On finite rank perturbations of selfadjoint operators in Krein spaces and eigenvalues in spectral gaps

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Abstract. It is shown that the finiteness of eigenvalues in a spectral gap of a definitizable or locally definitizable selfadjoint operator in a Krein space is preserved under finite rank perturbations. This results is applied to a class of singular Sturm-Liouville operators with an indefinite weight function.

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1. Introduction

Let A and B be bounded or unbounded operators in a Hilbert, Pontryagin or Krein space $\mathcal{H}$ which are selfadjoint with respect to the Hilbert, Pontryagin or Krein space inner product of $\mathcal{H}$, and assume that the resolvent difference

$$(B-\lambda_0)^{-1}-(A-\lambda_0)^{-1}, \qquad \lambda_0 \in \rho(A)\cap\rho(B), \tag{1.1}$$

has finite rank n. In the case that $\mathcal{H}$ is a Hilbert space it is well known that the dimensions of the spectral subspaces of A and B corresponding to a bounded interval $\Delta \subset \mathbb{R}$ differ at most by n; this result is particulary useful when considering the eigenvalues in gaps of the continuous spectrum. A similar result holds in a Pontryagin space with negative index κ where the spectral multiplicity can change by at most $n+2\kappa$. If $\mathcal{H}$ is a Krein space with infinite positive and negative index such an upper bound does not exist, moreover, a finite rank perturbation of an arbitrary selfadjoint operator will in general not preserve finiteness of eigenvalues in a gap of the continuous spectrum.

It is the main objective of this note to prove a result on the finiteness of eigenvalues in spectral gaps under finite rank perturbations for a particular class of selfadjoint operators in Krein spaces. Here we consider the class of definitizable and locally definitizable selfadjoint operators studied by

H. Langer in [30, 31] and P. Jonas in [16, 17, 19], respectively. Recall that a selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is said to be *definitizable* if $\rho(A) \neq \emptyset$ and there exists a polynomial p such that $p(A)$ is a nonnegative operator in $(\mathcal{H}, [\cdot,\cdot])$. The operator A is *locally definitizable* over some $\mathbb{R}$-symmetric domain Ω if, roughly speaking, A is the orthogonal sum of a definitizable operator and a selfadjoint operator which has spectrum outside Ω; cf. Definition 3.1. It is known from [3, 6, 21] that definitizability and local definitizability are preserved under finite rank perturbations in resolvent sense. In our main result in this note we show that for locally definitizable selfadjoint operators A and B such that (1.1) is of finite rank the number of distinct eigenvalues of A in a gap in the continuous spectrum is finite if and only if the same is true for the number of eigenvalues of B. It is convenient to formulate and prove this result for the slightly more general case of selfadjoint relations. We remark that our result is of somewhat different nature than the perturbation results in the Hilbert and Pontryagin space situation mentioned in the beginning. Namely, here only the number of distinct eigenvalues in the gap (instead of their multiplicities) is treated, but also eigenvalues of infinite multiplicity in the gap are allowed.

Definitizable and locally definitizable operators appear in various applications, e.g. in the spectral theory of indefinite Sturm-Liouville operators [5, 7, 9, 11, 12, 22, 23, 25, 26, 34] and in the theory of operator polynomials [1, 27, 28, 29, 30, 32]. We apply our main abstract perturbation result to a class of singular Sturm-Liouville operators with an indefinite weight function in an L^2-Krein space. This immediately yields nontrivial statements on the finiteness and accumulation properties of eigenvalues in the spectral gaps of the indefinite Sturm-Liouville operator in terms of the spectral properties of some underlying Sturm-Liouville operators which are selfadjoint in an L^2-Hilbert space. For the special case of left-definite Sturm-Liouville operators we refer the reader to [8] where more precise estimates were obtained.

