definitizable functions appear, matrix-valued functions will be used within the proof.

Let Ω be a domain as in the beginning of Section 2.2 and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued piecewise meromorphic function in $\Omega \backslash \overline{\mathbb{R}}$ which is symmetric with respect to the real axis, that is $\tau(\overline{\lambda}) = \tau(\lambda)^*$ for all points λ of holomorphy of τ. If, in addition, no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of nonreal poles of τ we write $\tau \in M^{n \times n}(\Omega)$. The set of the points of holomorphy of τ in $\Omega \backslash \overline{\mathbb{R}}$ and all points $\mu \in \Omega \cap \mathbb{R}$ such that τ can be analytically continued to μ and the continuations from $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(\tau)$.

The following definition of sets of positive and negative type with respect to matrix functions and Definition 2.4 below of locally definitizable matrix functions can be found in [21].

Definition 2.3. Let $\tau \in M^{n\times n}(\Omega)$. An open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is said to be of *positive type* with respect to τ if for every $x \in \mathbb{C}^n$ and every sequence (μ_k) of points in $\Omega \cap \mathbb{C}^+ \cap \mathfrak{h}(\tau)$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{k\to\infty} \operatorname{Im}\big(\tau(\mu_k)x, x\big) \geq 0.$$

An open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is said to be of *negative type* with respect to τ if Δ is of positive type with respect to $-\tau$. Δ is said to be of *definite type* with respect to τ if Δ is of positive or of negative type with respect to τ.

Definition 2.4. A function $\tau \in M^{n\times n}(\Omega)$ is called *definitizable in* Ω if the following holds.

(i) Every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to τ.

(ii) For every finite union Δ of open connected subsets in $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, there exist $m \geq 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that

$$\|\tau(\lambda)\| \leq M(1+|\lambda|)^{2m}\,|\operatorname{Im}\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U}\backslash\overline{\mathbb{R}}$.

The class of $\mathcal{L}(\mathbb{C}^n)$-valued definitizable functions in Ω will be denoted by $\mathcal{D}^{n\times n}(\Omega)$. In the case $n = 1$ we write $\mathcal{D}(\Omega)$ instead of $\mathcal{D}^{1\times 1}(\Omega)$ and we set

$$\widetilde{\mathcal{D}}(\Omega) := \mathcal{D}(\Omega) \cup \{d_\infty\},$$

where d_∞ denotes the relation $\big\{\binom{0}{c} \,|\, c \in \mathbb{C}\big\} \in \widetilde{\mathcal{C}}(\mathbb{C})$.

A function $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ which is definitizable in $\overline{\mathbb{C}}$ is called *definitizable*, see [20]. We note that $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ is definitizable if and only if there exists a rational function g symmetric with respect to the real axis such that the poles of g belong to $\mathfrak{h}(\tau) \cup \{\infty\}$ and $g\tau$ is the sum of a Nevanlinna function and a meromorphic function in $\overline{\mathbb{C}}$ (cf. [20]). For a comprehensive study of definitizable functions we refer to the papers [18, 19] of P. Jonas. We mention only that the generalized Nevanlinna class is a subclass of the definitizable functions. Recall that a function $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ is called a *generalized Nevanlinna function* if the kernel K_τ,

$$K_\tau(\lambda,\mu) = \frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}},$$

has finitely many negative squares (see [25] and [26]).

In [21] it is shown that a function $\tau \in M^{n\times n}(\Omega)$ is definitizable in Ω if and only if for every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$ such that $\overline{\Delta} \subset \Omega\cap\overline{\mathbb{R}}$, τ can be written as the sum

$$\tau = \tau_0 + \tau_{(0)} \tag{2.1}$$

of an $\mathcal{L}(\mathbb{C}^n)$-valued definitizable function τ_0 and an $\mathcal{L}(\mathbb{C}^n)$-valued function $\tau_{(0)}$ which is locally holomorphic on $\overline{\Delta}$. We say that a locally definitizable function

$\tau \in \mathcal{D}^{n\times n}(\Omega)$ is a *generalized Nevanlinna function in* Ω if the function τ_0 in (2.1) can be choosen as a generalized Nevanlinna function.

The class of $\mathcal{L}(\mathbb{C}^n)$-valued generalized Nevanlinna functions in Ω will be denoted by $\mathcal{N}^{n\times n}(\Omega)$. In the case $n = 1$ we write $\mathcal{N}(\Omega)$ instead of $\mathcal{N}^{1\times 1}(\Omega)$ and we set

$$\widetilde{\mathcal{N}}(\Omega) := \mathcal{N}(\Omega) \cup \{d_\infty\},$$

where d_∞ denotes the relation $\left\{ \binom{0}{c} \mid c \in \mathbb{C} \right\} \in \widetilde{\mathcal{C}}(\mathbb{C})$.

The following theorem is a consequence of [21, Propositions 2.8 and 3.4]. It establishes a connection between self-adjoint relations which are locally definitizable (locally of type π_+) and $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable functions (resp. local generalized Nevanlinna functions).

Theorem 2.5. *Let* Ω *be a domain as above and let* A_0 *be a self-adjoint relation in the Krein space* $\mathcal{K}$ *which is definitizable over* Ω*. Let* $\gamma \in \mathcal{L}(\mathbb{C}^n, \mathcal{K})$ *and* $S = S^* \in \mathcal{L}(\mathbb{C}^n)$*, fix some point* $\lambda_0 \in \rho(A_0) \cap \Omega$ *and define*

$$\tau(\lambda) := S + \gamma^+\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma$$

for all $\lambda \in \rho(A_0) \cap \Omega$*. Then the following holds.*

(i) *The function* τ *is definitizable in* Ω*,* $\tau \in \mathcal{D}^{n\times n}(\Omega)$*.*

(ii) *If* A_0 *is of type* π_+ *over* Ω*, then* τ *belongs to* $\mathcal{N}^{n\times n}(\Omega)$*.*

(iii) *An open subset* Δ *of* $\Omega \cap \overline{\mathbb{R}}$ *which is of positive type (negative type) with respect to* A_0 *is of positive type (resp. negative type) with respect to* τ*.*

In the sequel we shall often assume that the sign types of self-adjoint relations which are definitizable over Ω, and definitizable functions in Ω coincide outside of a discrete set in $\Omega \cap \overline{\mathbb{R}}$. A notion for this concept is introduced in the next definition, cf. [3, Definition 2.8].

