REVISITING THE WEAK COUPLING PHENOMENON FOR TWO-DIMENSIONAL SCHRÖDINGER OPERATORS

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Dedicated to Barry Simon on the occasion of his 80th birthday.

Abstract. We study the existence of negative eigenvalues for two-dimensional Schrödinger operators with real-valued potentials in the weak coupling regime. In his pioneering paper [30] from half a century ago, Simon was the first to describe the unique negative eigenvalue emerging from the threshold of the essential spectrum of one- and two-dimensional Schrödinger operators. The aim of this paper is to extend Simon's results in two dimensions to a broader class of potentials, allowing for both stronger singularities and slower decay at infinity, at the cost of losing uniqueness of weakly coupled eigenvalues.

1. Introduction and results

It is well-known that Schrödinger operators $-\Delta - V$ in $L^2(\mathbb{R}^d)$, $d = 1, 2$, have negative eigenvalues for arbitrarily small attractive potentials. More precisely, if $V : \mathbb{R}^d \to \mathbb{R}$ satisfies suitable integrability conditions and if the coupling $\varepsilon > 0$ is sufficiently small, then $-\Delta - \varepsilon V$ has a negative eigenvalue if and only if

$$V \not\equiv 0 \quad \text{and} \quad \int_{\mathbb{R}^d} V(x)\,\mathrm{d}x \geq 0. \tag{1.1}$$

In this case, the negative eigenvalue is unique and simple. This behaviour is specific to dimensions one and two; in contrast, for $d \geq 3$ the Cwikel-Lieb-Rozenblum bound guarantees the absence of negative eigenvalues of $-\Delta - \varepsilon V$ for real-valued potentials $V \in L^{d/2}(\mathbb{R}^d)$ and sufficiently small couplings $\varepsilon > 0$.

This so-called weak coupling phenomenon was first described by Simon in his celebrated paper [30], where he studied the weakly coupled eigenvalues, including existence, absence, uniqueness, asymptotic expansions, as well as their analyticity at zero, for Schrödinger operators in one and two dimensions. In $d = 1$ he considered potentials satisfying

$$(1 + |\cdot|^2)V \in L^1(\mathbb{R})$$

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and showed that the weakly coupled eigenvalue, if it exists, satisfies

$$\sqrt{-\lambda_\varepsilon} = \frac{\varepsilon}{2}\int_{\mathbb{R}} V(x)\,\mathrm{d}x - \frac{\varepsilon^2}{4}\int_{\mathbb{R}^2} V(x)|x-y|V(y)\,\mathrm{d}(x,y) + o(\varepsilon^2), \quad \varepsilon \to 0+.$$

One year later Klaus [19] extended Simon's results [30] for $d = 1$ to the broader class of potentials satisfying $(1+|\cdot|)V \in L^1(\mathbb{R})$. The case

$$V(x) \sim x^{-2+\delta}, \quad |x| \to \infty, \quad \text{for some} \quad \delta \in [0,1) \tag{1.2}$$

in one dimension was subsequently studied in [4]; for such potentials a sharp spectral transition occurs at $\delta = 0$. If V satisfies (1.1) and (1.2) with $\delta = 0$ one still finds that $-\Delta - \varepsilon V$ has exactly one negative eigenvalue as $\varepsilon \to 0+$. In contrast, if V satisfies (1.2) with $\delta \in (0,1)$, then $-\Delta - \varepsilon V$ has infinitely many negative eigenvalues even if $\int_{\mathbb{R}} V(x)\,\mathrm{d}x < 0$. In both cases, if $\int_{\mathbb{R}} V(x)\,\mathrm{d}x > 0$, the ground state admits the expansion

$$\sqrt{-\lambda_\varepsilon} = \frac{\varepsilon}{2}\int_{\mathbb{R}} V(x)\,\mathrm{d}x + o(\varepsilon), \quad \varepsilon \to 0+.$$

For $\delta = 0$, the latter can be further refined by isolating the second term, which is of order $\varepsilon^2 \ln \varepsilon$, and then estimating the new remainder, see [4].