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2. Linear relations in Krein spaces

Linear relations in a Hilbert or Krein space $\mathcal{H}$ are linear subspaces of the Cartesian product $\mathcal{H} \times \mathcal{H}$. Linear operators are always identified with linear relations via their graphs. For a linear relation A we write $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$ and $\operatorname{mul} A$ for the *domain*, *range*, *kernel* and *multivalued part* of A, respectively. For the usual definitions of the linear operations with relations, the inverse, product, and further details we refer to [2, 10, 13, 14]. Below we recall only a few notions which are used in the following.

A complex number λ is said to be an *eigenvalue* of a linear relation A when there is a nontrivial element x in $\ker(A - \lambda)$. The *root manifold* (or *algebraic eigenspace*) $\mathfrak{L}_\lambda(A)$ at λ of A is defined by $\cup_{i=1}^{\infty} \ker(A-\lambda)^i$. Similarly, ∞ is said to be an *eigenvalue* when there is a nontrivial element g in $\operatorname{mul} A$. The *root manifold* $\mathfrak{L}_\infty(A)$ at ∞ is defined by $\cup_{i=1}^{\infty} \operatorname{mul} A^i$. The dimension of $\mathfrak{L}_\lambda(A)$ or of $\mathfrak{L}_\infty(A)$, respectively, is called the *algebraic multiplicity.*

The *resolvent set* $\rho(A)$ of a closed linear relation A is the set of all points $\lambda \in \mathbb{C}$ such that $(A-\lambda)^{-1} \in \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ stands for the space of bounded linear operators defined on $\mathcal{H}$. The *spectrum* $\sigma(A)$ of A is the complement of $\rho(A)$ in $\mathbb{C}$. The *extended spectrum* $\widetilde{\sigma}(A)$ of A is defined by $\widetilde{\sigma}(A) = \sigma(A)$ if $A \in \mathcal{L}(\mathcal{H})$ and $\widetilde{\sigma}(A) = \sigma(A) \cup \{\infty\}$ otherwise. The *extended resolvent set* $\widetilde{\rho}(A)$ of A is defined by $\overline{\mathbb{C}} \setminus \widetilde{\sigma}(A)$. As usual $\overline{\mathbb{C}}$ and $\overline{\mathbb{R}}$ denote the compactifications $\mathbb{C} \cup \{\infty\}$ and $\mathbb{R} \cup \{\infty\}$ of $\mathbb{C}$ and $\mathbb{R}$, respectively.

3. Locally definitizable selfadjoint operators and relations

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space and let A be a selfadjoint linear relation in $\mathcal{K}$. A selfadjoint relation A is *definitizable* if the resolvent set of A is nonempty and there exists a real polynomial p such that

$$[g, f] \geq 0$$

holds for all $\binom{f}{g} \in p(A)$. In particular, if A is an operator, then A is definitizable if the resolvent set of A is nonempty and there exists a real polynomial p such that

$$[p(A)x, x] \geq 0$$

holds for all $x \in \operatorname{dom}(p(A))$. For a detailed study of definitizable selfadjoint operators and relations, we refer to the fundamental paper [31] of H. Langer and to [14, §4 and §5].

We recall briefly the notion of *locally definitizable selfadjoint operators* and *relations.* For this, let Ω be some domain in $\overline{\mathbb{C}}$ which is symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$, and the intersections of Ω with the upper and lower open half-planes are simply connected. Let A be a selfadjoint relation in the Krein space $\mathcal{K}$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the nonreal spectrum of A in Ω.

Definition 3.1. *Let A and Ω be as above. A is called* definitizable over Ω *if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a selfadjoint projection E in $\mathcal{K}$ such that A can be decomposed in*

$$A = \big(A \cap (E\mathcal{K})^2\big) \dotplus \big(A \cap ((1-E)\mathcal{K})^2\big)$$

and the following holds.