Definition 2.6. Let A_0 and A_1 be self-adjoint relations which are definitizable over Ω and let τ and $\widetilde{\tau}$ be $\mathcal{L}(\mathbb{C}^n)$-valued definitizable functions in Ω. We shall say that *the sign types of* A_0 *and* A_1 (A_0 *and* τ, τ *and* $\widetilde{\tau}$) *are d-compatible in* Ω if for every $\mu \in \Omega \cap \overline{\mathbb{R}}$ there exists an open connected neighbourhood $I_\mu \subset \Omega \cap \overline{\mathbb{R}}$ of μ such that each component of $I_\mu \backslash \{\mu\}$ is either of positive type with respect to A_0 and A_1 (resp. A_0 and τ, τ and $\widetilde{\tau}$) or of negative type with respect to A_0 and A_1 (resp. A_0 and τ, τ and $\widetilde{\tau}$).

If A_0 is definitizable over Ω and the function $\tau \in \mathcal{D}^{n\times n}(\Omega)$ is defined as in Theorem 2.5, then obviously the sign types of A_0 and τ are d-compatible in Ω. A typical nontrivial situation where d-compatibility of sign types of locally definitizable self-adjoint relations appears is shown in Theorem 2.7 below. For a proof we refer to [4, Theorem 3.2].

Theorem 2.7. *Let* A_0 *and* A_1 *be self-adjoint relations in* $\mathcal{K}$*, assume that the set* $\rho(A_0) \cap \rho(A_1) \cap \Omega$ *is non-empty and that* A_0 *is definitizable over* Ω*. If*

$$(A_1 - \mu)^{-1} - (A_0 - \mu)^{-1}$$

is a finite rank operator for some (and hence for all) $\mu \in \rho(A_0) \cap \rho(A_1) \cap \Omega$, then A_1 is definitizable over Ω and the sign types of A_0 and A_1 are d-compatible in Ω.

3. Generalized resolvents of a class of symmetric operators

3.1. Boundary value spaces and Weyl functions associated with symmetric relations in Krein spaces

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space, let J be a corresponding fundamental symmetry and let A be a closed symmetric relation in $\mathcal{K}$. We say that A has *defect* $m \in \mathbb{N} \cup \{\infty\}$, if both deficiency indices

$$n_\pm(JA) = \dim \ker((JA)^* - \overline{\lambda}), \qquad \lambda \in \mathbb{C}^\pm,$$

of the symmetric relation JA in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$ are equal to m. Here * denotes the Hilbert space adjoint. We remark, that this is equivalent to the fact that there exists a self-adjoint extension of A in $\mathcal{K}$ and that each self-adjoint extension $\hat{A}$ of A in $\mathcal{K}$ satisfies $\dim\big(\hat{A}/A\big) = m$.

We shall use the so-called boundary value spaces for the description of the self-adjoint extensions of closed symmetric relations in Krein spaces. The following definition is taken from [10].

Definition 3.1. Let A be a closed symmetric relation in the Krein space $\mathcal{K}$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary value space* for A^+ if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ are mappings such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G}^2$ is surjective, and the relation

$$[\![\Gamma\hat{f}, \Gamma\hat{g}]\!]_{\mathcal{G}^2} = [\![\hat{f}, \hat{g}]\!]_{\mathcal{K}^2}$$

holds for all $\hat{f}, \hat{g} \in A^+$.

In the following we recall some basic facts on boundary value spaces which can be found in e.g. [8] and [10]. For the Hilbert space case we refer to [17], [12] and [13]. Let A be a closed symmetric relation in $\mathcal{K}$. Then

$$\mathcal{N}_{\lambda,A^+} := \ker(A^+ - \lambda) = \operatorname{ran}\,(A - \overline{\lambda})^{[\perp]}$$

denotes the defect subspace of A at the point $\lambda \in r(A)$ and we set

$$\hat{\mathcal{N}}_{\lambda,A^+} := \left\{ \binom{f_\lambda}{\lambda f_\lambda} \big| f_\lambda \in \mathcal{N}_{\lambda,A^+} \right\}.$$

When no confusion can arise we write $\mathcal{N}_\lambda$ and $\hat{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_{\lambda,A^+}$ and $\hat{\mathcal{N}}_{\lambda,A^+}$.

If there exists a self-adjoint extension A' of A such that $\rho(A') \neq \emptyset$ then we have

$$A^+ = A' \dotplus \hat{\mathcal{N}}_\lambda \quad \text{for all} \quad \lambda \in \rho(A').$$

In this case there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that $\ker \Gamma_0 = A'$ (cf. [10]).

Let in the following A, $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ and Γ be as in Definition 3.1. It follows that the mappings Γ_0 and Γ_1 are continuous. The self-adjoint extensions

$$A_0 := \ker\Gamma_0 \qquad \text{and} \qquad A_1 := \ker\Gamma_1$$

of A are transversal, i.e. $A_0\cap A_1 = A$ and $A_0 \dotplus A_1 = A^+$. The mapping Γ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \bigl\{\hat f \in A^+ \,|\, \Gamma\hat f \in \Theta\bigr\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{3.1}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between the set of all closed linear relations $\widetilde{\mathcal{C}}(\mathcal{G})$ in $\mathcal{G}$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (3.1) gives a one-to-one correspondence between the closed symmetric (self-adjoint) extensions of A and the closed symmetric (resp. self-adjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker\bigl(\Gamma_1 - \Theta\Gamma_0\bigr). \tag{3.2}$$

Assume that $\rho(A_0) \neq \emptyset$ and denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}\times\mathcal{K}$. For every $\lambda \in \rho(A_0)$ we define the operators

$$\gamma(\lambda) := \pi_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G},\mathcal{K}) \quad\text{and}\quad M(\lambda) := \Gamma_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}).$$

The functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto M(\lambda)$ are called the *γ-field* and *Weyl function* corresponding to $\{\mathcal{G},\Gamma_0,\Gamma_1\}$. They are holomorphic on $\rho(A_0)$ and the relations

$$\gamma(\zeta) = (1 + (\zeta-\lambda)(A_0-\zeta)^{-1})\gamma(\lambda) \tag{3.3}$$

and

$$M(\lambda) - M(\zeta)^* = (\lambda-\overline{\zeta})\gamma(\zeta)^+\gamma(\lambda) \tag{3.4}$$

hold for all $\lambda,\zeta \in \rho(A_0)$ (cf. [10]). It follows that

$$\begin{aligned} M(\lambda) = \operatorname{Re} M(\lambda_0) + {}&\gamma(\lambda_0)^+\bigl((\lambda - \operatorname{Re}\lambda_0) \\ &+ (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A_0-\lambda)^{-1}\bigr)\gamma(\lambda_0) \end{aligned} \tag{3.5}$$

holds for a fixed $\lambda_0 \in \rho(A_0)$ and all $\lambda \in \rho(A_0)$. If, in addition, the condition $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(A_0)\}$ is fulfilled, then it follows from (3.3) and (3.4) that the function M is *strict*, that is

$$\bigcap_{\lambda\in\mathfrak{h}(M)} \ker\left(\frac{M(\lambda)-M(\mu)^*}{\lambda-\overline{\mu}}\right) = \{0\} \tag{3.6}$$

holds for some (and hence for all) $\mu \in \mathfrak{h}(\tau)$.

If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A (see (3.1)), then for every point $\lambda \in \rho(A_0)$ we have

$$\lambda \in \rho(A_\Theta) \qquad \text{if and only if} \qquad 0 \in \rho(\Theta - M(\lambda)). \tag{3.7}$$

For $\lambda \in \rho(A_\Theta)\cap\rho(A_0)$ the well-known resolvent formula

$$(A_\Theta-\lambda)^{-1} = (A_0-\lambda)^{-1} + \gamma(\lambda)\bigl(\Theta - M(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^+ \tag{3.8}$$

holds (for a proof see e.g. [10]).

3.2. A variant of the Krein-Naimark formula

We choose a domain Ω as in the beginning of Section 2.2. Let A be a (not necessarily densely defined) closed symmetric operator in the Krein space $\mathcal{K}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ and let $\mathcal{H}$ be a further Krein space. A self-adjoint extension $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ is said to be an *exit space extension of* A and $\mathcal{H}$ is called the *exit space*. The exit space extension $\widetilde{A}$ of A is said to be $\mathcal{K}$-*minimal* if $\rho(\widetilde{A}) \cap \Omega$ is non-empty and

$$\mathcal{K} \times \mathcal{H} = \operatorname{clsp}\left\{\mathcal{K}, (\widetilde{A} - \lambda)^{-1}\mathcal{K} \,|\, \lambda \in \rho(\widetilde{A}) \cap \Omega\right\}$$

holds. Note, that the defintion of $\mathcal{K}$-minimality depends on the domain Ω. The elements of $\mathcal{K} \times \mathcal{H}$ will be written in the form $\{k, h\}$, $k \in \mathcal{K}$, $h \in \mathcal{H}$. Let $P_{\mathcal{K}} : \mathcal{K} \times \mathcal{H} \to \mathcal{H}$, $\{k, h\} \mapsto k$, be the projection onto the first component of $\mathcal{K} \times \mathcal{H}$. Then the compression

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}, \qquad \lambda \in \rho(\widetilde{A}),$$

of the resolvent of $\widetilde{A}$ to $\mathcal{K}$ is called a *generalized resolvent* of A.

Theorem 3.2. *Let A be a closed symmetric operator of defect one in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, $A_0 = \ker \Gamma_0$, be a boundary value space for A^+ with corresponding γ-field γ and Weyl function M. Assume that A_0 is definitizable over Ω and that the condition $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_{\lambda,A^+} \,|\, \lambda \in \rho(A_0) \cap \Omega\}$ is fulfilled. Then the following assertions hold.*

(i) *Let $\widetilde{A}$ be a $\mathcal{K}$-minimal self-adjoint exit space extension of A in $\mathcal{K} \times \mathcal{H}$ which is definitizable over Ω and assume that the sign types of $\widetilde{A}$ and A_0 are d-compatible in Ω. Then there exists a locally definitizable function $\tau \in \widetilde{\mathcal{D}}(\Omega)$ such that the sign types of τ, $\widetilde{A}$ and A_0 are d-compatible in Ω,*

$$\rho(\widetilde{A}) \cap \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega$$

is a subset of $\mathfrak{h}((M + \tau)^{-1})$ and the formula

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{3.9}$$

holds for all $\lambda \in \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega$.

(ii) *Let $\tau \in \widetilde{\mathcal{D}}(\Omega)$ be a locally definitizable function such that $M(\mu) + \tau(\mu) \neq 0$ for some $\mu \in \Omega$, assume that the sign types of τ and A_0 are d-compatible in Ω and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. Then there exists a Krein space $\mathcal{H}$ and a $\mathcal{K}$-minimal self-adjoint exit space extension $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ which is definitizable over Ω', such that the sign types of $\widetilde{A}$, τ and A_0 are d-compatible in Ω',*

$$\rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big((M + \tau)^{-1}\big) \cap \Omega'$$

is a subset of $\rho(\widetilde{A})$ and formula (3.9) holds for all points λ belonging to $\rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((M + \tau)^{-1}) \cap \Omega'$.

Proof. The proof of assertion (i) consists of four steps. Let $\widetilde{A}$ be a $\mathcal{K}$-minimal self-adjoint exit space extension of A in $\mathcal{K}\times\mathcal{H}$ which is definitizable over Ω such that the sign types of $\widetilde{A}$ and A_0 are d-compatible in Ω.

