Spectral analysis of singular ordinary differential operators with indefinite weights

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Abstract

In this paper we develop a perturbation approach to investigate spectral problems for singular ordinary differential operators with indefinite weight functions. We prove a general perturbation result on the local spectral properties of selfadjoint operators in Krein spaces which differ only by finitely many dimensions from the orthogonal sum of a fundamentally reducible operator and an operator with finitely many negative squares. This result is applied to singular indefinite Sturm-Liouville operators and higher order singular ordinary differential operators with indefinite weight functions.

Key words: Sturm-Liouville operator, ordinary differential operator, indefinite weight, Krein space, definitizable operator, critical point, finite rank perturbation, Titchmarsh-Weyl theory

1 Introduction

Sturm-Liouville differential operators and higher order ordinary differential operators with indefinite weight functions have attracted a lot of attention in the recent past. In many situations it is possible to apply techniques from operator theory in indefinite inner product spaces and to obtain in this way

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information on the spectral structure of the indefinite differential operator, see, e.g., [3,6,9–11,19–25,29,34–37,39].

Let us consider the Sturm-Liouville differential expression

$$\ell = \frac{1}{w}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right), \tag{1.1}$$

where w, p^{-1} and q are real valued locally integrable functions on some bounded or unbounded interval (a,b) and $p(x) > 0$, $w(x) \neq 0$ almost everywhere. It will be assumed that the weight function w has different signs on subsets of positive Lebesgue measure of (a,b). In this case ℓ is said to be an *indefinite* Sturm-Liouville expression, and it is convenient also to consider the definite counterpart of ℓ,

$$\tau = \frac{1}{|w|}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right). \tag{1.2}$$

The operators associated to (1.1) and (1.2) act in the weighted L^2-space $L^2_{|w|}(a,b)$ which consists of all (equivalence classes) of complex valued measurable functions f on (a,b) such that $|f|^2|w|$ is integrable. Equipped with the scalar product

$$(f,g) = \int_a^b f(x)\,\overline{g(x)}\,|w(x)|\,dx, \qquad f,g \in L^2_{|w|}(a,b), \tag{1.3}$$

$L^2_{|w|}(a,b)$ is a Hilbert space and the definite Sturm-Liouville differential expression (1.2) is formally symmetric with respect to the positive definite inner product (1.3). The spectral properties of the selfadjoint realizations of τ in the Hilbert space $L^2_{|w|}(a,b)$ have been studied comprehensively, see, e.g., the monographs [47,48,50–52] as introductory texts and for further references.

In contrast to τ the indefinite Sturm-Liouville expression ℓ is not symmetric with respect to (1.3), but becomes symmetric with respect to the indefinite inner product

$$[f,g] = \int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx, \qquad f,g \in L^2_{|w|}(a,b), \tag{1.4}$$

and the challenging problem is now to investigate the spectral properties of the differential operators associated to ℓ which are selfadjoint with respect to (1.4). The Hilbert space scalar product (1.3) and the Krein space inner product (1.4) are connected via $[J\cdot,\cdot] = (\cdot,\cdot)$, where J is the multiplication operator with the function $x \mapsto \operatorname{sgn}(w(x))$. Formally, we have $J\tau = \ell$ and hence every selfadjoint realization A of τ in the Hilbert space $L^2_{|w|}(a,b)$ induces a J-selfadjoint realization JA of ℓ, i.e., an operator which is selfadjoint in the Krein space $(L^2_{|w|}(a,b),[\cdot,\cdot])$, and vice versa. We point out that the spectral properties of operators which are J-selfadjoint differ essentially from the spectral properties of selfadjoint operators in Hilbert spaces, e.g., the spectrum is in general not real and may even be emtpy or cover the whole complex plane.

Since the pioneering work [19] by B. Ćurgus and H. Langer in 1989 the spectral structure of the J-selfadjoint realizations of ℓ (and also of higher order ordinary differential operators with indefinite weights) in the regular case, i.e., the interval (a, b) is bounded and the coefficients are integrable up to the endpoints, is well understood. Namely, since every selfadjoint realization of the regular differential expression τ in the Hilbert space $L^2_{|w|}(a, b)$ is semibounded from below and the spectrum of such a differential operator A consists only of eigenvalues which accumulate to $+\infty$, it can be shown with the help of abstract perturbation arguments that the resolvent set of any J-selfadjoint realization of ℓ is nonempty, the spectrum of such a regular indefinite Sturm-Liouville operator JA is discrete and the form $[JA\cdot, \cdot] = (A\cdot, \cdot)$ has finitely many negative squares. It follows that the nonreal spectrum of JA consists of (at most) finitely many pairs of eigenvalues which are symmetric with respect to the real line and that the real eigenvalues accumulate to $+\infty$ and $-\infty$; cf. [19, § 1] and [44,45]. Under additional assumptions similar results also hold if both endpoints are in the limit circle case.

If at least one of the endpoints of the interval (a, b) is in the limit point case the situation becomes much more difficult. Let us consider the particularly interesting setting where both endpoints a and b are in the limit point case. Then it is well known that the maximal operator A associated to the definite Sturm-Liouville expression τ in (1.2) is selfadjoint in the Hilbert space $L^2_{|w|}(a, b)$ and hence the maximal operator JA associated to the indefinite Sturm-Liouville expression ℓ in (1.1) is J-selfadjoint. For simplicity, let us assume here in this paragraph that $(a, b) = \mathbb{R}$, that $p(x) = 1$, $w(x) = \operatorname{sgn}(x)$ holds for $|x| > c$ for some $c > 0$ and that the coefficient q admits limits at $+\infty$ and $-\infty$,

$$q_\infty := \lim_{x\to+\infty} q(x) \qquad \text{and} \qquad q_{-\infty} := \lim_{x\to-\infty} q(x).$$

In Corollaries 4.3 and 4.4 we shall treat this and more general cases. The operator A is then semibounded from below and the essential spectrum $\sigma_{\text{ess}}(A)$ coincides with the interval $[\mu, +\infty)$, where $\mu := \min\{q_\infty, q_{-\infty}\}$. If the lower bound of the spectrum $\sigma(A)$ is positive, then it is well known that the indefinite Sturm-Liouville operator JA has a nonempty resolvent set and is nonnegative with respect to the indefinite inner product $[\cdot, \cdot]$ in (1.4). This implies, in particular, that the spectrum of JA is real. There exists an extensive literature on such left-definite Sturm-Liouville problems. We mention only [12–15,40,41] and refer to the monograph [52] for further references and a detailed treatment of regular and singular left-definite problems. If only the lower bound μ of $\sigma_{\text{ess}}(A)$ is positive, then it can be shown that JA is an operator with finitely many negative squares and the resolvent set $\rho(JA)$ is nonempty, see, e.g., [19, Proposition 1.1], [39, Theorem 3.3] and Theorem 3.2. In particular, as in the regular case the nonreal spectrum consists only of finitely many pairs of eigenvalues which are symmetric with respect to the real axis. In the case that the lower bound of the essential spectrum of A becomes nonpositive serious difficulties arise, e.g., it does not even follow immediately that the resolvent set of the indefinite Sturm-Liouville operator JA is nonempty. However, under some additional assumptions a first result on the spectral structure of JA was proved in [6] with the help of a general perturbation result from [7], see also [37]. Namely, it was shown that JA is a so-called locally definitizable operator over $\overline{\mathbb{C}}\backslash[\mu, -\mu]$ in the sense of P. Jonas, see [30–32], with spectral properties similar to J-selfadjoint operators with finitely many negative squares outside any open neighborhood of the interval $[\mu, -\mu]$.

The first of our main objectives in this paper is to develop a perturbation approach to tackle spectral problems for singular ordinary differential operators with indefinite weight functions. For this we prove in Section 3 a general perturbation theorem which is directly applicable to the setting sketched above with $\mu \leq 0$ and much more general situations. In particular, our abstract result Theorem 3.5 ensures local definitizability of JA over $\overline{\mathbb{C}}\backslash[\mu, -\mu]$ as above. The second of our main objectives is to remove the additional assumption $\rho(JA) \neq \varnothing$ for a large class of J-selfadjoint singular indefinite Sturm-Liouville operators JA. In the special situation $w(x) = \operatorname{sgn}(x)$ and $p(x) = 1$ for a.e. $x \in \mathbb{R}$ it follows directly from the asymptotic behaviour of the Titchmarsh-Weyl functions associated to selfadjoint realizations of τ in $L^2(\mathbb{R}_\pm)$ that $\rho(JA)$ is nonempty, see [28] and [36,37], but for more general cases – e.g., when w has many turning points – this seems to be unknown. If the weight w has constant signs in a neighborhood of the singular endpoints a and b, and the definite Sturm-Liouville operator A is semibounded from below it will be shown in Theorem 4.5 that the resolvent set of the J-selfadjoint indefinite Sturm-Liouville operator JA is nonempty. For this we make use of local sign type properties of Titchmarsh-Weyl coefficients associated to indefinite Sturm-Liouville expressions on certain subintervals of (a, b).

Besides the indefinite Sturm-Liouville expression (1.1) we also study higher order differential expressions with an indefinite weight and real valued locally integrable coefficients of the form

$$\widehat{\ell} = \frac{1}{w}\left((-1)^n \frac{d^n}{dx^n} p_0 \frac{d^n}{dx^n} + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} p_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n\right) \tag{1.5}$$

on (a, b); cf. [19]. The corresponding definite differential expression $\widehat{\tau}$ is defined as $\widehat{\ell}$ in (1.5) with w replaced by $|w|$ and was already studied by M.G. Krein in [42], see also [48]. Again we are particularly interested in singular problems where the lower bound of the essential spectrum of a selfadjoint realization A of $\widehat{\tau}$ in $L^2_{|w|}(a, b)$ is nonpositive. Under the assumption $\rho(JA) \neq \varnothing$ our general perturbation result can be applied and yields local definitizability, information on sign type properties of the real spectrum and boundedness of the nonreal eigenvalues of the J-selfadjoint realization JA of $\widehat{\ell}$ in Theorem 5.1. We point out that it is not clear if $\rho(JA) \neq \varnothing$ holds in general. However, under additional assumptions on the weight function w and the boundary condition we make use of [19, Theorem 3.6] on the regularity of the critical point ∞ for J-selfadjoint realizations of the differential expression $\widehat{\ell} - \eta \operatorname{sgn} w$ (with suitable $\eta < 0$) to prove $\rho(JA) \neq \varnothing$ in Theorem 5.4.

The present paper is organized as follows. In Section 2 we collect the necessary preliminaries on J-selfadjoint operators, sign types of spectral points, and locally definitizable J-selfadjoint operators. The connection between the spectrum of a selfadjoint operator A in a Hilbert space and the spectrum of the corresponding J-selfadjoint operator JA is investigated in Section 3. The case that the lower bound of the spectrum or the essential spectrum of A is positive is recalled in Theorem 3.1 and Theorem 3.2; cf. [39, Theorems 1.1 and 1.2] and [16,19,44,45]. In both these cases JA is definitizable. The main result in this abstract part of the paper is Theorem 3.5. Here the lower bound of the essential spectrum of A is assumed to be nonpositive and it is shown under suitable assumptions that JA is still locally definitizable and the sign properties of the spectrum of JA are studied. This result is applied to singular indefinite Sturm-Liouville operators in Section 4 and to higher order singular ordinary differential operators with indefinite weights in Section 5.

2 Locally definitizable J-selfadjoint operators

Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$, let $J = J^* = J^{-1}$ be a bounded everywhere defined operator in $\mathcal{H}$ and define the inner product $[\cdot,\cdot]$ on $\mathcal{H}$ by

$$[h,k] := (Jh,k), \qquad h,k \in \mathcal{H}. \tag{2.1}$$

The inner product $[\cdot,\cdot]$ is in general indefinite, $(\mathcal{H},[\cdot,\cdot])$ is a so-called *Krein space* and J is the *fundamental symmetry* connecting the inner products $[\cdot,\cdot]$ and $(\cdot,\cdot)$. The orthogonal sum with respect to $(\cdot,\cdot)$ is denoted by $\oplus$. The fundamental symmetry J induces a fundamental decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \text{where} \quad \mathcal{H}_\pm = \ker(J \mp I),$$

of the Krein space $(\mathcal{H},[\cdot,\cdot])$. Here $(\mathcal{H}_+,[\cdot,\cdot])$ and $(\mathcal{H}_-,-[\cdot,\cdot])$ are Hilbert spaces and orthogonal to each other with respect to both $(\cdot,\cdot)$ and $[\cdot,\cdot]$. In the following all topological notions are to be understood with respect to the Hilbert space norm $\|\cdot\|$ induced by $(\cdot,\cdot)$. We refer the reader to [2,16,43] for further details on indefinite inner product spaces.