In this paper we focus on the results that Simon obtained in two dimensions. We recall them (including well-known spectral properties) in the following theorem.

Theorem 1.1 ([30, Thm. 3.4])**.** *Let $\varepsilon > 0$ and assume that $V \in L^1(\mathbb{R}^2)$ is real-valued and satisfies*

$$V \in L^{1+\eta}(\mathbb{R}^2) \quad \textit{for some} \quad \eta > 0 \tag{1.3}$$

and

$$|\cdot|^t V \in L^1(|x| > 1) \quad \textit{for some} \quad t > 0. \tag{1.4}$$

Then the following assertions hold.

(i) *The operator $H_\varepsilon = -\Delta - \varepsilon V$ defined in* (3.3) *is self-adjoint in $L^2(\mathbb{R}^2)$, bounded from below and one has $\sigma_{\mathrm{ess}}(H_\varepsilon) = [0,\infty)$.*

(ii) *If $V \not\equiv 0$ and $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x \geq 0$, then H_ε has exactly one negative eigenvalue λ_ε for all sufficiently small $\varepsilon > 0$, which is simple. If $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x > 0$, then this eigenvalue satisfies*

$$\ln(-\lambda_\varepsilon) = -4\pi \left[\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x\right]^{-1} \varepsilon^{-1} + o(\varepsilon^{-1}), \quad \varepsilon \to 0+;$$

if $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x = 0$, then this eigenvalue satisfies

$$\ln(-\lambda_\varepsilon) = -\frac{C}{\varepsilon^2} + o(\varepsilon^{-2}), \quad \varepsilon \to 0+, \tag{1.5}$$

where $C > 0$ is a constant.

(iii) *If $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x < 0$, then for all sufficiently small $\varepsilon > 0$ there are no negative eigenvalues of H_ε.*

After Simon's pioneering paper [30] many works followed that studied limiting absorption principles and the threshold behaviour of Schrödinger and other differential operators at the edge of the essential spectrum, see, e.g., [31, 4, 20, 24, 26, 21, 27, 16, 13, 5, 11], the monograph [29, Thm. XIII.11, p. 336-338] and for more recent developments [12, 22, 9, 8, 15, 25, 10, 32].

However, to the best of our knowledge, there are no results that relax the integrability conditions (1.4) and (1.3) on V in the two-dimensional case. As a small contribution to Simon's work in [30] we prove the following theorem.

Theorem 1.2. *Let* $\varepsilon > 0$ *and assume that* $V \in L^1(\mathbb{R}^2)$ *is real-valued and satisfies*

$$\int_{|x-y|<e} |V(x)|(\ln|x-y|)^2|V(y)|\,\mathrm{d}(x,y) < \infty \tag{1.6}$$

and

$$|\ln|\cdot||^s V \in L^1(|x|>1) \quad \text{for some} \quad s \in [0,1). \tag{1.7}$$

Then the following assertions hold.

(i) *The operator* $H_\varepsilon = -\Delta - \varepsilon V$ *defined in* (3.3) *is self-adjoint in* $L^2(\mathbb{R}^2)$*, bounded from below and one has* $\sigma_{\mathrm{ess}}(H_\varepsilon) = [0,\infty)$*.*

(ii) *If* $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x > 0$*, then* H_ε *has a negative eigenvalue* λ_ε *for all sufficiently small* $\varepsilon > 0$*. This eigenvalue is simple and satisfies*

$$\ln(-\lambda_\varepsilon) = -4\pi \left[\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x\right]^{-1} \varepsilon^{-1} + o(\varepsilon^{-1+s}), \quad \varepsilon \to 0+. \tag{1.8}$$

We explain below that our assumptions on V are weaker than those in Simon's classical work as they allow for stronger local singularities of V as well as a slower decay at infinity. A natural consequence of the latter, however, is the loss of uniqueness of the weakly coupled eigenvalue, see Example 1.3(i).