1. In this first step we prove assertion (i) for the case $\mathcal{H}=\{0\}$. Here $\widetilde{A}$ is a canonical extension of A and therefore, by (3.1), there exists a self-adjoint constant $\tau\in\mathbb{R}\cup\{d_\infty\}$ such that

$$(\widetilde{A}-\lambda)^{-1}=(A_0-\lambda)^{-1}-\gamma(\lambda)\big(M(\lambda)+\tau\big)^{-1}\gamma(\overline{\lambda})^{+}$$

holds (cf. (3.8)), that is, $\widetilde{A}$ coincides with the canonical self-adjoint extension $A_{-\tau}$ of A and by (3.7) we have $\rho(A_{-\tau})\cap\rho(A_0)\subset\mathfrak{h}((M+\tau)^{-1})$. Here each point in $\Omega\cap\overline{\mathbb{R}}$ is of positive as well as of negative type with respect to τ and hence assertion (i) follows.

2. In the following we assume $\mathcal{H}\neq\{0\}$. Following the lines of [11, §5] we define in this step a symmetric relation T in $\mathcal{H}$ and a special boundary value space for the adjoint T^+.

Below we will deal with direct products of linear relations. The following notation will be used. If U is a relation in $\mathcal{K}$ and V is a relation in $\mathcal{H}$ we shall write $U\times V$ for the direct product of U and V which is a relation in $\mathcal{K}\times\mathcal{H}$,

$$U\times V=\left\{\begin{pmatrix}\{f_1,f_2\}\\ \{f_1',f_2'\}\end{pmatrix}\;\Big|\;\begin{pmatrix}f_1\\ f_1'\end{pmatrix}\in U,\;\begin{pmatrix}f_2\\ f_2'\end{pmatrix}\in V\right\}.$$

For the pair $\binom{\{f_1,f_2\}}{\{f_1',f_2'\}}$ we shall also write $\{\hat{f}_1,\hat{f}_2\}$, where $\hat{f}_1=\binom{f_1}{f_1'}$ and $\hat{f}_2=\binom{f_2}{f_2'}$.

The linear relations

$$S:=\widetilde{A}\cap\mathcal{K}^2=\left\{\begin{pmatrix}k\\ k'\end{pmatrix}\;\Big|\;\begin{pmatrix}\{k,0\}\\ \{k',0\}\end{pmatrix}\in\widetilde{A}\right\}$$

and

$$T:=\widetilde{A}\cap\mathcal{H}^2=\left\{\begin{pmatrix}h\\ h'\end{pmatrix}\;\Big|\;\begin{pmatrix}\{0,h\}\\ \{0,h'\}\end{pmatrix}\in\widetilde{A}\right\}$$

are closed and symmetric in $\mathcal{K}$ and $\mathcal{H}$, respectively, and we have $A\subset S$. Let $J_\mathcal{K}$ and $J_\mathcal{H}$ be fundamental symmetries in the Krein spaces $\mathcal{K}$ and $\mathcal{H}$, respectively, and choose

$$J:=\begin{pmatrix}J_\mathcal{K}&0\\ 0&J_\mathcal{H}\end{pmatrix}\in\mathcal{L}(\mathcal{K}\times\mathcal{H})$$

as a fundamental symmetry in the Krein space $\mathcal{K}\times\mathcal{H}$. Then $J_\mathcal{K}S=J\widetilde{A}\cap\mathcal{K}^2$ and $J_\mathcal{H}T=J\widetilde{A}\cap\mathcal{H}^2$ are symmetric relations in the Hilbert spaces $(\mathcal{K},[J_\mathcal{K}\cdot,\cdot])$ and $(\mathcal{H},[J_\mathcal{H}\cdot,\cdot])$, respectively. It follows from [11, §5] that the deficiency indices of $J_\mathcal{K}S$ and $-J_\mathcal{H}T$ coincide. As $J_\mathcal{K}S$ is a symmetric extension of the symmetric operator $J_\mathcal{K}A$ in the Hilbert space $(\mathcal{K},[J_\mathcal{K}\cdot,\cdot])$ the deficiency indices $n_\pm(J_\mathcal{K}S)$ of $J_\mathcal{K}S$ are $(1,1)$ or $(0,0)$.

The case $n_\pm(J_\mathcal{K}S)=0$ is impossible here as otherwise also the relation $J_\mathcal{H}T$ would be self-adjoint in $(\mathcal{H},[J_\mathcal{H}\cdot,\cdot])$ and therefore $J\widetilde{A}$ would coincide with

$J_\mathcal{K}S \times J_\mathcal{H}T$. But as $\widetilde{A} = S \times T$ is by assumption a $\mathcal{K}$-minimal exit space extension of A we would obtain $\mathcal{H} = \{0\}$, a contradiction.

Therefore, it remains to consider the case $n_\pm(J_\mathcal{K}S) = 1$. Then the operators A and S coincide. Let us show that A^+ coincides with

$$R = \left\{ \begin{pmatrix} P_\mathcal{K}\{k,h\} \\ P_\mathcal{K}\{k',h'\} \end{pmatrix} \,\Big|\, \begin{pmatrix} \{k,h\} \\ \{k',h'\} \end{pmatrix} \in \widetilde{A} \right\} = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} \,\Big|\, \begin{pmatrix} \{k,h\} \\ \{k',h'\} \end{pmatrix} \in \widetilde{A} \right\}.$$

In fact, as $\widetilde{A}$ is self-adjoint we have

$$\big[g, P_\mathcal{K}\{k',h'\}\big] - \big[g', P_\mathcal{K}\{k,h\}\big] = \big[\{g,0\},\{k',h'\}\big] - \big[\{g',0\},\{k,h\}\big] = 0$$

for all $\binom{g}{g'} \in A$ and $\{\hat{k},\hat{h}\} \in \widetilde{A}$, $\hat{k} = \binom{k}{k'}$, $\hat{h} = \binom{h}{h'}$. Hence $A \subset R^+$. Similarly it follows that $R^+ \subset A$ holds. Therefore A^+ coincides with the closure of R and as A has finite defect and R is an extension of A we conclude $A^+ = R$. Replacing $P_\mathcal{K}$ by the projection $P_\mathcal{H}$ onto the second component of $\mathcal{K} \times \mathcal{H}$ the same arguments show

$$T^+ = \left\{ \begin{pmatrix} P_\mathcal{H}\{k,h\} \\ P_\mathcal{H}\{k',h'\} \end{pmatrix} \,\Big|\, \begin{pmatrix} \{k,h\} \\ \{k',h'\} \end{pmatrix} \in \widetilde{A} \right\} = \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\Big|\, \begin{pmatrix} \{k,h\} \\ \{k',h'\} \end{pmatrix} \in \widetilde{A} \right\}.$$