Let T be a densely defined linear operator in $\mathcal{H}$. The spectrum and resolvent set of T are denoted by $\sigma(T)$ and $\rho(T)$, respectively. The adjoint of T with respect to the scalar product $(\cdot,\cdot)$ is denoted by T^*. The adjoint of T with respect to the Krein space inner product $[\cdot,\cdot]$ is defined by $T^+ := JT^*J$, i.e.,

$$[Tx, y] = [x, T^+y] \quad \text{for all} \quad x \in \operatorname{dom} T,\ y \in \operatorname{dom} T^+ = J(\operatorname{dom} T^*).$$

The operator T is called *J-symmetric* (*J-selfadjoint*) if $T \subset T^+$ ($T = T^+$, respectively). Such operators are also said to be symmetric or selfadjoint in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Observe that T is J-symmetric (J-selfadjoint) if and only if JT is symmetric (selfadjoint, respectively) in the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ and that a J-symmetric operator has J-selfadjoint extensions in $\mathcal{H}$ if and only if the symmetric operator JT has selfadjoint extensions in $(\mathcal{H}, (\cdot,\cdot))$.

In the following we will briefly recall the definitions and basic properties of the so-called definitizable and locally definitizable J-selfadjoint operators. For a detailed exposition we refer to [30–32,44,45]. For a J-selfadjoint operator T in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ a point $\lambda \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{\mathrm{ap}}(T)$ of T if there exists a sequence $(x_n) \subset \operatorname{dom} T$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(T - \lambda)x_n\| \to 0$ if $n \to \infty$. If $\lambda \in \sigma_{\mathrm{ap}}(T)$ and each sequence $(x_n) \subset \operatorname{dom} T$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(T-\lambda)x_n\| \to 0$ for $n \to \infty$, satisfies

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \qquad \Big(\limsup_{n\to\infty} [x_n, x_n] < 0\Big),$$

then λ is called a *spectral point of positive type* (*negative type*, respectively) of T; cf. [32,46]. The J-selfadjointness of T implies that the spectral points of positive and negative type are real. An open set $\Delta \subset \mathbb{R}$ is said to be of *positive type* (*negative type*) with respect to T if $\Delta \cap \sigma(T)$ consists of spectral points of positive type (negative type, respectively) of T. We say that an open set $\Delta \subset \mathbb{R}$ is of *definite type* with respect to T if Δ is either of positive or of negative type with respect to T.

The next definition can be found in a more general form in, e.g., [31]. We denote the extended real line and extended complex plane by $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively.

Definition 2.1 *Let $I \subset \mathbb{R}$ be a closed connected set and let T be a J-selfadjoint operator in $\mathcal{H}$ such that $\sigma(T) \cap (\mathbb{C}\backslash\mathbb{R})$ consists of isolated points which are poles of the resolvent of T, and no point of $\overline{\mathbb{R}}\backslash I$ is an accumulation point of the nonreal spectrum of T. Then T is said to be* definitizable over $\overline{\mathbb{C}}\backslash I$, *if the following conditions* (i) *and* (ii) *hold:*

(i) *Every point $\mu \in \overline{\mathbb{R}}\backslash I$ has an open connected neighborhood $\mathcal{U}_\mu$ in $\overline{\mathbb{R}}$ such that both components of $\mathcal{U}_\mu\backslash\{\mu\}$ are of definite type with respect to T.*

(ii) *For every finite union* Δ *of open connected subsets of* $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \overline{\mathbb{R}}\backslash I$, *there exist* $m \geq 1$, $M > 0$ *and an open neighborhood* $\mathcal{O}$ *of* $\overline{\Delta}$ *in* $\overline{\mathbb{C}}$ *such that*

$$\|(T-\lambda)^{-1}\| \leq M(1+|\lambda|)^{2m-2}\,|\mathrm{Im}\,\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{O}\backslash\overline{\mathbb{R}}$.

A J*-selfadjoint operator* T *is said to be* nonnegative in a neighborhood of ∞ *if* T *is definitizable over* $\overline{\mathbb{C}}\backslash I$ *for some* $I \subset \mathbb{R}$ *as above and there exists* $\eta \geq 0$ *such that* $(\eta, +\infty)$ *is of positive type and* $(-\infty, -\eta)$ *is of negative type with respect to* T.

If T is a J-selfadjoint operator in $\mathcal{H}$ which is definitizable over $\overline{\mathbb{C}}$ $(= \overline{\mathbb{C}}\backslash\varnothing)$, then T is said to be *definitizable*. This is equivalent to the fact that there exists a polynomial p such that $[p(T)x, x] \geq 0$ holds for all $x \in \mathrm{dom}\, p(T)$ and $\rho(T) \neq \varnothing$; cf. [32, Theorem 4.7] and [44,45]. Such a polynomial is said to be definitizing for T.

Let $\kappa \in \mathbb{N}_0$. The J-selfadjoint operator T is said to have κ *negative squares* if $\rho(T) \neq \varnothing$ and the inner product $[T\cdot, \cdot]$ (on $\mathrm{dom}\, T$) has κ negative squares, i.e., there exists a subspace $M \subset \mathrm{dom}\, T$ with $\dim M = \kappa$ on which the inner product $[T\cdot, \cdot]$ is negative definite, and the dimension of every other subspace in $\mathrm{dom}\, T$ with this property is not greater than κ. A J-selfadjoint operator with a finite number κ of negative squares is definitizable and nonnegative in a neighborhood of ∞, see [45] and Theorem 3.2 below. If $\kappa = 0$, then the operator T is also called J*-nonnegative* or $[\cdot, \cdot]$*-nonnegative*.

Let T be a J-selfadjoint operator which is locally definitizable over $\overline{\mathbb{C}}\backslash I$. Then for every open set $\mathcal{O} \subset \mathbb{C}$ which contains I the operator T can be decomposed into the direct sum of a definitizable operator T_1 and a bounded operator T_2 with spectrum contained in $\overline{\mathcal{O}}$, see [32, Theorem 4.8]. If, in addition, T is nonnegative in a neighborhood of ∞, then $\mathcal{O}$ can be chosen such that T_1 is $[\cdot, \cdot]$-nonnegative, see, e.g., [8, §3.1]. This is due to the fact that T possesses a local spectral function $\delta \mapsto E_T(\delta)$ on $\overline{\mathbb{R}}\backslash I$ which is defined on all finite unions δ of connected subsets of $\overline{\mathbb{R}}\backslash I$ with endpoints in $\overline{\mathbb{R}}\backslash I$ which are either spectral points of definite type of T or belong to $\rho(T)$, see [32, Section 3.4 and Remark 4.9]. We note that an open interval $\Delta \subset \mathbb{R}\backslash I$ is of positive type (negative type) with respect to T if and only if for every set δ, $\overline{\delta} \subset \Delta$, for which $E_T(\delta)$ is defined, the spectral subspace $(E_T(\delta)\mathcal{H}, [\cdot, \cdot])$ $((E_T(\delta)\mathcal{H}, -[\cdot, \cdot])$, respectively) is a Hilbert space. Next we recall the notion of spectral points and intervals of type π_+ and type π_-; cf. [31]. The direct sum of subspaces in $\mathcal{H}$ is denoted by $\dot{+}$.

Definition 2.2 *Let* $I \subset \mathbb{R}$ *be a closed connected set and let* T *be a* J*-selfadjoint operator in* $\mathcal{H}$ *which is definitizable over* $\overline{\mathbb{C}}\backslash I$. *A point* $\lambda_0 \in \sigma(T) \cap (\overline{\mathbb{R}}\backslash I)$ *is called a spectral point of* type π_+ (type π_-) *of* T *if there exists an* *open interval* $\delta \subset \mathbb{R}\backslash I$ *with* $\lambda_0 \in \delta$ *such that both components of* $\delta\backslash\{\lambda_0\}$ *are of positive type* (*negative type, respectively*) *with respect to* T *and if* $\ker(T - \lambda_0) = \mathcal{K} \dotplus \mathcal{N}$, *where* $(\mathcal{K}, [\cdot,\cdot])$ $((\mathcal{K}, -[\cdot,\cdot])$, *respectively*) *is a Hilbert space and* $\dim \mathcal{N} < \infty$. *An open interval* $\Delta \subset \mathbb{R}\backslash I$ *is said to be of* type π_+ (type π_-) *with respect to* T *if every point in* $\sigma(T) \cap \Delta$ *is a spectral point of type* π_+ (*type* π_-, *respectively*) *of* T.

The above definition of spectral points of type π_+ and type π_- is equivalent to that in [31]. Spectral points of type π_+ and type π_- can also be characterized with the help of approximative eigensequences in a similar way as the spectral points of positive and negative type; cf. [5]. Observe that a spectral point of positive type (negative type) of T is at the same time a spectral point of type π_+ (type π_-, respectively) of T. Furthermore, the spectral function $E_T(\cdot)$ of the locally definitizable J-selfadjoint operator T can be used for the description of intervals of type π_+ and type π_- of T. More precisely, if T is definitizable over $\overline{\mathbb{C}}\backslash I$, then an open interval $\Delta \subset \mathbb{R}\backslash I$ is of type π_+ (type π_-) if and only if for every set δ, $\overline{\delta} \subset \Delta$, for which $E_T(\delta)$ is defined, the inner product $[\cdot,\cdot]$ has a finite number of negative (positive, respectively) squares on the spectral subspace $E_T(\delta)\mathcal{H}$.

Remark 2.3 *It is important to note that in an interval of type* π_+ (*type* π_-) *the set of points which are not of positive type* (*negative type, respectively*) *is discrete and can only accumulate to the endpoints of the interval; cf.* [5].

3 Spectral properties of a class of J-selfadjoint operators

Let throughout this section $(\mathcal{H}, (\cdot,\cdot))$ be a separable Hilbert space, let A be a bounded or unbounded selfadjoint operator in $\mathcal{H}$ and denote the spectral function of A by $E_A(\cdot)$. It will always be assumed that A is semibounded from below. Recall that the *essential spectrum* $\sigma_{\rm ess}(A)$ of A consists of the accumulation points of $\sigma(A)$ and the isolated eigenvalues of infinite multiplicity. For brevity we set

$$\nu := \min \sigma(A), \qquad \mu := \min \sigma_{\rm ess}(A), \tag{3.1}$$

and if the set $\sigma_{\rm ess}(A)$ is empty we define $\mu := +\infty$. Obviously, we then have the following inequality

$$-\infty < \nu \leq \mu \leq +\infty.$$

Suppose that a bounded linear operator $J = J^* = J^{-1}$ is given on $\mathcal{H}$ and let $[\cdot,\cdot]$ be the Krein space inner product induced by J and $(\cdot,\cdot)$, i.e. $[h,k] =$

(Jh, k), $h, k \in \mathcal{H}$. Then

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \qquad \mathcal{H}_\pm := \ker(J \mp I),$$

is the corresponding fundamental decomposition of $\mathcal{H}$. In the sequel we investigate the spectral properties of the J-selfadjoint operator JA. If the lower bound ν of the spectrum of A or the lower bound μ of the essential spectrum of A in (3.1) is positive, then it is known that JA is J-nonnegative or has a finite number of negative squares. For the convenience of the reader we recall these and some other facts in Theorem 3.1, Theorem 3.2 and Remark 3.3 below. The proofs of the statements are essentially contained in [16,19,44,45], see also [39, Theorem 3.3], [11, Theorem 3.1] and [21, Proposition 1.6].

Theorem 3.1 *Let A and J be as above and suppose $\nu = \min \sigma(A > 0$. Then the following holds for the J-selfadjoint operator JA:*

(i) *JA is J-nonnegative and $(\mathbb{C}\backslash\mathbb{R}) \cup \{0\} \subset \rho(JA)$;*
(ii) *$(0, \infty)$ is of positive and $(-\infty, 0)$ is of negative type with respect to JA;*
(iii) *JA is definitizable with definitizing polynomial $p(t) = t$.*

Recall that an eigenvalue λ of a closed operator T in $\mathcal{H}$ is said to be *normal* if λ is isolated and has finite algebraic multiplicity.