(i) *$A \cap (E\mathcal{K})^2$ is a selfadjoint definitizable relation in the Krein space $E\mathcal{K}$.*

(ii) *$\widetilde{\sigma}\big(A \cap ((1-E)\mathcal{K})^2\big) \cap \Omega' = \emptyset$.*

Locally definitizable operators were defined in a different way in [16, 17, 19], which is equivalent to the one given in Definition 3.1, see, e.g., [6, Theorem 1.3] and [19, Theorem 4.8]. We also note that by [19, Theorem 4.7] a selfadjoint relation A is definitizable over $\overline{\mathbb{C}}$ if and only if A is definitizable.

4. Finite rank perturbations and eigenvalues in spectral gaps

In this section we investigate finite dimensional perturbations in resolvent sense of locally definitizable operators and relations in separable Krein spaces. The main result is Theorem 4.3 below which states that a finite rank perturbation preserves the finiteness of eigenvalues in gaps of the continuous spectrum.

In the following let A and B be selfadjoint relations in the separable Krein space $\mathcal{K}$ and let Ω be a domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected. We assume that

$$\rho(A) \cap \rho(B) \cap \Omega \neq \emptyset \tag{4.1}$$

and that

$$(B - \lambda_0)^{-1} - (A - \lambda_0)^{-1} \quad \text{is of finite rank} \tag{4.2}$$

for some, and hence for all, $\lambda_0 \in \rho(A) \cap \rho(B)$.

The following proposition is well known for selfadjoint operators in Krein spaces; cf. [15] or [24]. In the case that A or B is a relation the statement follows, e.g. from [14, Corollary to Proposition 2.1 and Proposition 2.5] applied to the Cayley transforms of A and B, which also differ by a finite rank operator according to [13, Proposition 2.1 (v)].

Proposition 4.1. *Let A, B and Ω be as above and assume that* (4.1) *and* (4.2) *hold. Let Δ be an open connected set in $\overline{\mathbb{R}}$ such that $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$ and assume that the set $\widetilde{\sigma}(A) \cap \Delta$ consists of finitely many distinct eigenvalues.*

Then any eigenvalue of infinite algebraic multiplicity of A in Δ is also an eigenvalue of infinite algebraic multiplicity of B and the set $\widetilde{\sigma}(B) \cap \Delta$ consists of eigenvalues which may only accumulate at the eigenvalues of infinite algebraic multiplicity or at the boundary points of Δ in $\overline{\mathbb{R}}$. In particular,

$$\rho(A) \cap \rho(B) \cap \Delta \neq \emptyset. \tag{4.3}$$

The following theorem is one of the main results from [6] and is recalled here for the convenience of the reader; for the special case of definitizable operators see [3, 21].

Theorem 4.2. *Let A, B and Ω be as above and assume that* (4.1) *and* (4.2) *hold. Then A is definitizable over Ω if and only if B is definitizable over Ω.*

In the next theorem we assume that some open connected set $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ contains only finitely many spectral points of the unperturbed relation A. We point out that eigenvalues of A with infinite algebraic multiplicity are not excluded. In general, see Proposition 4.1, a finite rank perturbation in resolvent sense may lead to infinitely many eigenvalues in Δ. It is the content of the next theorem that in the case of (locally) definitizable relations this does not occur.

Theorem 4.3. *Let A, B and Ω be as above and assume that* (4.1) *and* (4.2) *hold. Suppose that A (or, equivalently, B) is definitizable over Ω and let Δ be an open connected set in $\overline{\mathbb{R}}$ such that $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$.*

Then the set $\widetilde{\sigma}(A) \cap \Delta$ consists of finitely many distinct eigenvalues if and only if the set $\widetilde{\sigma}(B) \cap \Delta$ consists of finitely many distinct eigenvalues. The eigenvalues of infinite multiplicity of A and B in Δ coincide and, in particular, the sum of the algebraic multiplicities of all eigenvalues of A in Δ is finite if and only if the sum of the algebraic multiplicities of all eigenvalues of B in Δ is finite.