We define the mappings $\widehat{P}_\mathcal{K}$ and $\widehat{P}_\mathcal{H}$ by

$$\widehat{P}_\mathcal{K} : \widetilde{A} \to A^+, \ \begin{pmatrix} \{k,h\} \\ \{k',h'\} \end{pmatrix} \mapsto \begin{pmatrix} k \\ k' \end{pmatrix}$$

and

$$\widehat{P}_\mathcal{H} : \widetilde{A} \to T^+, \ \begin{pmatrix} \{k,h\} \\ \{k',h'\} \end{pmatrix} \mapsto \begin{pmatrix} h \\ h' \end{pmatrix}.$$

In the sequel we denote the elements in A^+ and T^+ by $\hat{f}_1$ and $\hat{f}_2$, respectively. As the multivalued part of $\widehat{P}_\mathcal{H}^{-1}$ coincides with A it follows that $\Gamma_0 \widehat{P}_\mathcal{K} \widehat{P}_\mathcal{H}^{-1}$ and $\Gamma_1 \widehat{P}_\mathcal{K} \widehat{P}_\mathcal{H}^{-1}$ are operators. We define $\Gamma_0', \Gamma_1' : T^+ \to \mathbb{C}$ by

$$\Gamma_0' \hat{f}_2 := -\Gamma_0 \widehat{P}_\mathcal{K} \widehat{P}_\mathcal{H}^{-1} \hat{f}_2, \qquad \Gamma_1' \hat{f}_2 := \Gamma_1 \widehat{P}_\mathcal{K} \widehat{P}_\mathcal{H}^{-1} \hat{f}_2, \qquad \hat{f}_2 \in T^+,$$

cf. [11]. Taking into account that $\widetilde{A}$ is self-adjoint, one verifies that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ is a boundary value space for T^+. We set

$$T_0 := \ker \Gamma_0'. \tag{3.10}$$

An element $\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+$ belongs to $\widetilde{A}$ if and only if the set $\hat{f}_1 - \widehat{P}_\mathcal{K} \widehat{P}_\mathcal{H}^{-1} \hat{f}_2$ is contained in A. Therefore $\widetilde{A}$ is the canonical self-adjoint extension of the symmetric relation $A \times T$ in $\mathcal{K} \times \mathcal{H}$ given by

$$\widetilde{A} = \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = 0 \Big\}. \tag{3.11}$$

3. In order to show that T_0 has a non-empty resolvent set we construct in this step an auxiliary self-adjoint extension T_α of T in $\mathcal{H}$ such that $\rho(T_\alpha) \cap \Omega$ is non-empty and with the help of Theorem 2.7 we will show that T_α is definitizable over Ω and that the sign types of T_α are d-compatible with the sign types of $\widetilde{A}$ and A_0 in Ω.

It is easy to see that $\{\mathbb{C}^2, \Gamma_0'', \Gamma_1''\}$, where

$$\Gamma_0''\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \Gamma_0 \hat{f}_1 \\ \Gamma_0' \hat{f}_2 \end{pmatrix} \quad \text{and} \quad \Gamma_1''\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \Gamma_1 \hat{f}_1 \\ \Gamma_1' \hat{f}_2 \end{pmatrix}, \ \hat{f}_1 \in A^+, \ \hat{f}_2 \in T^+, \tag{3.12}$$

is a boundary value space for $A^+ \times T^+$. Setting

$$W := \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^4)$$

and

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} := W \begin{pmatrix} \Gamma_0'' \\ \Gamma_1'' \end{pmatrix} \tag{3.13}$$

we obtain a boundary value space $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $A^+ \times T^+$. This follows e.g. from the fact that W is unitary in the Krein space $(\mathbb{C}^4, [\![\cdot,\cdot]\!]_{\mathbb{C}^4})$, where

$$[\![\cdot,\cdot]\!]_{\mathbb{C}^4} := (\mathcal{J}\cdot,\cdot), \qquad \mathcal{J} = \begin{pmatrix} 0 & -iI_{\mathbb{C}^2} \\ iI_{\mathbb{C}^2} & 0 \end{pmatrix}, \tag{3.14}$$

(see Section 2.1). Here we have

$$\widetilde{A} = \ker \widetilde{\Gamma}_0. \tag{3.15}$$

As $\widetilde{A}$ is by assumption definitizable over Ω it follows from Theorem 2.5 and (3.5) that the Weyl function $\widetilde{M}$ corresponding to $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a definitizable function in Ω, $\widetilde{M} \in \mathcal{D}^{2\times 2}(\Omega)$, and the sign types of $\widetilde{M}$ and $\widetilde{A}$ are d-compatible in Ω.