Theorem 3.2 *Let A and J be as above and suppose $\mu = \min \sigma_{\text{ess}}(A) > 0$. Then the following holds for the J-selfadjoint operator JA:*

(i) *JA has κ negative squares, where*

$$\kappa = \dim E_A((-\infty, 0)) < \infty,$$

and the nonreal spectrum of JA consists of at most κ pairs of normal eigenvalues;
(ii) *$(0, \infty)$ is of type π_+ and $(-\infty, 0)$ is of type π_- with respect to JA;*
(iii) *JA is nonnegative in a neighborhood of ∞ and definitizable with definitizing polynomial $p(t) = tq(t)\overline{q(\bar{t})}$, where q is a monic polynomial with degree $\leq \kappa$.*

Remark 3.3 *We remark that for the J-selfadjoint operator JA in Theorem 3.2 the multiplicity of the nonreal eigenvalues, the positive eigenvalues which are not of positive type and the negative eigenvalues which are not of negative type can be estimated by the number $\kappa = \dim E_A((-\infty, 0))$. More precisely, if $\{\kappa_-(\lambda), \kappa_0(\lambda), \kappa_+(\lambda)\}$ denotes the signature of the inner product $[\cdot, \cdot]$ on the algebraic eigenspace corresponding to an eigenvalue λ of JA, then*

$$\sum_{\lambda \in (-\infty, 0)} (\kappa_+(\lambda) + \kappa_0(\lambda)) + \sum_{\lambda \in (0, \infty)} (\kappa_-(\lambda) + \kappa_0(\lambda)) + \sum_{\text{Im}\, \lambda > 0} \kappa_0(\lambda) \leq \kappa$$

and, if $0 \notin \sigma_p(JA)$, then equality holds; cf. [44,45] and [11, Theorem 3.1].

The following simple example shows that in the case $\mu \leq 0$ additional assumptions on A have to be imposed to obtain further information on the spectral structure of JA.

Example 3.4 *Let $\mathcal{K}$ be an infinite dimensional Hilbert space and let B be an unbounded nonnegative selfadjoint operator in $\mathcal{K}$. Consider the operators*

$$A := \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} 0 & I_{\mathcal{K}} \\ I_{\mathcal{K}} & 0 \end{pmatrix}$$

in the product Hilbert space $\mathcal{H} := \mathcal{K} \oplus \mathcal{K}$. Then A is a nonnegative selfadjoint operator in $\mathcal{H}$ with $\nu = \mu = 0$ and $J = J^* = J^{-1}$. Since $\operatorname{dom} B \neq \mathcal{K}$ *and*

$$JA = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \qquad \operatorname{dom} JA = \operatorname{dom} B \oplus \mathcal{K},$$

it follows that $\operatorname{ran}(JA - \lambda) \neq \mathcal{H}$ *for all $\lambda \in \mathbb{C}$, i.e., $\sigma(JA) = \mathbb{C}$.*

The next theorem is the main result of this section. Very roughly speaking it states that if A is such that JA differs by at most finitely many dimensions from the orthogonal sum of a so-called fundamentally reducible operator and an operator with finitely many negative squares, then JA is locally definitizable and nonnegative at ∞. Recall that the *essential spectrum* $\sigma_{\mathrm{ess}}(S)$ of a closed symmetric operator S in a Hilbert space consists of all points $\lambda \in \mathbb{C}$ such that $S - \lambda$ is not Fredholm, i.e., $\operatorname{ran}(S - \lambda)$ is not closed or $\dim(\ker(S - \lambda)) = \infty$. The set $r(S)$ of *points of regular type of S* is defined by $r(S) := \mathbb{C} \backslash (\sigma_{\mathrm{ess}}(S) \cup \sigma_p(S))$.

Theorem 3.5 *Let A be a selfadjoint operator in $(\mathcal{H}, (\cdot, \cdot))$ which is semibounded from below with $\mu = \min \sigma_{\mathrm{ess}}(A) \leq 0$ and let J and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be as above. Let $\mathcal{K}_\pm \subset \mathcal{H}_\pm$ and $\mathcal{K}_d$ be closed subspaces of $\mathcal{H}$ such that*

$$\mathcal{H} = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathcal{K}_d. \tag{3.2}$$

Assume that there exist linear subspaces $\mathcal{G}_i \subset \operatorname{dom} A \cap \mathcal{K}_i$, $i \in \{+, -, d\}$, such that the following holds:

(i) $\dim\Big(\operatorname{dom} A / (\mathcal{G}_+ \oplus \mathcal{G}_- \oplus \mathcal{G}_d)\Big) < \infty$;
(ii) $A\,\mathcal{G}_\pm \subset \mathcal{K}_\pm$, $A\,\mathcal{G}_d \subset \mathcal{K}_d$;
(iii) $\sigma_{\mathrm{ess}}(\overline{A \upharpoonright \mathcal{G}_d}) \cap [\mu, \varepsilon) = \varnothing$ *for some* $\varepsilon > 0$.

If, in addition, $\rho(JA) \neq \varnothing$, then the following statements hold for the J-selfadjoint operator JA:

(a) *JA is definitizable over $\overline{\mathbb{C}} \backslash [\mu, -\mu]$ and nonnegative in a neighborhood of ∞;*

(b) $(-\infty,\mu)$ *is of type* π_- *and* $(-\mu,\infty)$ *is of type* π_+ *with respect to* JA*;*
(c) $\sigma(JA)\cap(\mathbb{C}\backslash\mathbb{R})$ *is bounded and consists of normal eigenvalues with only possible accumulation points in* $[\mu,-\mu]$*.*

Remark 3.6 *Evidently, if* A *is bounded or boundedly invertible, then the condition* $\rho(JA)\neq\varnothing$ *in Theorem* 3.5 *is satisfied.*

Remark 3.7 *If* JA *satisfies* (a)-(c) *in Theorem 3.5 then the local spectral function can be used to decompose* JA *into an operator with finitely many negative squares and a bounded operator; cf. Section 2. More precisely, for every open neighborhood* $\mathcal{O}\subset\mathbb{C}$ *of the interval* $[\mu,-\mu]$ *there exists a decomposition* $\mathcal{H}_\infty\oplus\mathcal{H}_\mu$ *of* $\mathcal{H}$ *such that* JA *can be written in the form*

$$JA=\begin{pmatrix}(JA)_\infty & 0\\ 0 & (JA)_\mu\end{pmatrix},$$

where $(JA)_\infty$ *is a* $(J\restriction\mathcal{H}_\infty)$*-selfadjoint operator in* $\mathcal{H}_\infty$ *with* $\rho((JA)_\infty)\neq\varnothing$ *and finitely many negative squares, and* $(JA)_\mu$ *is a bounded* $(J\restriction\mathcal{H}_\mu)$*-selfadjoint operator in* $\mathcal{H}_\mu$ *with* $\sigma((JA)_\mu)\subset\overline{\mathcal{O}}$*.*

In the special case $\mathcal{K}_d=\{0\}$ Theorem 3.5 reduces to the following corollary.

Corollary 3.8 *Let* A*,* J *and* $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ *be as above and* $\mu\leq 0$*. Assume that there exists a subspace* $\mathcal{G}_+\oplus\mathcal{G}_-$*,* $\mathcal{G}_\pm\subset\mathcal{H}_\pm$*, in* $\operatorname{dom}A$ *such that* $\dim(\operatorname{dom}A/(\mathcal{G}_+\oplus\mathcal{G}_-))<\infty$ *and* $A\,\mathcal{G}_\pm\subset\mathcal{H}_\pm$ *holds. If, in addition,* $\rho(JA)\neq\varnothing$*, then the statements* (a)-(c) *in Theorem* 3.5 *hold.*

In the special cases $\mathcal{K}_-=\{0\}$ or $\mathcal{K}_+=\{0\}$ the assertions in Theorem 3.5 can be improved. We emphasize that, in particular, the assumption $\rho(JA)\neq\varnothing$ can be dropped. This is a consequence of [3, Theorem 2.2] and [4, Theorem 3.1], see also [33, Theorem 1].

Corollary 3.9 *Let* A*,* J *and* $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ *be as above and* $\mu\leq 0$*. Let* $\mathcal{K}_+\subset\mathcal{H}_+$ $(\mathcal{K}_-\subset\mathcal{H}_-)$ *and* $\mathcal{K}_d$ *be closed subspaces of* $\mathcal{H}$ *such that*

$$\mathcal{H}=\mathcal{K}_+\oplus\mathcal{K}_d\quad(\mathcal{H}=\mathcal{K}_-\oplus\mathcal{K}_d,\ \textit{respectively}).$$

Assume that there exist linear subspaces $\mathcal{G}_i\subset\operatorname{dom}A\cap\mathcal{K}_i$*,* $i\in\{+,d\}$ $(i\in\{-,d\}$*, respectively*$)$*, such that the following holds:*

(i)$^\pm$ $\dim(\operatorname{dom}A/(\mathcal{G}_+\oplus\mathcal{G}_d))<\infty$ $(\dim(\operatorname{dom}A/(\mathcal{G}_-\oplus\mathcal{G}_d))<\infty$*, respectively*$)$*;*
(ii)$^\pm$ $A\,\mathcal{G}_+\subset\mathcal{K}_+$ $(A\,\mathcal{G}_-\subset\mathcal{K}_-$*, respectively*$)$*,* $A\,\mathcal{G}_d\subset\mathcal{K}_d$*;*
(iii)$^\pm$ $\sigma_{\mathrm{ess}}(\overline{A\restriction\mathcal{G}_d})=\varnothing$*.*

Then $\rho(JA)\neq\varnothing$*,* $\sigma_{\mathrm{ess}}(JA)=\sigma_{\mathrm{ess}}(A)$ $(\sigma_{\mathrm{ess}}(JA)=\sigma_{\mathrm{ess}}(-A)$*, respectively*$)$ *and* JA *is definitizable* $($*over* $\overline{\mathbb{C}})$ *and nonnegative in a neighborhood of* ∞*.*

Proof of Theorem 3.5. The proof is divided into three steps. In the first step we construct a J-selfadjoint operator $J\widetilde{A}$ which differs only by finitely many dimensions from JA and admits a diagonal block operator matrix representation with respect to the decomposition (3.2). In step 2 we show that the assertions (a)-(c) are satisfied with JA replaced by $J\widetilde{A}$. Finally, a general perturbation result from [7] applied to the present situation shows that JA satisfies (a)-(c).

Step 1. Denote by $\widetilde{\mathcal{G}}_i$, $i \in \{+, -, d\}$, the closure of $\mathcal{G}_i$ with respect to the graph norm $\|\cdot\|_A$ of A. Since A is closed it follows without difficulties that $\widetilde{\mathcal{G}}_i \subset \operatorname{dom} A \cap \mathcal{K}_i$, $i \in \{+, -, d\}$, and that the assumptions (i)-(iii) are valid with $\mathcal{G}_i$ replaced by $\widetilde{\mathcal{G}}_i$ (in order to see that (iii) holds with $\widetilde{\mathcal{G}}_d$ instead of $\mathcal{G}_d$, note that $\overline{A \restriction \mathcal{G}_d} = A \restriction \widetilde{\mathcal{G}}_d$). Therefore, we may assume that $\mathcal{G}_+$, $\mathcal{G}_-$, $\mathcal{G}_d$ and also

$$\mathcal{G} := \mathcal{G}_+ \oplus \mathcal{G}_- \oplus \mathcal{G}_d$$

are closed in $(\operatorname{dom} A, \|\cdot\|_A)$. From this and assumption (ii) it follows that the operators

$$S := A \restriction \mathcal{G}, \quad S_+ := A \restriction \mathcal{G}_+, \quad S_- := A \restriction \mathcal{G}_- \quad \text{and} \quad S_d := A \restriction \mathcal{G}_d,$$

are (not necessarily densely defined) closed symmetric operators in the Hilbert spaces $\mathcal{H}$, $\mathcal{K}_+$, $\mathcal{K}_-$ and $\mathcal{K}_d$, respectively. Since A is a selfadjoint extension of S, the deficiency indices $n_\pm(S) = \dim (\operatorname{ran} (S \pm i))^\perp$ of S coincide and are finite by condition (i). Moreover, since S is closed, the fact that A is semibounded from below implies

$$\mathbb{C} \backslash [\nu, \infty) \subset r(S), \tag{3.3}$$

where ν is the lower bound of $\sigma(A)$; cf. (3.1). With respect to the decomposition (3.2) we have

$$S = \begin{pmatrix} S_+ & 0 & 0 \\ 0 & S_- & 0 \\ 0 & 0 & S_d \end{pmatrix}.$$

Therefore, $r(S) = r(S_+) \cap r(S_-) \cap r(S_d)$, and it follows from (3.3) that each of the symmetric operators S_+, S_- and S_d has equal finite deficiency indices. Note also that

$$n_\pm(S) = n_\pm(S_+) + n_\pm(S_-) + n_\pm(S_d) = \dim (\operatorname{dom} A/\mathcal{G}) \tag{3.4}$$

holds.