Remark 4.4. *In the special case that A (or, equivalenty) B in Theorem* 4.3 *is definitizable (over $\overline{\mathbb{C}}$) the assertion of the theorem holds for every open connected subset Δ in $\overline{\mathbb{R}}$.*

Proof. The proof of Theorem 4.3 is divided into four steps. In the first two steps the result is shown for an interval Δ and the special case of rank one perturbations in the resolvent sense. The first step is preparatory and mainly based on considerations in [6, Proof of Theorem 2.2], and the second step contains the main argument. In the third step a successive application of the rank one perturbation result yields the assertions in the theorem for the case of an interval Δ. The case that Δ is an open connected set with $\infty \in \Delta$ is discussed in the last step of the proof.

Step 1. Let us assume that the difference of the resolvents of A and B in (4.2) is a rank one operator. Then there exist $\varphi, \psi \in \mathcal{K}$, $\varphi, \psi \neq 0$, such that

$$(B - \lambda_0)^{-1} - (A - \lambda_0)^{-1} = [\cdot, \varphi]\psi \tag{4.4}$$

holds. As in [6, Proof of Theorem 2.2] one verifies the formula

$$(B - \lambda)^{-1} = (A - \lambda)^{-1} + \Gamma_\lambda\big(-\vartheta(\lambda)^{-1}\big)\Gamma_{\bar\lambda}^+ \tag{4.5}$$

for all $\lambda \in \rho(A) \cap \rho(B)$, where the mappings $\Gamma_\lambda \in \mathcal{L}(\mathbb{C}, \mathcal{K})$, $\lambda \in \rho(A)$, are defined by

$$\Gamma_\lambda : \mathbb{C} \to \mathcal{K}, \qquad z \mapsto z\big(1 + (\lambda - \lambda_0)(A - \lambda)^{-1}\big)\psi, \tag{4.6}$$

$\Gamma_{\bar\lambda}^+ : \mathcal{K} \to \mathbb{C}$ is the bounded functional $x \mapsto [x, (1 + (\bar\lambda - \lambda_0)(A - \bar\lambda)^{-1}\psi]$ and ϑ is a holomorphic function on $\rho(A)$ given by

$$\vartheta(\lambda) = \nu_0 + \big[\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \bar\lambda_0)(A - \lambda)^{-1}\big)\psi, \psi\big] \tag{4.7}$$

with some constant $\nu_0 \in \mathbb{R}$; cf. (2.10) and (2.16) in [6].

Step 2. Let $\Delta \subset \mathbb{R}$ be an interval and assume that $\sigma(A) \cap \Delta$ consists of finitely many distinct eigenvalues. Then by Proposition 4.1 the set $\sigma(B) \cap \Delta$ consists of eigenvalues which are either poles of the resolvent of B or have infinite algebraic multiplicity, in which case they are also eigenvalues of A. Hence there are at most finitely many eigenvalues of infinite algebraic multiplicity of B in Δ. According to (4.5) and (4.6) the poles of the resolvent of B are poles of the resolvent of A or zeros of the function ϑ in (4.7). Therefore it remains to show that the function ϑ has at most finitely many zeros in $\rho(A) \cap \Delta$.

According to [19, Theorem 3.18] (see also [20, Section 3.1] and [6, Theorem 1.7]) the function ϑ, and hence by [4, Theorem 2.5] also the function $-\vartheta^{-1}$ is definitizable over Ω, that is, for every domain Ω' with the same properties as Ω such that $\overline{\Omega'} \subset \Omega$ the function $-\vartheta^{-1}$ (and, analogously ϑ) can be written as the sum

$$-\vartheta^{-1} = \tau_0 + \tau_{(0)}, \tag{4.8}$$

where $\tau_{(0)}$ is locally holomorphic on $\overline{\Omega'}$ and τ_0 satisfies

$$r\,\tau_0 = N + g \tag{4.9}$$

with some rational functions r and g which are symmetric with respect to the real line, and a Nevanlinna function N; cf. [18, 20] for more details on definitizable and locally definitizable functions. The rational function r can be assumed to have poles only in $\{\mu, \bar{\mu}\}$, where μ is an arbitrary point of holomorphy of τ_0 with $\operatorname{Im}\mu \neq 0$.