It is not difficult to verify that

$$\operatorname{ran}\big(P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}|_{\mathcal{K}}\big) = \ker(T^+ - \lambda) = \mathcal{N}_{\lambda,T^+}, \qquad \lambda \in \rho(\widetilde{A}),$$

holds. Since $\widetilde{A}$ is an $\mathcal{K}$-minimal exit space extension of A we have

$$\begin{aligned} \mathcal{H} &= \operatorname{clsp}\big\{\operatorname{ran}\big(P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}|_{\mathcal{K}}\big) \mid \lambda \in \rho(\widetilde{A}) \cap \Omega\big\} \\ &= \operatorname{clsp}\big\{\mathcal{N}_{\lambda,T^+} \mid \lambda \in \rho(\widetilde{A}) \cap \Omega\big\} \end{aligned} \tag{3.16}$$

and from the assumption $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_{\lambda,A^+} \mid \lambda \in \rho(A_0) \cap \Omega\}$ we obtain

$$\mathcal{K} \times \mathcal{H} = \operatorname{clsp}\big\{\mathcal{N}_{\lambda,A^+\times T^+} \mid \lambda \in \rho(A_0) \cap \rho(\widetilde{A}) \cap \Omega\big\}.$$

This implies that the function $\widetilde{M} \in \mathcal{D}^{2\times 2}(\Omega)$ is strict (see (3.6)). We claim that there exists $\alpha \in \mathbb{R}$ such that the function

$$\lambda \mapsto \widetilde{M}(\lambda) - \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$$

is invertible for some $\lambda' \in \rho(\widetilde{A}) \cap \Omega$. Indeed, let $\widetilde{M}(\lambda) = (m_{ij}(\lambda))_{i,j=1}^2$ and suppose that for all $\lambda \in \rho(\widetilde{A}) \cap \Omega$ and every $\alpha \in \mathbb{R}$ we have

$$\det\left(\widetilde{M}(\lambda) - \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\right) = m_{11}(\lambda)(m_{22}(\lambda) - \alpha) - m_{21}(\lambda)m_{12}(\lambda) = 0.$$

This implies $m_{11}(\lambda) = m_{12}(\lambda)m_{21}(\lambda) = 0$ and since m_{12} and m_{21} are piecewise meromorphic functions in $\Omega\backslash\overline{\mathbb{R}}$ and $\widetilde{M}$ is symmetric with respect to the real axis we conclude $m_{12}(\lambda) = m_{21}(\lambda) = 0$, $\lambda \in \rho(\widetilde{A}) \cap \Omega$, which contradicts the strictness of $\widetilde{M}$.

It is straightforward to check that the matrix

$$V := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -\alpha & \alpha & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^4) \tag{3.17}$$

is unitary in $(\mathbb{C}^4, [\![\cdot,\cdot]\!]_{\mathbb{C}^4})$, cf. (3.14). Let $\{\mathbb{C}^2, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ be the boundary value space for $A^+ \times T^+$ defined by

$$\begin{pmatrix} \widehat{\Gamma}_0 \\ \widehat{\Gamma}_1 \end{pmatrix} := V \begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} = VW \begin{pmatrix} \Gamma''_0 \\ \Gamma''_1 \end{pmatrix}, \tag{3.18}$$

(see (3.13)). From

$$VW = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

we obtain

$$\widehat{\Gamma}_0\{\hat{f}_1, \hat{f}_2\} = \begin{pmatrix} \Gamma_0 \hat{f}_1 \\ \Gamma'_1 \hat{f}_2 - \alpha\Gamma'_0 \hat{f}_2 \end{pmatrix}, \qquad \hat{f}_1 \in A^+, \hat{f}_2 \in T^+,$$

and

$$\widehat{\Gamma}_1\{\hat{f}_1, \hat{f}_2\} = \begin{pmatrix} \Gamma_1 \hat{f}_1 \\ -\Gamma'_0 \hat{f}_2 \end{pmatrix}, \qquad \hat{f}_1 \in A^+, \hat{f}_2 \in T^+.$$

We denote the self-adjoint extension $\ker(\Gamma'_1 - \alpha\Gamma'_0) \in \widetilde{\mathcal{C}}(\mathcal{H})$ of T in $\mathcal{H}$ by T_α. Then the self-adjoint extension $\ker \widehat{\Gamma}_0$ of $A \times T$ in $\mathcal{K} \times \mathcal{H}$ coincides with $A_0 \times T_\alpha$.

Since (3.18) and (3.17) imply

$$\begin{aligned} A_0 \times T_\alpha = \ker \widehat{\Gamma}_0 &= \ker\left(\begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix} \widetilde{\Gamma}_0 + \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \widetilde{\Gamma}_1\right) \\ &= \ker\left(\widetilde{\Gamma}_1 - \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \widetilde{\Gamma}_0\right), \end{aligned}$$

we find from (3.15), (3.2) and (3.7) that a point $\lambda \in \rho(\widetilde{A})$ belongs to the set $\rho(A_0 \times T_\alpha)$ if and only if 0 belongs to the resolvent set of

$$\widetilde{M}(\lambda) - \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}.$$

But we have chosen α such that this function is invertible for some $\lambda' \in \rho(\widetilde{A}) \cap \Omega$, therefore λ' belongs to $\rho(A_0 \times T_\alpha)$. In particular $\lambda' \in \rho(A_0) \cap \rho(T_\alpha)$ and

$$\rho(T_\alpha) \cap \rho(A_0) \cap \rho(\widetilde{A}) \cap \Omega \neq \emptyset.$$

As $A \times T$ is a symmetric relation of defect two and $\widetilde{A}$ and $A_0 \times T_\alpha$ are self-adjoint extensions of $A \times T$ in $\mathcal{K} \times \mathcal{H}$ we have

$$\dim\bigl(\mathrm{ran}\ \bigl((\widetilde{A} - \lambda)^{-1} - ((A_0 \times T_\alpha) - \lambda)^{-1}\bigr)\bigr) \leq 2$$

for all $\lambda \in \rho(\widetilde{A}) \cap \rho(A_0) \cap \rho(T_\alpha) \cap \Omega$. Since $\widetilde{A}$ is definitizable over Ω we obtain from Theorem 2.7 that also the self-adjoint relation $A_0 \times T_\alpha$ is definitizable over Ω and that the sign types of $\widetilde{A}$ and the sign types of $A_0 \times T_\alpha$ are d-compatible in Ω.