Let A_+, A_- and A_d be selfadjoint extensions of S_+, S_- and S_d in the Hilbert spaces $\mathcal{K}_+$, $\mathcal{K}_-$ and $\mathcal{K}_d$, respectively. Although the domains $\mathcal{G}_+$, $\mathcal{G}_-$ and $\mathcal{G}_d$ of the symmetric operators S_+, S_- and S_d, respectively, may not be dense it is no restriction to assume that A_+, A_- and A_d are operators (instead of linear relations). Then the operator

$$\widetilde{A} := \begin{pmatrix} A_+ & 0 & 0 \\ 0 & A_- & 0 \\ 0 & 0 & A_d \end{pmatrix}$$

is a selfadjoint extension of S in the Hilbert space $\mathcal{H} = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathcal{K}_d$. Obviously both operators A and $\widetilde{A}$ coincide on $\mathcal{G} = \operatorname{dom} S$ and therefore, by (3.4) and condition (i),

$$\dim \operatorname{ran}\Big((A-\lambda)^{-1} - (\widetilde{A}-\lambda)^{-1}\Big) \le n_\pm(S) < \infty \tag{3.5}$$

holds for all $\lambda \in \rho(A) \cap \rho(\widetilde{A})$. Well known perturbation results for selfadjoint operators in Hilbert spaces imply $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(\widetilde{A})$ and, in particular, $\min \sigma_{\mathrm{ess}}(\widetilde{A}) = \mu$. From the definition of ν and μ it follows that for all $\lambda \in [\nu, \mu)$ the finite-dimensional restriction S of A satisfies

$$\dim \ker (S-\lambda) < \infty \quad \text{and} \quad \overline{\operatorname{ran}(S-\lambda)} = \operatorname{ran}(S-\lambda). \tag{3.6}$$

Moreover, the eigenvalues of S in $[\nu, \mu)$ are discrete with μ as only possible accumulation point and hence this is also true for S_d. As a consequence of assumption (iii) (3.6) with S replaced by S_d holds also for all $\lambda \in [\nu, \varepsilon)$. Hence, by well known properties of Fredholm operators we find that for all $\varepsilon_0 \in (0, \varepsilon)$ the interval $(-\infty, \varepsilon_0)$, with the possible exception of at most finitely many eigenvalues with finite multiplicities, is contained in $r(S_d)$. Therefore, since S_d is semibounded from below the spectrum of the finite dimensional selfadjoint extension A_d in $(-\infty, \varepsilon_0)$ consists of at most finitely many eigenvalues with finite multiplicities. This also implies $\min \sigma_{\mathrm{ess}}(A_d) \ge \varepsilon$ and hence $\mu = \min \sigma_{\mathrm{ess}}(A_+)$ or $\mu = \min \sigma_{\mathrm{ess}}(A_-)$.

Set $J_\pm := J \upharpoonright \mathcal{K}_\pm$. As $\mathcal{K}_\pm \subset \mathcal{H}_\pm$, we have $J_\pm = \pm I_{\mathcal{K}_\pm}$ and thus, $\mathcal{K}_d$ is J-invariant. Therefore, with respect to the decomposition (3.2) the fundamental symmetry J has the form

$$J = \begin{pmatrix} I_{\mathcal{K}_+} & 0 & 0 \\ 0 & -I_{\mathcal{K}_-} & 0 \\ 0 & 0 & J_d \end{pmatrix}, \tag{3.7}$$

where $J_d := J \upharpoonright \mathcal{K}_d$. Hence, with respect to (3.2) the representation of the J-selfadjoint operator $J\widetilde{A}$ is as follows:

$$J\widetilde{A} = \begin{pmatrix} A_+ & 0 & 0 \\ 0 & -A_- & 0 \\ 0 & 0 & J_d A_d \end{pmatrix}. \tag{3.8}$$

Note that A_+ is a selfadjoint operator in the Hilbert space $(\mathcal{K}_+, (\cdot,\cdot))$ and that $-A_-$ is a selfadjoint operator in $(\mathcal{K}_-, -(\cdot,\cdot))$. In particular, $\sigma(A_+)$ and $\sigma(-A_-)$ are real.

Step 2. In this step we show that statements (a)–(c) in the theorem hold with JA replaced by the operator $J\widetilde{A}$ in (3.8). Similar arguments have also been used in [3,6,49]. Observe first that by Theorem 3.2 the J_d-selfadjoint operator J_dA_d in the Krein space $(\mathcal{K}_d, (J_d\cdot,\cdot))$ is an operator with finitely many negative squares, $\sigma(J_dA_d)\cap(\mathbb{C}\backslash\mathbb{R})$ consists of at most finitely many normal eigenvalues, the interval $(0,\infty)$ is of type π_+ and the interval $(-\infty,0)$ is of type π_- with respect to J_dA_d. Furthermore J_dA_d is nonnegative in a neighborhood of ∞ and since J_dA_d is definitizable the resolvent satisfies the growth condition

$$\|(J_dA_d-\lambda)^{-1}\| \leq M_d(1+|\lambda|)^{2m_d-2}|\mathrm{Im}\,\lambda|^{-m_d} \tag{3.9}$$

for some $m_d \geq 1$, $M_d > 0$ and all nonreal $\lambda \in \rho(J_dA_d)$ near $\overline{\mathbb{R}}$; cf. [32,44,45] and Definition 2.1.

In order to verify definitizability of $J\widetilde{A}$ over the set $\overline{\mathbb{C}}\backslash[\mu,-\mu]$ observe first that $\sigma(J\widetilde{A})\cap(\mathbb{C}\backslash\mathbb{R})$ coincides with $\sigma(J_dA_d)\cap(\mathbb{C}\backslash\mathbb{R})$ and hence $\sigma(J\widetilde{A})\cap(\mathbb{C}\backslash\mathbb{R})$ consists of (at most finitely many) isolated points which are poles of the resolvent of $J\widetilde{A}$, and, in particular, no point of $\overline{\mathbb{R}}\backslash[\mu,-\mu]$ is an accumulation point of the nonreal eigenvalues of $J\widetilde{A}$. The growth condition in Definition 2.1 on the resolvent of $J\widetilde{A}$ is satisfied, since $\|(J_dA_d-\lambda)^{-1}\|$ can be estimated as in (3.9) and the norm of the resolvents of the selfadjoint operators A_+ and $-A_-$ can be estimated by $|\mathrm{Im}\,\lambda|^{-1}$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

We show that condition (i) in Definition 2.1 holds for all $\lambda \in \overline{\mathbb{R}}\backslash[\mu,-\mu]$. For this assume first $\lambda \in (-\mu,\infty)$. We have to check that there exists some $\delta > 0$ such that the intervals $(\lambda-\delta,\lambda)$ and $(\lambda,\lambda+\delta)$ are of definite type with respect to $J\widetilde{A}$. If $\lambda \in \rho(J\widetilde{A})$ this is clear. If $\lambda \in \sigma(J\widetilde{A})$, then, from $\sigma_{\mathrm{ess}}(-A_-) \subset (-\infty,-\mu]$ we conclude $\lambda \notin \sigma_{\mathrm{ess}}(-A_-)$, i.e., there exists $\delta > 0$ such that $(\lambda-\delta,\lambda+\delta)\backslash\{\lambda\} \subset \rho(-A_-)$. In particular, the set $(\lambda-\delta,\lambda+\delta)\backslash\{\lambda\}$ is of positive type with respect to $-A_-$. Furthermore, since the interval $(0,\infty)$ is of type π_+ with respect to the definitizable operator J_dA_d we can assume that δ is chosen such that $(\lambda-\delta,\lambda+\delta)\backslash\{\lambda\}$ is also of positive type with respect to J_dA_d, see Remark 2.3. Clearly, as A_+ is a selfadjoint operator in the Hilbert space $(\mathcal{K}_+,(\cdot,\cdot))$ any real spectral point is of positive type with respect to A_+ and hence $(\lambda-\delta,\lambda+\delta)\backslash\{\lambda\}$ is of positive type with respect to A_+. Now it is not difficult to see that $(\lambda-\delta,\lambda+\delta)\backslash\{\lambda\}$ is also of positive type with respect to the orthogonal sum of the operators A_+, $-A_-$ and J_dA_d, that is, $(\lambda-\delta,\lambda+\delta)\backslash\{\lambda\}$ is of positive type with respect to $J\widetilde{A}$.

In the case $\lambda\in(-\infty,\mu)$ a similar reasoning applies. In fact, as $\lambda\notin\sigma_{\text{ess}}(A_+)$ there exists $\delta'>0$ such that $(\lambda-\delta',\lambda+\delta')\backslash\{\lambda\}\subset\rho(A_+)$ and according to Theorem 3.2 we can assume that $(\lambda-\delta',\lambda+\delta')\backslash\{\lambda\}$ is of negative type with respect to J_dA_d. Furthermore, every spectral point of the selfadjoint operator $-A_-$ in $(\mathcal{K}_-,-(\cdot,\cdot))$ is of negative type and hence $(\lambda-\delta',\lambda+\delta')\backslash\{\lambda\}$ is of negative type with respect to $-A_-$. Therefore $(\lambda-\delta',\lambda+\delta')\backslash\{\lambda\}$ is of negative type with respect to $J\widetilde{A}$.

It remains to discuss the point $\lambda=\infty$. For this choose $\eta>0$ such that $(\eta,\infty)\subset\rho(-A_-)$ and (η,∞) is of positive type with respect to J_dA_d, which is possible according to Theorem 3.2. As (η,∞) is also of positive type with respect to A_+ it follows that (η,∞) is of positive type with respect to the orthogonal sum $J\widetilde{A}$. Analogous arguments show that for some $\eta'>0$ sufficiently large the interval $(-\infty,-\eta')$ is of negative type with respect to $J\widetilde{A}$. We have shown that $J\widetilde{A}$ is definitizable over $\overline{\mathbb{C}}\backslash[\mu,-\mu]$ and that $J\widetilde{A}$ is nonnegative in a neighborhood of ∞, i.e., assertion (a) holds for $J\widetilde{A}$.

Let us show that assertion (b) holds for $J\widetilde{A}$. In the above arguments it was already shown that the interval $(-\mu,\infty)$, with the possible exception of a discrete set that can only accumulate to $-\mu$, is of positive type with respect to $J\widetilde{A}$. The exceptional points are eigenvalues of $-A_-$ or spectral points of J_dA_d which are not of positive type. Since $(-\mu,\infty)\cap\sigma_{\text{ess}}(-A_-)=\varnothing$ and $(0,\infty)$ is of type π_+ with respect to J_dA_d it follows directly from

$$\ker(J\widetilde{A}-\lambda)=\ker(A_+-\lambda)\oplus\ker(-A_--\lambda)\oplus\ker(J_dA_d-\lambda)$$

and Definition 2.2 that the spectral points of $J\widetilde{A}$ in $(-\mu,\infty)$ which are not of positive type are of type π_+ with respect to $J\widetilde{A}$. A similar reasoning shows that $(-\infty,\mu)$ is of type π_- with respect to $J\widetilde{A}$.

Finally, assertion (c) is true since the nonreal eigenvalues of $J\widetilde{A}$ are the (at most finitely many) nonreal eigenvalues of the operator J_dA_d. Observe that by Theorem 3.2 these are normal eigenvalues.