Assume that ϑ has infinitely many zeros in some open interval I contained in $\rho(A) \cap \Delta$ and that Ω' is chosen such that $\overline{I} \subset \Omega'$ and (4.8) holds. Then $-\vartheta^{-1}$ has infinitely many poles in I. With (4.9) we write $-\vartheta^{-1}$ in (4.8) in the form

$$-\vartheta^{-1} = \frac{1}{r}N + \frac{g}{r} + \tau_{(0)}.$$

Then also the Nevanlinna function N has infinitely many poles in I. There exists a finite set c in I such that the rational functions r^{-1} and gr^{-1} have no poles and no zeros in $I \setminus c$. Then $I \setminus c$ consists of finitely many open intervals, and there is at least one open subinterval I_0 of $I \setminus c$ such that N has infinitely many poles in I_0. Between two consecutive poles the Nevanlinna function N is strictly increasing. Hence N takes all values in $\mathbb{R}$ on I_0 infinitely many times and this remains true for $-\vartheta^{-1}$ as $\tau_{(0)}$ is holomorphic in $\overline{I} \subset \Omega'$ and there are no poles and no sign changes of the rational functions r^{-1} and gr^{-1} in I_0. Therefore the function $-\vartheta^{-1}$ in (4.8) has infinitely many zeros in I_0. But the function ϑ in (4.7) is holomorphic in $\rho(A)$, hence $-\vartheta^{-1}$ has no zero in $I_0 \subset \rho(A)$, a contradiction. Thus ϑ has at most finitely many zeros in any open subinterval of $\rho(A) \cap \Delta$ and as $\sigma(A) \cap \Delta$ is finite the function ϑ has at most finitely many zeros in $\rho(A) \cap \Delta$. The theorem is proved for intervals Δ under the additional assumption (4.4).

Step 3. Let us assume that the difference of the resolvents of A and B in (4.2) is of finite rank. According to (4.3) it is no restriction to assume that λ_0 in (4.2) is a real point in $\rho(A) \cap \rho(B) \cap \Omega$, so that the resolvent difference in (4.2) is a selfadjoint operator in $\mathcal{K}$. Then there exist $n \in \mathbb{N}$ and bounded selfadjoint rank one operators F_j, $j = 1, \dots, n$, such that

$$(B - \lambda_0)^{-1} - (A - \lambda_0)^{-1} = \sum_{j=1}^{n} F_j.$$

We define linear relations A_k, $k = 0, \dots, n$, by

$$A_0 := A \quad \text{and} \quad A_k := \left((A - \lambda_0)^{-1} + \sum_{j=1}^{k} F_j \right)^{-1} + \lambda_0.$$

Then we have $A_n = B$ and by

$$(A_k - \lambda_0)^{-1} = (A - \lambda_0)^{-1} + \sum_{j=1}^{k} F_j$$

each A_k, $k = 1, \dots, n$, is a selfadjoint relation with $\lambda_0 \in \rho(A_k)$. Moreover, for all $k = 1, \dots, n$, the resolvent difference

$$(A_k - \lambda_0)^{-1} - (A_{k-1} - \lambda_0)^{-1} = F_k$$

is a rank one operator. Hence a successive application of the rank one perturbation result from Step 1 and 2 completes the proof in the case of an interval Δ.