It is a simple consequence from Definition 2.1 that

$$\bigl(\sigma_{++}(A_0 \times T_\alpha) \cap \sigma_{ap}(T_\alpha)\bigr) \subset \sigma_{++}(T_\alpha)$$

and

$$\bigl(\sigma_{--}(A_0 \times T_\alpha) \cap \sigma_{ap}(T_\alpha)\bigr) \subset \sigma_{--}(T_\alpha)$$

holds. Hence, real points from $\sigma_{++}(A_0 \times T_\alpha)$ ($\sigma_{--}(A_0 \times T_\alpha)$) belong to $\rho(T_\alpha)$ or to $\sigma_{++}(T_\alpha)$ (resp. $\sigma_{--}(T_\alpha)$). Therefore T_α is definitizable over Ω and the sign types of T_α in Ω are d-compatible with the sign types of $A_0 \times T_\alpha$ and, hence, with the sign types of $\widetilde{A}$ and A_0 in Ω.

4. In this step we show that also T_0 in (3.10) has a non-empty resolvent set and that formula (3.9) holds with the Weyl function τ corresponding to the boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$. Moreover, we show that τ is locally definitizable and that its sign types are d-compatible with the sign types of A_0 and $\widetilde{A}$ in Ω.

It is straightforward to verify that $\{\mathbb{C}, \Gamma_1' - \alpha\Gamma_0', -\Gamma_0'\}$ is a boundary value space for T^+ and we have $T_\alpha = \ker(\Gamma_1' - \alpha\Gamma_0')$ and $T_0 = \ker(-\Gamma_0')$. The corresponding Weyl function τ_α is defined for all $\lambda \in \rho(T_\alpha)$. As T_α is definitizable over Ω the function τ_α belongs to the class $\mathcal{D}(\Omega)$ and the sign types of τ are d-compatible with the sign types of T_α, $\widetilde{A}$ and A_0 in Ω (cf. Theorem 2.5 and (3.5) or [3, Proposition 3.2]). Relation (3.16) implies that τ_α is strict and in particular τ_α is not identically equal to zero.

Then, by (3.8), for $\lambda \in \rho(T_\alpha) \cap \mathfrak{h}(\tau_\alpha^{-1})$ we have

$$(T_0 - \lambda)^{-1} = (T_\alpha - \lambda)^{-1} - \gamma_\alpha'(\lambda)\frac{1}{\tau_\alpha(\lambda)}\gamma_\alpha'(\overline{\lambda})^+,$$

where γ_α' is the γ-field of the boundary value space $\{\mathbb{C}, \Gamma_1' - \alpha\Gamma_0', -\Gamma_0'\}$. Therefore the set $\rho(T_\alpha) \cap \rho(T_0) \cap \Omega$ is non-empty and by Theorem 2.7 the self-adjoint relation

T_0 is definitizable over Ω and the sign types of T_0 and T_α are d-compatible in Ω. The Weyl function τ corresponding to the boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ satisfies

$$\tau(\lambda) = -\tau_\alpha(\lambda)^{-1} + \alpha, \qquad \lambda \in \mathfrak{h}(\tau_\alpha^{-1}) \cap \rho(T_\alpha).$$

and is holomorphic on $\rho(T_0)$.

It follows from Theorem 2.5 and (3.5) that τ belongs to the class $\mathcal{D}(\Omega)$ and that its sign types are d-compatible with the sign types of T_0 and T_α and hence also with the sign types of $\widetilde{A}$ and A_0. The γ-field corresponding to $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ will be denoted by γ'.

Since A_0 and T_0 are both definitizable over Ω the set $\rho(A_0)\cap\rho(T_0)\cap\Omega$ is nonempty. The γ-field γ'' and the Weyl function M'' corresponding to the boundary value space $\{\mathbb{C}^2, \Gamma_0'', \Gamma_1''\}$ defined in (3.12) are given by

$$\lambda \mapsto \gamma''(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma'(\lambda) \end{pmatrix}, \quad \lambda \in \rho(A_0) \cap \rho(T_0) \cap \Omega \tag{3.19}$$

and

$$\lambda \mapsto M''(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \quad \lambda \in \rho(A_0) \cap \rho(T_0) \cap \Omega, \tag{3.20}$$

respectively. The relation

$$\Theta := \left\{ \begin{pmatrix} \{u, -u\} \\ \{v, v\} \end{pmatrix} \,\Big|\, u, v \in \mathbb{C} \right\} \in \widetilde{\mathcal{C}}(\mathbb{C}^2) \tag{3.21}$$

is self-adjoint and the corresponding self-adjoint extension of $A \times T$ is given by

$$\begin{pmatrix} \Gamma_0'' \\ \Gamma_1'' \end{pmatrix}^{-1} \Theta = \left\{ \{\hat f_1, \hat f_2\} \in A^+ \times T^+ \,|\, \Gamma_0 \hat f_1 + \Gamma_0' \hat f_2 = \Gamma_1 \hat f_1 - \Gamma_1' \hat f_2 = 0 \right\} \tag{3.22}$$

and coincides with $\widetilde{A}$ (see (3.11)).

By (3.7) a point $\lambda \in \rho(A_0 \times T_0)$ belongs to $\rho(\widetilde{A})$ if and only if

$$0 \in \rho(\Theta - M''(\lambda)).$$

Hence, for $\lambda \in \rho(A_0 \times T_0) \cap \rho(\widetilde{A}) \cap \Omega = \rho(A_0) \cap \mathfrak{h}(\tau) \cap \rho(\widetilde{A}) \cap \Omega$

$$\big(\Theta - M''(\lambda)\big)^{-1} = \left\{ \begin{pmatrix} \{v - M(\lambda)u, v + \tau(\lambda)u\} \\ \{u, -u\} \end{pmatrix} \,\Big|\, u, v \in \mathbb{C} \right\}$$

is an operator. Therefore $(M(\lambda) + \tau(\lambda))u = 0$ implies $u = 0$ and we conclude that the set $\rho(\widetilde{A}) \cap \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega$ is a subset of $\mathfrak{h}((M+\tau)^{-1})$. Setting $x = v - M(\lambda)u$ and $y = v + \tau(\lambda)u$ we obtain

$$u = -\big(M(\lambda) + \tau(\lambda)\big)^{-1} x + \big(M(\lambda) + \tau(\lambda)\big)^{-1} y$$

for $\lambda \in \rho(A_0 \times T_0) \cap \rho(\widetilde{A}) \cap \Omega$. This implies

$$\big(\Theta - M''(\lambda)\big)^{-1} = \begin{pmatrix} -(M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \\ (M(\lambda) + \tau(\lambda))^{-1} & -(M(\lambda) + \tau(\lambda))^{-1} \end{pmatrix}. \tag{3.23}$$