In more detail, Assumption (1.6) is weaker than (1.3); note that both conditions, roughly speaking, control the local singularities of V. Indeed, if $V \in L^{1+\eta}(\mathbb{R}^2)$ for some $\eta > 0$, then an application of Fubini's theorem and Hölder's inequality yields

$$\begin{aligned} &\int_{|x-y|<e} |V(x)|(\ln|x-y|)^2|V(y)|\,\mathrm{d}(x,y) \\ &\quad = \int_{\mathbb{R}^2} |V(x)| \int_{|x-y|<e} (\ln|x-y|)^2|V(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\quad \le \left(\int_{|u|<e} |\ln|u||^{2+\frac{2}{\eta}}\,\mathrm{d}u\right)^{\frac{\eta}{1+\eta}} \|V\|_{L^1(\mathbb{R}^2)}\|V\|_{L^{1+\eta}(\mathbb{R}^2)} < \infty. \end{aligned} \tag{1.9}$$

Next, it is not difficult to check that Assumption (1.4) for some $t > 0$ implies Assumption (1.7) for any $s \ge 0$ and that both conditions are essentially decay restrictions on V. However, (1.7) also covers the case $s = 0$, which includes potentials $V \in L^1(\mathbb{R}^2)$ without any additional decay. We also note that potentials $V \in L^1(\mathbb{R}^2)$ satisfying (1.6) are relatively form compact perturbations of $-\Delta$ in $L^2(\mathbb{R}^2)$, see Corollary 2.3.

Example 1.3. (i) Consider the potential

$$V_\infty(x) := \frac{\mathbb{1}_{\{|x|>3\}}(x)}{|x|^2(\ln|x|)^{1+\delta}(\ln\ln|x|)^2}, \quad \delta \in [0,1). \tag{1.10}$$

Clearly, $V_\infty \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, but (1.4) does not hold for any $t > 0$. However, it is simple to check that V_∞ satisfies (1.7) for $s = \delta$ and hence Theorem 1.2 implies that for all sufficiently small $\varepsilon > 0$ there exists a simple negative eigenvalue of H_ε that satisfies (1.8). At the same time a variational argument (see, e.g., [6]) shows that H_ε has infinitely many negative eigenvalues for any $\varepsilon > 0$.

(ii) Next, consider the potential

$$V_0(x) := \frac{\mathbb{1}_{\{|x|<\frac{1}{3}\}}(x)}{|x|^2(\ln|x|)^4}. \tag{1.11}$$

It is simple to check that V_0 belongs to $L^1(\mathbb{R}^2)$ and satisfies both (1.4) and (1.7) for any $t > 0$ and $s \geq 0$, respectively. Moreover, V_0 satisfies (1.6) (see Section 4 for details) but $V_0 \notin L^{1+\eta}(\mathbb{R}^2)$ for any $\eta > 0$, i.e., (1.3) does not hold. However, Theorem 1.2 still implies that for every sufficiently small $\varepsilon > 0$ there exists a simple negative eigenvalue of H_ε that satisfies (1.8).

As a final remark let us also comment on the situation when the potential V satisfies (1.7) with $s \geq 1$. In this case the expansion (1.8) can be further refined; similarly as in [3, 22], the second term can be derived together with corresponding estimates on the remainder. In addition, one can then also consider potentials of zero mean, i.e., $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x = 0$, for which the existence of a simple bound state satisfying (1.5) can be shown by using arguments analogous to [30].