Step 3. It is clear that $J\widetilde{A}$ and JA are both finite-dimensional J-selfadjoint extensions of the J-symmetric operator JS. Therefore, as $\rho(JA)\neq\varnothing$ by assumption and $\sigma(J\widetilde{A})\backslash\mathbb{R}=\sigma(J_dA_d)\backslash\mathbb{R}$ consists of at most finitely many points we conclude that $\rho(JA)\cap\rho(J\widetilde{A})$ is nonempty and that

$$\dim\operatorname{ran}\Big((JA-\lambda)^{-1}-(J\widetilde{A}-\lambda)^{-1}\Big)\le n_\pm(S)<\infty$$

holds for all $\lambda \in \rho(JA) \cap \rho(J\widetilde{A})$; cf. (3.5). From [7, Theorem 3.2] we now obtain that JA is also locally definitizable over $\overline{\mathbb{C}} \setminus [\mu, -\mu]$, the interval $(-\mu, \infty)$ is of type π_+ and the interval $(-\infty, \mu)$ is of type π_- with respect to JA, and JA is nonnegative in a neighborhood of ∞. □

Remark 3.10 *In general the region of definitizability of the operator JA in Theorem* 3.5 *can not be enlarged. However, easy examples show that the statement is not always optimal. Suppose, e.g., that $\mu = \min \sigma_{\text{ess}}(A) < 0$ is an isolated point of $\sigma_{\text{ess}}(A)$ and that $(\mu, -\mu]$ is contained in $\rho(A)$. Then a slight variation of the proof of Theorem* 3.5 *shows that the diagonal operator $J\widetilde{A}$ in* (3.8) *is definitizable (over $\overline{\mathbb{C}}$) and hence JA is definitizable (over $\overline{\mathbb{C}}$). Furthermore, at least one of the points μ or $-\mu$ belongs to $\sigma_{\text{ess}}(JA)$. If $\mu \in \sigma_{\text{ess}}(JA)$ ($-\mu \in \sigma_{\text{ess}}(JA)$), then it can be shown that μ ($-\mu$) is an isolated spectral point of JA which is not of type π_- (type π_+, respectively).*

4 Indefinite Sturm-Liouville operators

In this section we study the spectral properties of J-selfadjoint operators associated to the indefinite Sturm-Liouville differential expression

$$\ell = \frac{1}{w}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right), \tag{4.1}$$

on an open interval (a, b) where $-\infty \leq a < b \leq \infty$, the coefficients w, p^{-1}, q are real valued and locally integrable, $w(x) \neq 0$ and $p(x) > 0$ for a.e. $x \in (a, b)$, and the weight function w changes its sign on (a, b). Under certain natural assumptions on the differential expression ℓ we apply our general perturbation result from the previous section to a differential operator T associated to ℓ. For this it is necessary to ensure that the resolvent set of this J-selfadjoint operator T is nonempty; cf. Theorem 3.5. Since this fact seems to be known only for a special class of indefinite differential expressions, see, e.g., [34,36,37], we investigate this problem in Theorem 4.5. Making use of local sign type properties of Titchmarsh-Weyl functions associated to Sturm-Liouville expressions on subintervals of (a, b) we verify that under the conditions (I) and (II) below the resolvent set of the J-selfadjoint operator T is in fact always nonempty.

We will consider the case where the weight function w in (4.1) has different signs at the endpoints of the bounded or unbounded interval (a, b). More precisely, we will assume that the following condition (I) holds:

(I) There exist $\alpha, \beta \in (a, b)$, $\alpha < \beta$, such that $w(x) < 0$ for a.e. $x \in (a, \alpha)$ and $w(x) > 0$ for a.e. $x \in (\beta, b)$.

Let $L^2_{|w|}(a,b)$ be the space of all equivalence classes of complex valued measurable functions f defined on (a,b) such that $\int_a^b |f(x)|^2 |w(x)|\, dx$ is finite and let $(\cdot,\cdot)$ be be the usual Hilbert space scalar product in $L^2_{|w|}(a,b)$ from (1.3). We also equip $L^2_{|w|}(a,b)$ with the indefinite inner product (1.4). Since by assumption $w(x) \neq 0$ for a.e. $x \in (a,b)$ the space $(L^2_{|w|}(a,b), [\cdot,\cdot])$ is a Krein space and the fundamental symmetry

$$(Jf)(x) := (\operatorname{sgn} w(x)) f(x), \qquad x \in (a,b), \quad f \in L^2_{|w|}(a,b), \tag{4.2}$$

connects the indefinite inner product $[\cdot,\cdot]$ in (1.4) with the Hilbert space scalar product $(\cdot,\cdot)$ in (1.3), i.e., $[Jf,g] = (f,g)$ for all $f,g \in L^2_{|w|}(a,b)$.

Besides the indefinite Sturm-Liouville expression ℓ in (4.1) we shall also make use of its definite counterpart $\tau = J\ell$ in (1.2). Denote by $\mathcal{D}_{\max}$ the set of all functions $f \in L^2_{|w|}(a,b)$ such that f and pf' are absolutely continuous and $\tau(f) \in L^2_{|w|}(a,b)$ (or, equivalenty, $\ell(f) \in L^2_{|w|}(a,b)$) and define the maximal Sturm-Liouville differential operator A associated to the definite differential expression τ by

$$Af := \tau(f) = \frac{1}{|w|}\Big(-(pf')' + qf\Big), \qquad \operatorname{dom} A = \mathcal{D}_{\max}. \tag{4.3}$$

Later we shall sometimes write $\mathcal{D}_{\max}(a,b)$ instead of $\mathcal{D}_{\max}$ to emphasize that the functions are defined on the interval (a,b). It will be assumed that the following condition (II) holds for A:

(II) The operator A is selfadjoint in the Hilbert space $(L^2_{|w|}(a,b), (\cdot,\cdot))$ and semibounded from below.

Remark 4.1 *If the differential expression τ is regular or in the limit circle case at the endpoint a or b, then suitable boundary conditions have to be imposed on the functions in $\mathcal{D}_{\max}$ in order to ensure the selfadjointness in condition* (II)*, see, e.g.,* [48,50–52]*. However, we are mainly interested in singular differential expressions where both endpoints are in the limit point case since our methods yield only new insights for semibounded differential operators A with nonempty essential spectrum. It is well known that A is selfadjoint if and only if τ is in the limit point case at a and b. We refer to* [48,50–52] *for conditions on the coefficients w, p and q that imply semiboundedness of A, see also Corollaries* 4.3 *and* 4.4*.*

The maximal operator T associated to the indefinite differential expression ℓ is defined as $T := JA$, i.e.,

$$Tf = JAf = \frac{1}{w}\Big(-(pf')' + qf\Big), \qquad \operatorname{dom} T = \operatorname{dom} A. \tag{4.4}$$

Since by condition (II) A is selfadjoint in the Hilbert space $L^2_{|w|}(a,b)$ it is clear that T is J-selfadjoint.

For the case that the lower bound of the spectrum or the essential spectrum of A is positive Theorem 3.1 and Theorem 3.2 imply that T is J-nonnegative or T has finitely many negative squares, respectively. These facts are well known, see, e.g., [19,39]. The next theorem deals with the more difficult case $\min \sigma_{\text{ess}}(A) \leq 0$ and generalizes earlier results from [3,6,11,36,37]. With the exception of the statement $\rho(T) \neq \varnothing$ the proof is a consequence of the abstract perturbation result Theorem 3.5.

Theorem 4.2 *Suppose that conditions* (I) *and* (II) *are satisfied, let* $\mu = \min \sigma_{\text{ess}}(A) \leq 0$ *and let* T *be the* J*-selfadjoint indefinite Sturm-Liouville operator from* (4.4)*. Then the following statements hold:*

- (a) T *is definitizable over* $\overline{\mathbb{C}}\backslash[\mu, -\mu]$ *and nonnegative in a neighborhood of* ∞*;*
- (b) $(-\infty, \mu)$ *is of type* π_- *and* $(-\mu, \infty)$ *is of type* π_+ *with respect to* T*;*
- (c) $\sigma(T) \cap (\mathbb{C}\backslash\mathbb{R})$ *is bounded and consists of normal eigenvalues with only possible accumulation points in* $[\mu, -\mu]$.

Proof. Let us verify that the conditions (i)-(iii) in Theorem 3.5 are fulfilled. Observe first, that in the present situation the fundamental decomposition of $L^2_{|w|}(a,b)$ is given by $L^2_{|w|}(a,b) = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_\pm = L^2_{|w|}(\Delta_\pm) \quad \text{and} \quad \Delta_\pm := \{x \in (a,b) : \pm w(x) > 0\}.$$

Let $\alpha < \beta$ be as in condition (I) and define

$$\mathcal{K}_- := L^2_{|w|}(a,\alpha), \quad \mathcal{K}_d := L^2_{|w|}(\alpha,\beta), \quad \text{and} \quad \mathcal{K}_+ := L^2_{|w|}(\beta,b);$$

here the index function $|w|$ is the corresponding restriction of $|w|$ onto the interval (a,α), (α,β) and (β,b), respectively, and the weighted L^2-spaces and their inner products are defined in the same way as $L^2_{|w|}(a,b)$ and the inner products in (1.3) and (1.4). By condition (I) we have $(\beta,b) \subset \Delta_+$ and $(a,\alpha) \subset \Delta_-$ and therefore $\mathcal{K}_\pm \subset \mathcal{H}_\pm$. Furthermore, it is clear that $\mathcal{K}_+$, $\mathcal{K}_-$ and $\mathcal{K}_d$ are closed subspaces of $L^2_{|w|}(a,b)$ and that the decomposition

$$L^2_{|w|}(a,b) = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathcal{K}_d$$

holds, i.e., (3.2) is valid.

Denote the sets of functions that are restrictions of elements in $\mathcal{D}_{\max}$ onto the subintervals (a,α), (α,β) and (β,b) by $\mathcal{D}_{\max}(a,\alpha)$, $\mathcal{D}_{\max}(\alpha,\beta)$ and $\mathcal{D}_{\max}(\beta,b)$, respectively. These are the maximal domains of the differential operators associated to ℓ and τ in the spaces $L^2_{|w|}(a,\alpha)$, $L^2_{|w|}(\alpha,\beta)$ and $L^2_{|w|}(\beta,b)$, respectively.

Let now

$$\begin{aligned}\mathcal{G}_- &:= \Big\{ g \in \mathcal{D}_{\max}(a,\alpha) : g(\alpha) = (pg')(\alpha) = 0 \Big\}, \\ \mathcal{G}_d &:= \Big\{ h \in \mathcal{D}_{\max}(\alpha,\beta) : h(\alpha) = (ph')(\alpha) = h(\beta) = (ph')(\beta) = 0 \Big\}, \\ \mathcal{G}_+ &:= \Big\{ k \in \mathcal{D}_{\max}(\beta,b) : k(\beta) = (pk')(\beta) = 0 \Big\}.\end{aligned}$$

Then we have $\mathcal{G}_i \subset \operatorname{dom} A \cap \mathcal{K}_i$, $i \in \{+,-,d\}$, and it is easy to see that the conditions (i) and (ii) of Theorem 3.5 are fulfilled. As the restriction of the differential expression τ onto (α,β) is regular at α and β any selfadjoint realization in the Hilbert space $L^2_{|w|}(\alpha,\beta)$ has a compact resolvent and hence the closed minimal operator $A \upharpoonright \mathcal{G}_d$ has no essential spectrum; cf. [48,50–52].

The fact that the resolvent set $\rho(T)$ of the J-selfadjoint operator $T = JA$ is nonempty will be shown in Theorem 4.5 below. Hence, all conditions of Theorem 3.5 are satisfied and therefore the assertions of Theorem 4.2 follow. □

If condition (I) is replaced by the condition

(I′) There exist $\alpha,\beta \in (a,b)$, $\alpha < \beta$, such that $w(x) > 0$ for a.e. $x \in (a,\alpha)$ and $w(x) < 0$ for a.e. $x \in (\beta,b)$;

then the assertions in Theorem 4.2 remain true. It is also not difficult to see that in the case that the signs of w on (a,α) and (β,b) coincide and that (II) holds the operator T is definitizable (over $\overline{\mathbb{C}}$); cf. Corollary 3.9.

The following corollary is an immediate consequence of Theorem 4.2 and well known spectral properties of selfadjoint Sturm-Liouville differential operators in Hilbert spaces. We leave it to the reader to formulate a variant of Corollary 4.3 for the case $\mu > 0$; cf. Theorems 3.1 and 3.2, and [19,39].