Step 4. Assume now that Δ is an open connected set with $\infty \in \Delta$ such that $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$. Then $\Delta \setminus \{\infty\}$ decomposes into two open (unbounded) intervals I_1 and I_2. We apply the perturbation result from Step 1-3 to the intervals I_1 and I_2. Together with Proposition 4.1 applied to the point ∞, the assertions of Theorem 4.3 follow. □

5. Spectral gaps of singular Sturm-Liouville operators with an indefinite weight

In this section we apply Theorem 4.3 to definitizable and locally definitizable differential operators associated with the singular indefinite Sturm-Liouville expression

$$\ell = \frac{1}{w} \left(-\frac{d}{dx} p \frac{d}{dx} + q \right) \tag{5.1}$$

on $\mathbb{R}$. Here the coefficients w, p^{-1}, q are real valued and locally integrable, $w(x) \neq 0$ and $p(x) > 0$ for almost all $x \in \mathbb{R}$. The weight function w changes its sign and it will be assumed that the following condition (I) holds:

(I) There exist $\alpha < \beta$, such that $w(x) < 0$ for almost all $x \in (-\infty, \alpha)$ and $w(x) > 0$ for almost all $x \in (\beta, \infty)$.

Let $L^2_{|w|}(\mathbb{R})$ be the space of all equivalence classes of complex valued measurable functions f on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|^2 |w(x)|\, dx$ is finite and let

$$(f, g) = \int_{\mathbb{R}} f(x)\, \overline{g(x)}\, |w(x)|\, dx, \qquad f, g \in L^2_{|w|}(\mathbb{R}), \tag{5.2}$$

be the usual (weighted) Hilbert space scalar product in $L^2_{|w|}(\mathbb{R})$. We also equip $L^2_{|w|}(\mathbb{R})$ with the indefinite inner product

$$[f,g] = \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,w(x)\,dx, \qquad f,g \in L^2_{|w|}(\mathbb{R}). \tag{5.3}$$

Then $(L^2_{|w|}(\mathbb{R}), [\cdot,\cdot])$ is a Krein space and the fundamental symmetry J,

$$(Jf)(x) := (\operatorname{sgn} w(x))f(x), \qquad x \in \mathbb{R}, \quad f \in L^2_{|w|}(\mathbb{R}),$$

connects the indefinite inner product $[\cdot,\cdot]$ in (5.3) with the Hilbert space scalar product $(\cdot,\cdot)$ in (5.2) via $[Jf,g] = (f,g)$ for all $f,g \in L^2_{|w|}(\mathbb{R})$.

Besides the indefinite Sturm-Liouville expression ℓ in (5.1), we also consider the definite counterpart

$$\tau = J\ell = \frac{1}{|w|}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right),$$

and it will be assumed that following condition (II) holds for τ:

(II) The differential expression τ is in the limit point case at ∞ and $-\infty$.

Denote by $\mathcal{D}_{\max}$ the set of all functions $f \in L^2_{|w|}(\mathbb{R})$ such that f and pf' are absolutely continuous and $\tau(f) \in L^2_{|w|}(\mathbb{R})$ (or, equivalently, $\ell(f) \in L^2_{|w|}(\mathbb{R})$) and define the maximal Sturm-Liouville differential operator T associated to the definite differential expression τ by

$$Tf := \tau(f) = \frac{1}{|w|}\big(-(pf')' + qf\big), \qquad \operatorname{dom} T = \mathcal{D}_{\max}. \tag{5.4}$$

It follows from (II) that T is selfadjoint in the Hilbert space $(L^2_{|w|}(\mathbb{R}), (\cdot,\cdot))$. The restriction of τ onto $(-\infty,\alpha)$ and (β,∞) induces a family of selfadjoint operators in the corresponding Hilbert spaces $L^2_{|w|}(-\infty,\alpha)$ and $L^2_{|w|}(\beta,\infty)$. Let us fix the selfadjoint Dirichlet realizations T_- and T_+ in $L^2_{|w|}(-\infty,\alpha)$ and $L^2_{|w|}(\beta,\infty)$, respectively. Recall that the *essential spectrum* $\sigma_{\mathrm{ess}}(T_\pm)$ is defined as the set of spectral points which are not isolated eigenvalues (of finite multiplicity) of $T_\pm$. We impose the following condition:

(III) The operators T_- and T_+ are semibounded from below; the minima of their essential spectra are denoted by m_- and m_+, respectively. If T_+ (T_-) has no essential spectrum, we set $m_+ = \infty$ ($m_- = \infty$, respectively).