For all $\lambda \in \rho(A_0 \times T_0) \cap \rho(\widetilde{A}) \cap \Omega$ the relation

$$(\widetilde{A} - \lambda)^{-1} = \big((A_0 \times T_0) - \lambda\big)^{-1} + \gamma''(\lambda)\big(\Theta - M''(\lambda)\big)^{-1}\gamma''(\overline{\lambda})^+ \tag{3.24}$$

holds (cf. (3.8)) and it follows from (3.24), (3.19) and (3.23) that the formula

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+$$

holds. This completes the proof of assertion (i).

5. Assertion (ii) was already proved in [3] in a slightly different form. For the convenience of the reader we sketch the proof.

If τ is identically equal to a real constant, then $A_{-\tau} := \ker(\Gamma_1 + \tau\Gamma_0)$ is a canonical self-adjoint extension of A. As the Weyl function M corresponding to A^+ and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is strict we obtain $\rho(A_{-\tau}) \cap \Omega \neq \emptyset$ and Theorem 2.7 implies that $A_{-\tau}$ is definitizable over Ω and that the sign types of A_0, $A_{-\tau}$ and $\tau \in \mathbb{R}$ are d-compatible. By (3.8)

$$(A_{-\tau} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau\big)^{-1}\gamma(\overline{\lambda})^+$$

holds for all $\lambda \in \rho(A_0) \cap ((M+\tau)^{-1})$. In the case $\tau = d_\infty = \big\{\big(\begin{smallmatrix}0\\c\end{smallmatrix}\big) \mid c \in \mathbb{C}\big\}$ we have $A_{-\tau} = A_0$.

Assume now that $\tau \in \mathcal{D}(\Omega)$ is not equal to a constant and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. With the help of [21, Theorem 3.8] it was shown in [3, Theorem 3.3] that there exists a Krein space $\mathcal{H}$, a closed symmetric operator T of defect one in $\mathcal{H}$ and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that $T_0 := \ker\Gamma_0'$ is definitizable over Ω', the sign types of τ and T_0 are d-compatible and τ coincides with the Weyl function corresponding to $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ on $\Omega' \cap \rho(T_0)$. Moreover the condition

$$\mathcal{H} = \operatorname{clsp}\big\{\gamma'(\lambda) \mid \lambda \in \rho(T_0) \cap \Omega'\big\} \tag{3.25}$$

is fulfilled. We choose the boundary value space $\{\mathbb{C}^2, \Gamma_0'', \Gamma_1''\}$ for $A^+ \times T^+$ as in (3.12) with γ-field and Weyl function given by (3.19) and (3.20), respectively. The self-adjoint extension corresponding to Θ in (3.21) via (3.1) is denoted by $\widetilde{A}$. Then $\widetilde{A}$ has the form (3.22) and the relation (3.24) holds for all $\lambda \in \Omega'$ which belong to $\rho(A_0 \times T_0)$ and fulfil $0 \in \rho(\Theta - M''(\lambda))$. From (3.23) we conclude

$$\rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big((M+\tau)^{-1}\big) \cap \Omega' \subset \rho(\widetilde{A})$$

and (3.24) implies that the formula (3.9) holds. Since the minimality condition (3.25) is fulfilled it follows from (3.24) that $\widetilde{A}$ is a $\mathcal{K}$-minimal exit space extension of A. As $A_0 \times T_0$ is definitizable over Ω' the relation (3.24) and Theorem 2.7 imply that $\widetilde{A}$ is also definitizable over Ω' and the sign types of $\widetilde{A}$, A_0 and τ are d-compatible. □

The next theorem is a variant of the Krein-Naimark formula for the case that A_0 and $\widetilde{A}$ are locally of type π_+ and τ is a local generalized Nevanlinna function.

The proof of Theorem 3.3 below is essentially the same as the proof of Theorem 3.2. Instead of the result on finite rank perturbations of locally definitizable self-adjoint relations from [4], cf. Theorem 2.7, one has to use [5, Theorem 2.4] on the stability of self-adjoint operators and relations locally of type π_+ under compact perturbations in resolvent sense. We leave the details to the reader.

Theorem 3.3. *Let A be a closed symmetric operator of defect one in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ with corresponding γ-field γ and Weyl function M. Assume that $A_0 = \ker \Gamma_0$ is of type π_+ over Ω and that the condition $\mathcal{K} = \text{clsp}\,\{\mathcal{N}_{\lambda,A^+} \mid \lambda \in \rho(A_0) \cap \Omega\}$ is fulfilled. Then the following assertions hold.*

(i) *For every $\mathcal{K}$-minimal self-adjoint exit space extension $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ which is of type π_+ over Ω there exists a function $\tau \in \widetilde{\mathcal{N}}(\Omega)$ such that*

$$\rho(\widetilde{A}) \cap \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega$$

is a subset of $\mathfrak{h}((M+\tau)^{-1})$ and the formula

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{3.26}$$

holds for all $\lambda \in \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega$.

(ii) *Let $\tau \in \widetilde{\mathcal{N}}(\Omega)$ be a local generalized Nevanlinna function such that $M(\mu) + \tau(\mu) \neq 0$ for some $\mu \in \Omega$ and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. Then there exists a Krein space $\mathcal{H}$ and a $\mathcal{K}$-minimal self-adjoint exit space extension $\widetilde{A}$ of A in $\mathcal{K} \times \mathcal{H}$ which is of type π_+ over Ω', such that*

$$\rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big((M+\tau)^{-1}\big) \cap \Omega$$

is a subset of $\rho(\widetilde{A})$ and formula (3.26) holds for all points λ belonging to $\rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((M+\tau)^{-1}) \cap \Omega$.

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