Corollary 4.3 *Let $(a,b) = \mathbb{R}$ and suppose that condition* (I) *holds. Assume, in addition, that the functions $s_\pm(x) = \int_0^x \frac{1}{p(t)}dt$, $x \in \mathbb{R}^\pm$, do not belong to $L^2_{|w|}(\mathbb{R}^\pm)$ and that*

$$Q_{\pm\infty} := \liminf_{x\to\pm\infty} \frac{q(x)}{|w(x)|} > -\infty. \tag{4.5}$$

Then condition (II) *is valid and* $\min\{Q_{-\infty}, Q_{+\infty}\} \le \min \sigma_{\mathrm{ess}}(A)$ *holds. In the case* $\mu = \min \sigma_{\mathrm{ess}}(A) \le 0$ *the statements* (a)-(c) *in Theorem* 4.2 *hold for the J-selfadjoint indefinite Sturm-Liouville operator T in* (4.4).

Proof. Since $s_\pm \notin L^2_{|w|}(\mathbb{R}^\pm)$ and (4.5) holds it follows that both singular endpoints $+\infty$ and $-\infty$ are in the limit point case by [51, Satz 13.24], and hence the operator A in (4.3) is selfadjoint in $L^2_{|w|}(a,b)$. Let $\gamma_\pm < Q_{\pm\infty}$ be real numbers. It is no restriction to assume that α and β in condition (I) are chosen such that

$$\frac{q(x)}{|w(x)|} \geq \gamma_+, \quad x \in (\beta, \infty), \quad \text{and} \quad \frac{q(x)}{|w(x)|} \geq \gamma_-, \quad x \in (-\infty, \alpha),$$

hold. Let $\mathcal{G}_-$, $\mathcal{G}_d$ and $\mathcal{G}_+$ be defined as in the proof of Theorem 4.2 and denote the restrictions of A onto these subspaces by S_-, S_d and S_+, respectively. Then S_-, S_d and S_+ coincide with the minimal operators associated to the definite differential expression τ in $L^2_{|w|}(-\infty, \alpha)$, $L^2_{|w|}(\alpha, \beta)$ and $L^2_{|w|}(\beta, \infty)$, respectively. As in the proof of Theorem 4.2 we have $\sigma_{\mathrm{ess}}(S_d) = \varnothing$ and S_d is semibounded from below. According to [50, Theorem 6.A.1 (page 104)] the operators S_+ and S_- are semibounded from below by γ_+ and γ_-, respectively. As $S := S_- \oplus S_d \oplus S_+ \subset A$ and S has finite defect it follows that A is semibounded from below. Furthermore, we have

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(S_-) \cup \sigma_{\mathrm{ess}}(S_+) \subset [\min\{\gamma_-, \gamma_+\}, \infty).$$

With $\gamma_\pm \to Q_{\pm\infty}$ the corollary is proved. □

In the special case that the coefficient q admits limits $q_{\pm\infty}$ at $\pm\infty$ and $p(x) = 1$, $w(x) = \operatorname{sgn}(x)$ outside of a compact subinterval it is easy to see that $\sigma_{\mathrm{ess}}(A) = [\min\{q_{+\infty}, q_{-\infty}\}, \infty)$ holds. In this case Theorem 4.2 reduces to the following statement; cf. [6, Corollary 3.4] where an additional condition was imposed to ensure $\rho(T) \neq \varnothing$.

Corollary 4.4 *Let $(a, b) = \mathbb{R}$ and suppose that $p(x) = 1$ and $w(x) = \operatorname{sgn}(x)$ for a.e. $x \in (-\infty, \alpha) \cup (\beta, +\infty)$ for some numbers $\alpha < \beta$. Assume, in addition, that the limits*

$$q_{-\infty} := \lim_{x \to -\infty} q(x) \qquad \text{and} \qquad q_{+\infty} := \lim_{x \to +\infty} q(x)$$

exist and satisfy $\mu := \min\{q_{-\infty}, q_{+\infty}\} \leq 0$. Then conditions (I) *and* (II) *are valid, $[\mu, \infty) = \sigma_{\mathrm{ess}}(A)$, the statements* (a)-(c) *in Theorem* 4.2 *hold for the J-selfadjoint differential operator T in* (4.4) *and $\sigma_{\mathrm{ess}}(T) = \mathbb{R}$.*

We mention that the statements in Corollary 4.4 can be slighly improved, namely, the assertions (a)-(c) in Theorem 4.2 even hold for the possibly smaller interval $[q_{+\infty}, -q_{-\infty}]$; cf. Remark 3.10.

The following theorem completes the proof of Theorem 4.2 but is also of independent interest.

Theorem 4.5 *Assume that conditions* (I) *and* (II) *are satisfied. Then the resolvent set of the J-selfadjoint indefinite Sturm-Liouville operator $T = JA$ in* (4.4) *is nonempty.*

Proof. Let ℓ be as in (4.1) and β as in condition (I), set

$$J_{a\beta} := J \upharpoonright L^2_{|w|}(a,\beta) \quad \text{and} \quad J_{\beta b} := J \upharpoonright L^2_{|w|}(\beta,b),$$

and define two operators $T_{a\beta}$ and $T_{\beta b}$ in $L^2_{|w|}(a,\beta)$ and $L^2_{|w|}(\beta,b)$, respectively, by

$$T_{a\beta}g := \ell(g), \qquad \operatorname{dom} T_{a\beta} := \{g \in \mathcal{D}_{\max}(a,\beta) : g(\beta) = 0\},$$

and

$$T_{\beta b}k := \ell(k), \qquad \operatorname{dom} T_{\beta b} := \{k \in \mathcal{D}_{\max}(\beta,b) : k(\beta) = 0\}.$$

Then $T_{a\beta}$ is $J_{a\beta}$-selfadjoint in $L^2_{|w|}(a,\beta)$ and as $J_{\beta b}$ is the identity operator in $L^2_{|w|}(\beta,b)$ the definite and indefinite differential expressions τ and ℓ coincide on (β,b). Thus the operator $T_{\beta b}$ is a selfadjoint Sturm-Liouville operator in $L^2_{|w|}(\beta,b)$.

The rest of the proof is devided into four steps. The idea is as follows: We show first that the $J_{a\beta}$-selfadjoint operator $T_{a\beta}$ is definitizable and nonnegative in a neighborhood of ∞. Nonnegativity at ∞ is also reflected in local sign type properties of the Titchmarsh-Weyl coefficient $m_{a\beta}$ associated to $T_{a\beta}$. If $\rho(T)$ was empty, the Titchmarsh-Weyl coefficient $m_{\beta b}$ associated to the selfadjoint Sturm-Liouville operator $T_{\beta b}$ would coincide with $-m_{a\beta}$, which together with the fact that $m_{\beta b}$ is a Nevanlinna function leads to a contradiction.

Step 1. In this step of the proof we show that the operator $T_{a\beta}$ is definitizable and nonnegative in a neighborhood of ∞. In fact, this is a simple consequence of Corollary 3.9. To see this, we set $A_{a\beta} := J_{a\beta}T_{a\beta}$, $\mathcal{H} := L^2_{|w|}(a,\beta)$ and

$$\mathcal{H}_\pm := L^2_{|w|}(\delta_\pm), \quad \text{where} \quad \delta_\pm := \{x \in (a,\beta) : \pm w(x) > 0\}.$$

Then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is the fundamental decomposition induced by the fundamental symmetry $J_{a\beta}$ and $A_{a\beta}$ is a selfadjoint operator in the Hilbert space $\mathcal{H}$. Since $A_{a\beta}$ is a one dimensional extension of the minimal symmetric operator associated to τ on (a,β) and this symmetric operator is contained in A it follows from condition (II) that $A_{a\beta}$ is semibounded from below. Furthermore, let α be as in condition (I) and define the subspaces $\mathcal{K}_-$, $\mathcal{K}_d$ and the linear manifolds $\mathcal{G}_-$ and $\mathcal{G}_d$ as in the proof of Theorem 4.2. Now, it is easy to see that the conditions (i)$^-$- (iii)$^-$ of Corollary 3.9 are satisfied with A replaced by $A_{a\beta}$. Hence, the operator $T_{a\beta}$ is definitizable and nonnegative in a neighborhood of ∞.

Step 2. Let $\lambda \in \mathbb{C}$ and denote by ϕ_λ and ψ_λ the unique solutions of the equation $(\ell - \lambda)u = 0$ which satisfy

$$\begin{aligned} \phi_\lambda(\beta) &= 1, \quad (p\phi_\lambda')(\beta) = 0, \\ \psi_\lambda(\beta) &= 0, \quad (p\psi_\lambda')(\beta) = 1, \end{aligned} \tag{4.6}$$

i.e., ϕ_λ, ψ_λ, $p\phi_\lambda'$ and $p\psi_\lambda'$ are locally absolutely continuous on (a,b), ϕ_λ and ψ_λ solve the equation

$$-(pu')' + (q - \lambda w)u = 0,$$

and (4.6) holds. For $\lambda \in \rho(T_{a\beta})$ and $\mu \in \rho(T_{\beta b})$ the (values of the) Titchmarsh-Weyl coefficients corresponding to ℓ on (a,β) and (β,b) are defined as the unique complex numbers $m_{a\beta}(\lambda)$ and $m_{\beta b}(\mu)$, respectively, such that

$$\phi_\lambda - m_{a\beta}(\lambda)\psi_\lambda \in L^2_{|w|}(a,\beta) \quad \text{and} \quad \phi_\mu + m_{\beta b}(\mu)\psi_\mu \in L^2_{|w|}(\beta,b),$$

respectively. The existence and uniqueness of $m_{a\beta}(\lambda)$ follow from the fact that the definite differential expression τ is in the limit point case at a by assumption (II) and the definiteness of w on (a,α). We also remark that $m_{\beta b}$ is the usual Titchmarsh-Weyl coefficient on (β,b) corresponding to τ and the selfadjoint operator $T_{\beta b}$. In this step we will show the following implication:

$$\rho(T) = \varnothing \implies -m_{a\beta}(\lambda) = m_{\beta b}(\lambda) \text{ for all } \lambda \in \rho(T_{a\beta}) \cap \rho(T_{\beta b}). \tag{4.7}$$

Let us assume that $\rho(T) = \varnothing$. Then

$$\rho(T_{a\beta}) \cap \rho(T_{\beta b}) \subset \sigma_p(T). \tag{4.8}$$

To see this let $S := T \upharpoonright \{f \in \operatorname{dom} T : f(\beta) = 0\}$. Obviously, the operator S is contained in both T and $T_{a\beta} \oplus T_{\beta b}$ (with respect to the decomposition $L^2_{|w|}(a,b) = L^2_{|w|}(a,\beta) \oplus L^2_{|w|}(\beta,b)$) and has defect one. Now, if $\lambda \in \rho(T_{a\beta}) \cap \rho(T_{\beta b})$ then $\lambda \in \rho(T_{a\beta} \oplus T_{\beta b})$, and thus $\operatorname{ran}(S - \lambda)$ has codimension one and $\ker(S-\lambda) = \{0\}$. By writing $\operatorname{dom} T = \operatorname{dom} S \dotplus \operatorname{span}\{g\}$ with some $g \in \operatorname{dom} T$ it is not difficult to see that λ is an eigenvalue of T as $\lambda \notin \rho(T)$ by assumption.

Let $\lambda \in \rho(T_{a\beta}) \cap \rho(T_{\beta b})$. Then, by (4.8) there exists a nontrivial solution f of $(\ell - \lambda)u = 0$ which is an element of $L^2_{|w|}(a,b)$ and hence the restrictions of f onto (a,β) and (β,b) belong to $L^2_{|w|}(a,\beta)$ and $L^2_{|w|}(\beta,b)$, respectively. It is no restriction to assume $f(\beta) = 1$. Due to the definition of the Titchmarsh-Weyl coefficients we have

$$f(x) = \begin{cases} \phi_\lambda(x) - m_{a\beta}(\lambda)\psi_\lambda(x), & x \in (a,\beta), \\ \phi_\lambda(x) + m_{\beta b}(\lambda)\psi_\lambda(x), & x \in (\beta,b). \end{cases}$$

From this and (4.6) we conclude

$$-m_{a\beta}(\lambda) = \lim_{x\uparrow\beta}(pf')(x) = \lim_{x\downarrow\beta}(pf')(x) = m_{\beta b}(\lambda)$$

which proves (4.7). We note that in the abstract framework of boundary triplets for symmetric operators in Hilbert and Krein spaces (see [24,26]) the Titchmarsh-Weyl coefficients $m_{a\beta}$ and $m_{\beta b}$ can be viewed as the Weyl functions of suitable boundary triplets for the closed minimal operators on (a,β) and (β,b) associated to the Sturm-Liouville differential expressions ℓ and τ. From this point of view the implication (4.7) is a simple consequence of general properties of boundary triplets and their Weyl functions.