The maximal operator A associated to the indefinite differential expression ℓ is defined as $A := JT$, i.e.,

$$Af = JTf = \frac{1}{w}\big(-(pf')' + qf\big), \qquad \operatorname{dom} A = \operatorname{dom} T. \tag{5.5}$$

Since T is selfadjoint in the Hilbert space $L^2_{|w|}(\mathbb{R})$, A is selfadjoint in the Krein space $(L^2_{|w|}(\mathbb{R}), [\cdot,\cdot])$. The statements on local definitizability in (i) and (ii) of the following theorem are slight generalizations of [5, Theorem 3.2], see also [9, 23].

Theorem 5.1. *Suppose that the conditions* (I)-(III) *are satisfied. Then the following assertions* (i) *and* (ii) *hold.*

(i) *If* $m_+ > -m_-$ *then* A *is a definitizable operator and* $\sigma(A) \cap (-m_-, m_+)$ *consists of isolated eigenvalues with finite algebraic multiplicity. They accumulate at* $-m_-$ (m_+) *if and only if the eigenvalues of the operator* T_- *in* $(-\infty, m_-)$ $(T_+$ *in* $(-\infty, m_+))$ *accumulate at* m_- $(m_+$*, respectively).*

(ii) *If* $m_+ \leq -m_-$ *then* A *is definitizable over* $\overline{\mathbb{C}} \setminus [m_+, -m_-]$.

Moreover, the following holds for any open interval Δ.

(a) *If* $\overline{\Delta} \subset (-m_-, \infty)$ *and* $\Delta \cap \sigma(T_+)$ *is finite, then also* $\Delta \cap \sigma(A)$ *is finite.*

(b) *If* $\overline{\Delta} \subset (-\infty, m_+)$ *and* $\Delta \cap \sigma(-T_-)$ *is finite, then also* $\Delta \cap \sigma(A)$ *is finite.*

In the cases (a) *and* (b) *the set* $\Delta \cap \sigma(A)$ *consists of at most finitely many eigenvalues with finite algebraic multiplicity and the kernels* $\ker(A - \lambda)$, $\lambda \in \sigma_p(A)$*, have dimension one.*

Proof. The restriction of ℓ onto (α, β) induces a family of selfadjoint operators in the corresponding Krein space $(L^2_{|w|}(\alpha, \beta), [\cdot,\cdot])$. Let us fix the selfadjoint Dirichlet realization in the Krein space $(L^2_{|w|}(\alpha, \beta), [\cdot,\cdot])$ and denote it by $A_{\alpha\beta}$. By [11], $A_{\alpha\beta}$ is a definitizable operator and the hermitian form $[A_{\alpha\beta}\,\cdot,\cdot]$ has a finite number of negative squares. Moreover, the spectrum of $A_{\alpha\beta}$ consists of isolated eigenvalues with finite algebraic multiplicities which accumulate only to $\pm\infty$ and the set $\mathbb{C} \setminus \mathbb{R}$, with the possible exception of finitely many eigenvalues of $A_{\alpha\beta}$, belongs to $\rho(A_{\alpha\beta})$.

