Definitizability of a Class of J-Selfadjoint Operators with Applications

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We formulate an abstract result concerning the definitizability of J-selfadjoint operators which, roughly speaking, differ by at most finitely many dimensions from the orthogonal sum of a J-selfadjoint operator with finitely many negative squares and a semibounded selfadjoint operator in a Hilbert space. The general perturbation result is applied to a class of singular Sturm-Liouville operators with indefinite weight functions.

1 Introduction

Let $(\mathcal{H}, (\cdot,\cdot))$ be a Hilbert space and let J be a bounded and boundedly invertible linear operator in $\mathcal{H}$ with $J = J^{-1} = J^*$. Such an operator induces an additional inner product $[\cdot,\cdot] = (J\cdot,\cdot)$ in $\mathcal{H}$. The linear operators in $\mathcal{H}$ which are selfadjoint with respect to $[\cdot,\cdot]$ are called *J-selfadjoint*. The spectrum of J-selfadjoint operators is known to be symmetric with respect to the real axis and it is easy to see that the mapping $A \mapsto JA$ establishes a one-to-one correspondence between the selfadjoint operators and the J-selfadjoint operators in $\mathcal{H}$.

Among the J-selfadjoint operators is the particularly interesting class of *definitizable* operators. A J-selfadjoint operator T is called *definitizable* if $\rho(T) \neq \varnothing$ and if there exists a polynomial p such that $[p(T)f, f] \geq 0$ for all $f \in \operatorname{dom} p(T)$. Such a polynomial is called *definitizing* for T. If T is a definitizable J-selfadjoint operator, then the nonreal spectrum of T consists of at most finite many eigenvalues and T possesses a spectral function E on $\overline{\mathbb{R}}$ with singularities; cf. [5]. We say that a definitizable operator T is *nonnegative in a neighbourhood of* ∞ if there exists some $c > 0$ such that $[Tf, f] \geq 0$ for all $f \in \operatorname{dom} T \cap E(\overline{\mathbb{R}} \setminus (-c, c))\mathcal{H}$. This is the case, if, e.g., the poynomial $p(t) = tq(t)\overline{q}(t)$, where q is a monic polynomial, is definitizing for T, see [5].

The following result is well known, see, e.g. [3, Remark 1.3 and Proposition 1.1]. Recall that the essential spectrum $\sigma_{\text{ess}}(A)$ of a selfadjoint operator A in $\mathcal{H}$ is the set of all spectral points which are no isolated eigenvalues of finite multiplicity.

Theorem 1.1 *Let A be a selfadjoint operator in $\mathcal{H}$ which is semibounded from below. If $\sigma_{\text{ess}}(A) \cap (-\infty, 0] = \varnothing$, then JA is definitizable and nonnegative in a neighbourhood of ∞ with definitizing polynomial $p(t) = tq(t)\overline{q}(t)$, where q is a monic polynomial.*

We mention that the statement of Theorem 1.1 does not hold in general if $\min \sigma_{\text{ess}}(A) \leq 0$. Simple examples show that even $\rho(JA) = \varnothing$ may happen. The main purpose of the present note is to show that under additional assumptions on A the definitizability and nonnegativity of JA are preserved if $\min \sigma_{\text{ess}}(A) \leq 0$. In Section 3 we apply this result to a class of indefinite Sturm-Liouville operators which are regular at one endpoint.

2 A perturbation result

Throughout this section let $(\mathcal{H}, (\cdot,\cdot))$ be a Hilbert space, let J be a bounded operator in $\mathcal{H}$ with $J = J^{-1} = J^*$. Let $\mathcal{H}_\pm$ be the eigenspace of J with respect to the eigenvalue ± 1. Then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. For a closed operator B in $\mathcal{H}$ denote by $\sigma_{\text{ess}}(B)$ the set of points $\lambda \in \mathbb{C}$ such that $B - \lambda$ is not Fredholm. Note that for a selfadjoint operator in $\mathcal{H}$ this coincides with the definition above. In the next theorem we show that under suitable assumptions also in the case $\min \sigma_{\text{ess}}(A) \leq 0$ the assertions in Theorem 1.1 remain true. Although the statement is essentially a consequence of the results in [4] and [5] we give a short proof for the convenience of the reader.