Step 3. By step 1 of the proof the operator $T_{a\beta}$ is (definitizable and) nonnegative in a neighborhood of ∞. This implies, in particular, that there exists a number $R>0$ such that the interval (R,∞) is of positive type with respect to $T_{a\beta}$. In this step we prove that for any sequence $(\lambda_n)\subset\mathbb{C}^+\cap\rho(T_{a\beta})$ which converges to some $\lambda>R$ we have

$$\liminf_{n\to\infty}\,\operatorname{Im} m_{a\beta}(\lambda_n)\geq 0. \tag{4.9}$$

Let us suppose that this is not true. Then there exists some $\lambda>R$, a sequence $(\lambda_n)\subset\mathbb{C}^+\cap\rho(T_{a\beta})$ converging to λ and some $\varepsilon>0$ such that

$$\operatorname{Im} m_{a\beta}(\lambda_n)\leq -\varepsilon \tag{4.10}$$

holds for all $n\in\mathbb{N}$. For $\mu\in\rho(T_{a\beta})$ define the function

$$g_\mu := \phi_\mu - m_{a\beta}(\mu)\psi_\mu \in L^2_{|w|}(a,\beta).$$

The indefinite and definite inner products in $L^2_{|w|}(a,\beta)$ are defined in the same way as in (1.4) and (1.3) and will also be denoted by $[\cdot,\cdot]$ and $(\cdot,\cdot)$, respectively. By $\ell g_\mu=\mu g_\mu$, the Lagrange identity, (4.6) and the fact that τ is in the limit point case at a we have

$$\begin{aligned}(\mu-\bar\mu)\,[g_\mu,g_\mu] &= [\ell g_\mu,g_\mu]-[g_\mu,\ell g_\mu]=(\tau g_\mu,g_\mu)-(g_\mu,\tau g_\mu)\\ &= g_\mu(\beta)\overline{(pg_\mu')(\beta)}-(pg_\mu')(\beta)\overline{g_\mu(\beta)}=m_{a\beta}(\mu)-\overline{m_{a\beta}(\mu)},\end{aligned}$$

and hence $\operatorname{Im} m_{a\beta}(\mu)=(\operatorname{Im}\mu)\,[g_\mu,g_\mu]$. Thus, (4.10) implies

$$-\varepsilon\geq(\operatorname{Im}\lambda_n)\,[g_{\lambda_n},g_{\lambda_n}],\qquad\text{where}\quad g_{\lambda_n}=\phi_{\lambda_n}-m_{a\beta}(\lambda_n)\psi_{\lambda_n}, \tag{4.11}$$

for all $n\in\mathbb{N}$. In particular, this yields $[g_{\lambda_n},g_{\lambda_n}]\to-\infty$ and thus $\|g_{\lambda_n}\|\to\infty$ as $n\to\infty$. Let $\nu\in\rho(T_{a\beta})$ be fixed and $g_\nu=\phi_\nu-m_{a\beta}(\nu)\psi_\nu$. Then the functions

$$f_n := \|g_{\lambda_n}\|^{-1}\,(g_{\lambda_n}-g_\nu)\in\mathcal{D}_{\max}(a,\beta)$$

satisfy $f_n(\beta)=0$ and hence $f_n\in\operatorname{dom}T_{a\beta}$ for all $n\in\mathbb{N}$. From $\|f_n\|\to 1$,

$$\begin{aligned}(T_{a\beta}-\lambda)f_n &= \|g_{\lambda_n}\|^{-1}\,(\ell-\lambda)\,(g_{\lambda_n}-g_\nu)\\ &= (\lambda_n-\lambda)\|g_{\lambda_n}\|^{-1}g_{\lambda_n}-\|g_{\lambda_n}\|^{-1}(\nu-\lambda)g_\nu\to 0\end{aligned}$$

as $n \to \infty$ we obtain $\lambda \in \sigma_{\rm ap}(T_{a\beta})$. Since $\lambda > R$ it follows that λ is a spectral point of positive type of $T_{a\beta}$, hence

$$\liminf_{n\to\infty} [f_n, f_n] > 0. \tag{4.12}$$

If we now set $h_n := \|g_{\lambda_n}\|^{-1} g_{\lambda_n}$ then it follows from (4.11) that $[h_n, h_n] \leq 0$ for all $n \in \mathbb{N}$. But as $\|h_n - f_n\| = \|g_{\lambda_n}\|^{-1}\|g_\nu\| \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} [f_n, f_n] = \liminf_{n\to\infty} [h_n, h_n] \leq 0$$

which contradicts (4.12) and hence (4.9) holds.

Step 4. In this last step we complete the proof. Recall first that the Titchmarsh-Weyl coefficient $m_{\beta b}$ is an analytic function on $\rho(T_{\beta b})$ which maps the upper half plane into itself and, hence, admits the integral representation

$$m_{\beta b}(\lambda) = c + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\sigma(t), \tag{4.13}$$

where $c \in \mathbb{R}$ and σ is a nondecreasing function such that $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$; cf. [1,17,38]. We can assume that σ is normalized by $\sigma(t) = \frac{1}{2}(\sigma(t+0) - \sigma(t-0))$, so that σ can be expressed in terms of $m_{\beta b}$ via the inversion formula [38, (S1.1.7)],

$$\sigma(t_2) - \sigma(t_1) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{t_1}^{t_2} \operatorname{Im} m_{\beta b}(\lambda + i\varepsilon)\, d\lambda, \quad t_1, t_2 \in \mathbb{R}. \tag{4.14}$$

Let us suppose that $\rho(T) = \varnothing$. Let $(\lambda_n) \subset \mathbb{C}^+ \cap \rho(T_{a\beta})$ be a sequence which converges to some $\lambda > R$ where R is chosen such that (R, ∞) is of positive type with respect to $T_{a\beta}$. By step 3 we have

$$\liminf_{n\to\infty} \operatorname{Im} m_{a\beta}(\lambda_n) \geq 0.$$

Moreover, $m_{a\beta}(\mu) = -m_{\beta b}(\mu)$ holds for all $\mu \in \rho(T_{a\beta}) \cap \rho(T_{\beta b})$ by step 2 and hence

$$\limsup_{n\to\infty} \operatorname{Im} m_{\beta b}(\lambda_n) \leq 0. \tag{4.15}$$

As $m_{\beta b}$ maps the upper half plane into itself (4.15) implies that

$$\lim_{n\to\infty} \operatorname{Im} m_{\beta b}(\lambda_n) = 0 \tag{4.16}$$

holds for all sequences $(\lambda_n) \subset \mathbb{C}^+ \cap \rho(T_{a\beta})$ with $\lim_{n\to\infty} \lambda_n = \lambda > R$. For all $t_1, t_2 \in \mathbb{R}$ such that $R < t_1 < t_2 < \infty$ relation (4.16) together with (4.14) and the Lebesgue convergence theorem shows $\sigma(t_1) = \sigma(t_2)$. Hence, the function

σ is constant on (R,∞) and the integral representation (4.13) implies that $m_{\beta b}$ admits an analytic extension on (R,∞); cf. [38, §1.2]. But the domain of holomorphy of the Titchmarsh-Weyl coefficient $m_{\beta b}$ coincides with the set $\rho(T_{\beta b})$ and hence $(R,\infty)\subset\rho(T_{\beta b})$. As the selfadjoint Sturm-Liouville operator $T_{\beta b}$ is also bounded from below it follows that $T_{\beta b}$ is bounded, a contradiction. This completes the proof of Theorem 4.5. □

5 Higher order singular ordinary differential operators with indefinite weight functions

As in [19, Section 2] we consider the formal differential expression $\widehat{\ell}$ of order $2n$ on the interval (a,b), $-\infty\le a<b\le\infty$, given by

$$\widehat{\ell}(f)=\frac{1}{w}\Big((-1)^n(p_0f^{(n)})^{(n)}+(-1)^{n-1}(p_1f^{(n-1)})^{(n-1)}+\cdots+p_nf\Big),\qquad(5.1)$$

where $w,p_0^{-1},p_1,\dots,p_n\in L^1_{\rm loc}(a,b)$ are assumed to be real valued functions such that $w(x)\neq 0$ and $p_0(x)>0$ for a.e. $x\in(a,b)$. With the help of the quasi-derivatives

$$\begin{aligned}&f^{[0]}:=f, && f^{[k]}:=\frac{d^kf}{dx^k},\quad k=1,2,\dots,n-1,\\ &f^{[n]}:=p_0\frac{d^nf}{dx^n}, && f^{[n+k]}:=p_k\frac{d^{n-k}f}{dx^{n-k}}-\frac{d}{dx}f^{[n+k-1]},\quad k=1,2,\dots,n;\end{aligned}\qquad(5.2)$$

cf. [42,48], formula (5.1) can be written as

$$\widehat{\ell}(f)=\frac{1}{w}f^{[2n]}.\qquad(5.3)$$

It will be assumed that the weight function w satisfies condition (I) or (I′) from Section 4. Let $L^2_{|w|}(a,b)$ be the weighted L^2-space as in the previous section and equip $L^2_{|w|}(a,b)$ with the Hilbert space scalar product (1.3), the indefinite inner product (1.4) and let J be the fundamental symmetry from (4.2). Besides the indefinite differential expression (5.3) we also introduce the definite differential expression $\widehat{\tau}$ by

$$\widehat{\tau}(f)=\frac{1}{|w|}f^{[2n]};\qquad(5.4)$$

cf. (1.2). The maximal operator $A_{\max}f=\widehat{\tau}(f)$ associated to (5.4) is defined on the dense subspace $\mathfrak{D}_{\max}$ consisting of all functions $f\in L^2_{|w|}(a,b)$ which have locally absolutely continuous quasi derivatives $f^{[0]},f^{[1]},\dots,f^{[2n-1]}$ such that $\widehat{\tau}(f)\in L^2_{|w|}(a,b)$. The restriction $A^0_{\min}$ of $A_{\max}$ to functions with compact support is a densely defined symmetric operator in the Hilbert space $L^2_{|w|}(a,b)$. The minimal operator $A_{\min}$ is defined as the closure of $A^0_{\min}$. Then $A_{\min}$ is a symmetric operator with equal deficiency indies (m,m), $0 \le m \le 2n$, and $A^*_{\min} = A_{\max}$ holds; cf. [19,48]. In particular, the selfadjoint realizations of $\widehat{\tau}$ in the Hilbert space $L^2_{|w|}(a,b)$ are finite dimensional extensions of $A_{\min}$.

The following theorem is the analogue of Theorem 4.2 for the more general class of differential operators considered here. In contrast to Theorem 4.2 we have to impose the assumption that the resolvent set of the J-selfadjoint realization of $\widehat{\ell}$ is nonempty. In special cases it is known that this holds, see [3,19,39], Corollary 5.3 and Theorem 5.4. In Theorem 5.1 below it is assumed that the selfadjoint realizations of $\widehat{\tau}$ have a nonempty essential spectrum, which also implies that the deficiency indices of $A_{\min}$ are smaller than $2n$; cf. [48]. Although the proof of Theorem 5.1 is similar to the proof of Theorem 4.2 a short sketch is given for the convenience of the reader.