We identify the product $L^2_{|w|}(-\infty, \alpha)[\dot{+}]L^2_{|w|}(\alpha, \beta)[\dot{+}]L^2_{|w|}(\beta, \infty)$ with $L^2_{|w|}(\mathbb{R})$ and consider therein the diagonal operator

$$B := \begin{pmatrix} -T_- & 0 & 0 \\ 0 & A_{\alpha\beta} & 0 \\ 0 & 0 & T_+ \end{pmatrix}, \tag{5.6}$$

where the domain of B is given as the product of the domains of the operators T_-, $A_{\alpha\beta}$ and T_+, considered as a (dense) subset of $L^2_{|w|}(\mathbb{R})$. Then B is a selfadjoint operator in the Krein space $(L^2_{|w|}(\mathbb{R}), [\cdot,\cdot])$ and, with the possible exception of finitely many eigenvalues of $A_{\alpha\beta}$, $\mathbb{C}\setminus\mathbb{R}$ belongs to $\rho(B)$. Conditions (I)-(III) and [9, Theorem 4.5] imply that the maximal operator A in (5.5) has also non-empty resolvent set. As A and B coincide on the intersection of their domains, we conclude

$$\operatorname{rank}\left((A-\lambda)^{-1} - (B-\lambda)^{-1}\right) = 2, \qquad \lambda \in \rho(A) \cap \rho(B). \tag{5.7}$$

The spectrum of $A_{\alpha\beta}$, the spectrum of $-T_-$ in $(-m_-, \infty)$ and the spectrum of T_+ in $(-\infty, m_+)$ consist only of isolated eigenvalues with finite algebraic multiplicities. Therefore, it is easy to see that the statements (i), (ii), (a) and (b) of Theorem 5.1 hold for A replaced by B. Then, by (5.7), Theorem 4.2, [21, Theorem 1] and Theorem 4.3, the assertions of Theorem 5.1 follow. □

Remark 5.2. *In Theorem* 5.1 *statement (ii) can be made more precise in the case* $m_+ = -m_-$. *In this situation, A is a definitizable operator if and only if* $(-\infty, m_+) \cap \sigma(T_+)$ *and* $(-\infty, m_+) \cap \sigma(T_-)$ *are finite. This follows from the fact that in this case B in* (5.6) *is a definitizable operator and* [21, Theorem 1].

Example 5.3. *We consider the case that T_+ is periodic. More precisely, assume that the conditions* (I)-(III) *are satisfied and that the coefficients* $|w|$, p *and* q *of the definite Sturm-Liouville expression* τ *in* (5.4) *are* γ*-periodic on the interval* (β, ∞) *for some* $\gamma > 0$. *Let* $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ *be the eigenvalues of the selfadjoint operator associated with the differential expression* τ *restricted to functions in* $L^2_{|w|}(\beta, \beta+\gamma)$ *with the boundary conditions*

$$\begin{pmatrix} f(\beta) \\ (pf')(\beta) \end{pmatrix} = \begin{pmatrix} f(\beta+\gamma) \\ (pf')(\beta+\gamma) \end{pmatrix}.$$

Furthermore, let $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$ *be the eigenvalues of the selfadjoint operator associated with* τ *restricted to functions in* $L^2_{|w|}(\beta, \beta+\gamma)$ *with the boundary conditions*

$$\begin{pmatrix} f(\beta) \\ (pf')(\beta) \end{pmatrix} = -\begin{pmatrix} f(\beta+\gamma) \\ (pf')(\beta+\gamma) \end{pmatrix}.$$

Then one has $\lambda_1 < \mu_1 \leq \mu_2 < \lambda_2 \leq \lambda_3 < \mu_3 \dots$ *and the essential spectrum of* T_+ *is given by (see, e.g.,* [33, § 12]*),*

$$\sigma_{\text{ess}}(T_+) = [\lambda_1, \mu_1] \cup [\mu_2, \lambda_2] \cup [\lambda_3, \mu_3] \cup [\mu_4, \lambda_4] \dots.$$

Let $n' \in \mathbb{N}$ *such that* $\mu_{2n'-1} > -m_-$. *Then by Theorem* 5.1 *(a) the indefinite Sturm-Liouville operator A in* (5.5) *has at most finitely many eigenvalues with finite algebraic multiplicity in the spectral gaps*

$$(\mu_{2n-1}, \mu_{2n}) \quad \text{and} \quad (\lambda_{2n}, \lambda_{2n+1}), \qquad n \geq n'.$$

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