Theorem 2.1 *Let J and $\mathcal{H}_\pm$ be as above and let A be a selfadjoint operator in $\mathcal{H}$ which is semibounded from below with $\mu := \min \sigma_{\text{ess}}(A) \leq 0$. Suppose there are linear manifolds $\mathcal{D}_+$ and $\mathcal{D}_0$ in $\operatorname{dom} A$ with $\mathcal{D}_+ \subset \mathcal{H}_+$ and $\overline{\mathcal{D}_0} = \mathcal{D}_+^\perp$ such that $\dim(\operatorname{dom} A/(\mathcal{D}_+ \oplus \mathcal{D}_0)) < \infty$, $A\mathcal{D}_0 \subset \overline{\mathcal{D}_0}$ and $\sigma_{\text{ess}}(\overline{A \upharpoonright \mathcal{D}_0}) = \varnothing$. Then the operator JA is definitizable and nonnegative in a neighbourhood of ∞ with definitizing polynomial $p(t) = (t - \mu)q(t)\overline{q}(t)$, where q is a monic polynomial.*

P r o o f. It is no restriction to assume that $\mathcal{D}_+$ and $\mathcal{D}_0$ are closed with respect to the graph norm of A. From the selfadjointness of A we conclude $A\mathcal{D}_+ \subset \overline{\mathcal{D}_+}$. Set $\mathcal{K}_+ := \overline{\mathcal{D}_+}$, $\mathcal{K}_0 := \overline{\mathcal{D}_0} = \mathcal{D}_+^\perp$, $S_+ := A \upharpoonright \mathcal{D}_+$ and $S_0 := A \upharpoonright \mathcal{D}_0$. Then S_0 and S_+ are densely defined closed symmetric operators in $\mathcal{K}_0$ and $\mathcal{K}_+$, respectively. Since every $\lambda < \min \sigma(A)$ is a point of regular

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type of both S_+ and S_0, these operators admit selfadjoint extensions A_+ and A_0 in the Hilbert spaces $\mathcal{K}_+$ and $\mathcal{K}_0$, respectively. Then, by assumption, $A_+ \oplus A_0$ and A both are finite dimensional extensions of $S_+ \oplus S_0$. Hence $\sigma_{\text{ess}}(A_+) \cap (-\infty, \mu) = \varnothing$ and $\sigma_{\text{ess}}(A_0) = \sigma_{\text{ess}}(S_0) = \varnothing$. Therefore there exists a monic polynomial q_+ such that $(A_+ - \mu)q_+(A_+)\overline{q_+}(A_+)$ is a nonnegative operator in the Hilbert space $\mathcal{K}_+$. From Theorem 1.1, applied to A_0, we see that J_0A_0 is definitizable and since $\sigma_{\text{ess}}(A_0) = \varnothing$ it can be shown that $p_0(t) := (t-\mu)q_0(t)\overline{q_0}(t)$ is a definitizing polynomial for J_0A_0, where q_0 is some monic polynomial. We have $J(A_+ \oplus A_0) = A_+ \oplus J_0A_0$. With $p(t) := (t-\mu)q_0(t)q_+(t)\overline{q_0}(t)\overline{q_+}(t)$ and $f = f_+ \oplus f_0$, where $f_+ \in \operatorname{dom} p(A_+)$ and $f_0 \in \operatorname{dom} p(J_0A_0)$, we obtain

$$[p(A_+ \oplus J_0A_0)f, f] = ((A_+ - \mu)q_+(A_+)\overline{q_+}(A_+)q_0(A_+)f_+, q_0(A_+)f_+) + [p_0(J_0A_0)q_+(J_0A_0)f_0, q_+(J_0A_0)f_0] \geq 0.$$

Hence, $J(A_+ \oplus A_0)$ is definitizable. From [2, Theorem 2.2], [1, Theorem 3.1] and the proof of [4, Theorem 1] it follows that JA is definitizable with a definitizing polynomial of the desired form, hence JA is nonnegative in a neighbourhood of ∞. □