Theorem 5.1 *Suppose that condition* (I) *or* (I′) *in Section* 4 *holds and that A is a selfadjoint realization of $\widehat{\tau}$ in $L^2_{|w|}(a,b)$ which is semibounded from below such that $\mu := \min \sigma_{\mathrm{ess}}(A) \le 0$. Then $T := JA$ is a J-selfadjoint realization of $\widehat{\ell}$, and if $\rho(T) \neq \varnothing$, then the following statements hold:*

(a) *T is definitizable over $\overline{\mathbb{C}}\backslash[\mu,-\mu]$ and nonnegative in a neighborhood of ∞;*
(b) *$(-\infty,\mu)$ is of type π_- and $(-\mu,\infty)$ is of type π_+ with respect to T;*
(c) *$\sigma(T) \cap (\mathbb{C}\backslash\mathbb{R})$ is bounded and consists of normal eigenvalues with only possible accumulation points in $[\mu,-\mu]$.*

Proof. Suppose that condition (I) holds. The arguments in the case that condition (I′) holds are almost the same. Let us verify that the conditions (i)-(iii) in Theorem 3.5 are fulfilled. For this we decompose $L^2_{|w|}(a,b)$ in the same way as in the proof of Theorem 4.2,

$$L^2_{|w|}(a,b) = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad \text{and} \qquad L^2_{|w|}(a,b) = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathcal{K}_d.$$

With numbers α and β as in (I) let $\mathfrak{D}_{\max}(a,\alpha)$, $\mathfrak{D}_{\max}(\alpha,\beta)$ and $\mathfrak{D}_{\max}(\beta,b)$ be the sets of functions that are restrictions of elements in $\mathfrak{D}_{\max}$ onto the subintervals (a,α), (α,β) and (β,b), respectively. These are the maximal domains of the differential operators associated to $\widehat{\ell}$ and $\widehat{\tau}$ in the spaces $L^2_{|w|}(a,\alpha)$, $L^2_{|w|}(\alpha,\beta)$ and $L^2_{|w|}(\beta,b)$, respectively. Let now

$$\begin{aligned} \mathcal{G}_- &:= \Big\{ g \in \mathfrak{D}_{\max}(a,\alpha) : g^{[0]}(\alpha) = \cdots = g^{[2n-1]}(\alpha) = 0 \Big\}, \\ \mathcal{G}_d &:= \Big\{ h \in \mathfrak{D}_{\max}(\alpha,\beta) : h^{[0]}(\alpha) = \cdots = h^{[2n-1]}(\alpha) = 0, \\ &\qquad\qquad h^{[0]}(\beta) = \cdots = h^{[2n-1]}(\beta) = 0 \Big\}, \\ \mathcal{G}_+ &:= \Big\{ k \in \mathfrak{D}_{\max}(\beta,b) : k^{[0]}(\beta) = \cdots = k^{[2n-1]}(\beta) = 0 \Big\}. \end{aligned}$$

Then we have $\mathcal{G}_i \subset \operatorname{dom} A \cap \mathcal{K}_i$, $i \in \{+,-,d\}$, and the conditions (i) and (ii) of Theorem 3.5 are fulfilled. As the restriction of the differential expression $\widehat{\tau}$ onto (α, β) is regular at α and β the essential spectrum of the closed minimal operator $A \upharpoonright \mathcal{G}_d$ is empty; cf. [48]. Hence, all conditions of Theorem 3.5 are satisfied and therefore the assertions in Theorem 5.1 follow. □

Assume that $\widehat{\ell}$ in (5.1) is defined on $\mathbb{R}$, that $w(x) = \operatorname{sgn}(x)$ and that the coefficients $p_0, p_1, \ldots, p_n$, $p_0 > 0$, are constant outside of some bounded interval. Then it follows that $A := A_{\max}$ is selfadjoint and from [27, XIII.7. Corollary 14] together with Glazman's decomposition principle one obtains $\sigma_{\text{ess}}(A) = [\mu, \infty)$, where

$$\mu = \min\{\mathfrak{p}(t) : t \in \mathbb{R}\}, \quad \mathfrak{p}(t) = p_0 t^{2n} + p_1 t^{2(n-1)} + p_2 t^{2(n-2)} + \cdots + p_n. \tag{5.5}$$

This implies the following statement which is similar to Corollary 4.4.

Corollary 5.2 *Let $(a, b) = \mathbb{R}$, suppose that $w(x) = \operatorname{sgn}(x)$ and the functions $p_0, p_1, \ldots, p_n$ are constant for a.e. $x \in (-\infty, \alpha) \cup (\beta, +\infty)$ for some numbers $\alpha < \beta$, and assume that $\mu \leq 0$ in* (5.5). *Then $T = JA$ is a J-selfadjoint realization of $\widehat{\ell}$, and if $\rho(T) \neq \varnothing$, then the statements* (a)-(c) *in Theorem* 5.1 *hold and $\sigma_{\text{ess}}(T) = \mathbb{R}$.*

In general it is not clear if the resolvent set of the J-selfadjoint operator $T = JA$ in Theorem 5.1 is nonempty; cf. also Theorem 5.4. However, if at least one endpoint of the interval (a, b) is regular, that is, one endpoint is finite and the coefficients $w, p_0^{-1}, p_1, \ldots, p_n$ are integrable up to this point, then Corollary 3.9 implies the following statement.

Corollary 5.3 *Suppose that condition* (I) *or* (I′) *in Section* 4 *holds and that A is a selfadjoint realization of $\widehat{\tau}$ in $L^2_{|w|}(a, b)$ which is semibounded from below such that $\mu = \min \sigma_{\text{ess}}(A) \leq 0$. Assume, in addition, that $\widehat{\tau}$ is regular at a or b. Then $T = JA$ is a J-selfadjoint realization of $\widehat{\ell}$ with $\rho(T) \neq \varnothing$ and T is definitizable* (*over $\overline{\mathbb{C}}$*) *and nonnegative in a neighborhood of ∞. Furthermore,*

(i) *if* (I) *holds and a* (*b*) *is regular, then $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(A)$ ($\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(-A)$, respectively*).
(ii) *if* (I′) *holds and a* (*b*) *is regular, then $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(-A)$ ($\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(A)$, respectively*).

Our next goal is to find a sufficient condition for $\rho(T) \neq \varnothing$ in terms of the behaviour of the weight function w at its turning points and the properties of the functions in the form domain of A. Theorem 5.4 below is inspired by [19, Theorem 3.6] where a sufficient condition for the regularity of the critical point ∞ of J-selfadjoint realizations associated to ordinary differential expressions with indefinite weights is proved, see also [18]. Recall first that the *form domain* $\mathrm{dom}\,[A]$ of a selfadjoint operator A which is semibounded from below consists of all f such that there exists a sequence $(f_n) \subset \mathrm{dom}\,A$ with $f_n \to f$ and $(A(f_n - f_m), (f_n - f_m)) \to 0$ for $n, m \to \infty$. Next we recall the notion of n-simplicity of turning points used in [19]. For this let $v : (a, b) \to \mathbb{R}$ and let $x_0 \in (a, b)$. If for some $\delta > 0$ the function v is nonnegative (nonpositive) on $[x_0, x_0 + \delta]$ and there exists $\xi \in C^n[x_0, x_0 + \delta]$, $\xi(x_0) > 0$ and $\xi'(x_0+) = \cdots = \xi^{(n-1)}(x_0+) = 0$, such that

$$v(x) = (x - x_0)^\tau \xi(x) \qquad \Big(v(x) = -(x - x_0)^\tau \xi(x),\ \text{ respectively}\Big)$$

holds for some $\tau > -1$ and a.e. $x \in [x_0, x_0 + \delta]$, then v is said to be *n-simple from the right at x_0*. The function v is said to be *n-simple from the left at x_0* if the function $x \mapsto v(-(x - x_0) + x_0)$ is n-simple from the right at x_0. The function v is said to be *n-simple at x_0* if v is n-simple from the right and n-simple from the left at x_0.

Theorem 5.4 *Let A be a selfadjoint realization of $\hat\tau$ in $L^2_{|w|}(a, b)$ which is semibounded from below such that $\mu = \min \sigma_{\mathrm{ess}}(A) \le 0$ and assume that the following conditions hold.*

(i) *If $f \in \mathrm{dom}\,[A]$ coincides in a neighborhood of the endpoint a (b) with a function $g \in L^2_{|w|}(a, b)$, where $g, g', \dots, g^{(n-1)}$ are locally absolutely continuous on (a, b), $g^{(n)}\sqrt{p_0} \in L^2_{\mathrm{loc}}(a, b)$, and $g = 0$ in a neighborhood of the other endpoint b (a, respectively), then $g \in \mathrm{dom}\,[A]$;*
(ii) *w changes its sign at $2k + 1$ points, is n-simple at each of these turning points and p_0 and p_0^{-1} are essentially bounded in neighborhoods of the turning points.*

Then $T = JA$ is a J-selfadjoint realization of $\hat\ell$ with $\rho(T) \neq \varnothing$ and the statements (a)-(c) *in Theorem* 5.1 *hold.*

Proof. Let A be a selfadjoint realization of $\hat\tau$ which is semibounded from below. It will be shown that the resolvent set of the J-selfadjoint operator $T = JA$ is nonempty. For this fix some $\eta \in (-\infty, \nu)$, where $\nu = \min \sigma(A)$, and consider the differential expression

$$\hat\tau_\eta f := \frac{1}{|w|} f^{\langle 2n \rangle},$$

where the quasi-derivatives $f^{\langle k \rangle}$ are defined by

$$f^{\langle k \rangle} := f^{[k]},\ k = 0, \dots, 2n-1, \quad \text{and} \quad f^{\langle 2n \rangle} := (p_n - \eta|w|)f - \frac{d}{dx} f^{\langle 2n-1 \rangle};$$

cf. (5.2). Then $f^{\langle 2n \rangle} = f^{[2n]} - \eta|w|f$ and $\hat\tau_\eta f = \hat\tau f - \eta f$. Therefore $A_{\max} - \eta$ is the maximal operator corresponding to $\hat\tau_\eta$ on (a, b). From this and the choice of η we conclude that the operator $A - \eta$ is a uniformly positive selfadjoint realization of $\widehat{\tau}_\eta$ and that, by condition (i), $\operatorname{dom}[A-\eta] = \operatorname{dom}[A]$ is separated in the sense of [19, page 53]. Since by Theorem 3.1 the J-selfadjoint operator

$$T_\eta := J(A-\eta) = T - \eta J$$

is J-nonnegative with $0 \in \rho(T_\eta)$ it follows from [19, Theorem 3.6] that the point ∞ (the only critical point of T_η) is a regular critical point of T_η. Hence, T_η admits a spectral function $E_{T_\eta}(\cdot)$ on $\mathbb{R}$ such that the spectral projections $E_+ := E_{T_\eta}((0,\infty))$ and $E_- := E_{T_\eta}((-\infty,0))$ exist, see, e.g., [44,45].

Let $[\cdot,\cdot]$ be the indefinite inner product in (1.4) and let $[\dot{+}]$ be the direct $[\cdot,\cdot]$-orthogonal sum in $\mathcal{H} := L^2_{|w|}(a,b)$. Since T_η is J-nonnegative and $0 \in \rho(T_\eta)$ the spectral subspaces $(E_+\mathcal{H},[\cdot,\cdot])$ and $(E_-\mathcal{H},-[\cdot,\cdot])$ are Hilbert spaces, $E_+ + E_- = I$ and with respect to the decomposition $\mathcal{H} = E_+\mathcal{H}\,[\dot{+}]E_-\mathcal{H}$ the J-selfadjoint operator T_η can be written in the form

$$T_\eta = \begin{pmatrix} T_{\eta,+} & 0 \\ 0 & T_{\eta,-} \end{pmatrix},$$

where $T_{\eta,+}$ and $T_{\eta,-}$ are selfadjoint in the Hilbert spaces $(E_\pm\mathcal{H},\pm[\cdot,\cdot])$. The Hilbert space scalar product

$$(f,g)_\sim := [E_+f,E_+g] - [E_-f,E_-g], \qquad f,g \in \mathcal{H},$$

is connected with the usual Hilbert space scalar product $(\cdot,\cdot)$ in (1.3) by

$$(f,g)_\sim = \big(J(E_+ - E_-)f,g\big), \qquad f,g \in \mathcal{H},$$

and as $J(E_+ - E_-)$ is an isomorphism in $\mathcal{H}$ the norms $\|\cdot\|_\sim$ and $\|\cdot\|$ induced by $(\cdot,\cdot)_\sim$ and $(\cdot,\cdot)$, respectively, are equivalent. Fix some $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ such that $|\operatorname{Im}\lambda_0| > \|\eta J\|_\sim$. Then, as T_η is selfadjoint with respect to $(\cdot,\cdot)_\sim$, we have

$$\|\eta J(T_\eta - \lambda_0)^{-1}\|_\sim \le \|\eta J\|_\sim \frac{1}{|\operatorname{Im}\lambda_0|} < 1,$$

and it follows that

$$T - \lambda_0 = T_\eta - \lambda_0 + \eta J = \Big(I + \eta J(T_\eta - \lambda_0)^{-1}\Big)(T_\eta - \lambda_0)$$

is boundedly invertible in $(\mathcal{H},(\cdot,\cdot)_\sim)$ and hence in $(\mathcal{H},(\cdot,\cdot))$, i.e., $\lambda_0 \in \rho(T)$.

According to (ii) the weight function w has an odd number of turning points and therefore condition (I) or (I′) holds. Hence we can apply Theorem 5.1 and the statements (a)-(c) hold. □

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