3 An application to Sturm-Liouville Operators with indefinite weight functions

We apply Theorem 2.1 to indefinite Sturm-Liouville operators associated to differential expressions of the form

$$\ell = \frac{1}{w}\left(-\frac{d}{dx}\,p\,\frac{d}{dx} + q\right)$$

on an interval (a,b), $-\infty < a < b \leq \infty$, with real coefficients w, p^{-1}, q integrable over (a,c) for every $c \in (a,b)$ such that $p > 0$ and $w \neq 0$ a.e. on (a,b). Note that the differential expression ℓ is thus assumed to be regular at the endpoint a whereas b is in general a singular endpoint. The definite counterpart $\tau = |w|^{-1}(-\frac{d}{dx}p\frac{d}{dx} + q)$ of ℓ generates selfadjoint operators in the weighted L^2-space $L^2((a,b),|w|)$ which are one or two dimensional restrictions of the maximal differential operator

$$A_{\text{max}}(a,b)f = \tau f, \quad f \in \mathcal{D}_{\text{max}}(a,b) = \{h \in L^2((a,b),|w|) : h, ph' \text{ locally absolutely continuous}, \tau h \in L^2((a,b),|w|)\},$$

associated to τ. These *selfadjoint realizations* of τ in $L^2((a,b),|w|)$ can also be viewed as finite dimensional extensions of the minimal operator $A_{\text{min}}(a,b) = A_{\text{max}}(a,b)^*$, where * denotes the adjoint with respect to the scalar product in $L^2((a,b),|w|)$. It is well known that $A_{\text{min}}(a,b)$ is a densely defined closed symmetric operator in $L^2((a,b),|w|)$ with equal deficiency indices $(1,1)$ (if τ is in the limit point case at b) or $(2,2)$ (if τ is in the limit circle case at b), see, e.g. [6].

The multiplication operator $J = \operatorname{sign}(w)$ satisfies $J = J^{-1} = J^*$ and connects the definite and indefinite Sturm-Liouville expressions τ and ℓ, i.e., $\ell = J\tau$, or, more precisely, the J-selfadjoint realizations T of ℓ in $L^2((a,b),|w|)$ are in one-to-one correspondence with the selfadjoint realizations A of τ via $A \mapsto T = JA$. We note that the indefinite inner product induced by J is $[f,g] = (Jf,g) = \int_a^b f\overline{g}\,w\,dx$, where $(\cdot,\cdot)$ is the scalar product in $L^2((a,b),|w|)$.

Theorem 3.1 *Suppose that the weight function w is positive near the endpoint b and that $A_{\text{min}}(a,b)$ is semibounded from below. Then every J-selfadjoint realization of ℓ in $L^2((a,b),|w|)$ is definitizable and nonnegative in a neighbourhood of ∞.*

P r o o f. Let A be a selfadjoint realization of τ in $L^2((a,b),|w|)$. Since $A_{\text{min}}(a,b)$ has finite deficiency indices the extension A is semibounded from below. In the case $\min \sigma_{\text{ess}}(A) > 0$ the assertions of Theorem 3.1 follow from Theorem 1.1. Hence, let $\min \sigma_{\text{ess}}(A) \leq 0$. According to the assumption on w there exists $c \in (a,b)$ such that $w > 0$ a.e. on (c,b). Denote by $A_{\text{min}}(a,c)$ and $A_{\text{min}}(c,b)$ the minimal operators associated to τ in $L^2((a,c),|w|)$ and, $L^2((c,b),|w|)$, respectively, and let

$$\mathcal{D}_0 := \operatorname{dom} A_{\text{min}}(a,c) \qquad \text{and} \qquad \mathcal{D}_+ := \operatorname{dom} A_{\text{min}}(c,b).$$

Then we have $\overline{\mathcal{D}_0} = L^2((a,c),|w|)$ and $\overline{\mathcal{D}_+} = L^2((c,b),|w|)$. Thus $\overline{\mathcal{D}_0} = \mathcal{D}_+^\perp$, where $\mathcal{D}_0$ and $\mathcal{D}_+$ are canonically embedded in $L^2((a,b),|w|)$. As $J \upharpoonright \mathcal{D}_+ = 1$ it follows that $\mathcal{D}_+$ is a subspace of $\mathcal{H}_+ := \ker(J-1)$. Moreover, $\sigma_{\text{ess}}(A \upharpoonright \mathcal{D}_0) = \sigma_{\text{ess}}(A_{\text{min}}(a,c)) = \varnothing$ follows from the fact that τ is regular at a and c, and all conditions of Theorem 2.1 are satisfied. □

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