2. Definition and properties of $-\Delta - V$

In the following we define a self-adjoint realization H of $-\Delta - V$ in $L^2(\mathbb{R}^2)$ assuming that $V \in L^1(\mathbb{R}^2)$ is real-valued and satisfies (1.6); note that the condition (1.7) is not used until Section 3. Under these assumptions it turns out that the Birman-Schwinger operator

$$Q(\alpha) := \overline{|V|^{\frac{1}{2}}(-\Delta + \alpha^2)^{-1}V^{\frac{1}{2}}}, \quad V^{\frac{1}{2}} := |V|^{\frac{1}{2}}\operatorname{sgn}(V), \quad \alpha > 0, \tag{2.1}$$

is a well-defined Hilbert-Schmidt operator in $L^2(\mathbb{R}^2)$, see Proposition 2.2 below. Following Kato's construction in [17] we set $R_0(\alpha) := (-\Delta + \alpha^2)^{-1}$, $\alpha > 0$, and

$$\begin{aligned} R(\alpha) = R_0(\alpha) + \overline{R_0(\alpha)V^{\frac{1}{2}}}[I - Q(\alpha)]^{-1}|V|^{\frac{1}{2}}R_0(\alpha), \\ \alpha \in \{\beta > 0 : 1 \in \rho(Q(\beta))\}, \end{aligned} \tag{2.2}$$

and conclude in Proposition 2.4 below that $R(\alpha)$ is the resolvent of a semibounded self-adjoint operator H (which is an extension of the symmetric operator $-\Delta - V$ defined on $\operatorname{dom}(-\Delta - V) = H^2(\mathbb{R}^2) \cap \operatorname{dom} V$) in $L^2(\mathbb{R}^2)$.

As a first step we analyze the Birman-Schwinger operator $Q(\alpha)$ given by (2.1). Recall that for $\alpha > 0$ the resolvent $(-\Delta + \alpha^2)^{-1}$ of the free Laplacian in $L^2(\mathbb{R}^2)$ is an integral operator with kernel

$$\mathcal{G}(x, y; \alpha) = \frac{1}{2\pi} K_0(\alpha|x-y|), \quad x, y \in \mathbb{R}^2, \quad x \neq y, \tag{2.3}$$

where $K_0 : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the modified Bessel function of second kind of order zero (see [1, Chap. 9.6] for more details). Recall that K_0 is monotonically decreasing and has the expansions

$$K_0(w) = -\ln w + \ln 2 - \gamma + \mathcal{O}(w^2 \ln w), \quad w \to 0+, \tag{2.4}$$

$$K_0(w) = \left(\frac{\pi}{2w}\right)^{\frac{1}{2}} e^{-w} \left(1 + \mathcal{O}(w^{-1})\right), \quad w \to +\infty, \tag{2.5}$$

where γ is the Euler-Mascheroni constant, see [1, Eq. (9.6.13) and (9.7.2)], respectively.

In the next lemma we collect useful estimates for the Green's function (2.3).

Lemma 2.1. *Let* $\mathcal{G}(x, y; \alpha)$ *be given by* (2.3). *Then there exists a constant* $C > 0$ *such that for all* $\alpha \in (0, \frac{1}{e})$, *all* $s \in [0, 2]$ *and all* $x, y \in \mathbb{R}^2$ *with* $x \neq y$ *the following inequalities hold.*

- (i) $\left|\mathcal{G}(x, y; \alpha) + \frac{\ln \alpha}{2\pi}\right|^s \leq C(1 + |\ln|x-y||^s)$.
- (ii) $\left|\mathcal{G}(x, y; \alpha) + \frac{\ln \alpha}{2\pi}\right|^s \leq C|\ln \alpha|^s$ *if* $\alpha|x-y| \geq 1$.
- (iii) $\mathcal{G}(x, y; \alpha)^2 \leq C((\ln \alpha)^2 + (\ln|x-y|)^2)$.

Proof. In the following C denotes a positive constant that may change between estimates. For ease of notation we set

$$w := \alpha|x-y| \tag{2.6}$$

for $x, y \in \mathbb{R}^2$ with $x \neq y$; note that $w > 0$ and that the kernel we estimate reads

$$\mathcal{G}(x, y; \alpha) + \frac{\ln \alpha}{2\pi} = \frac{K_0(w) + \ln \alpha}{2\pi}. \tag{2.7}$$

Note also that it suffices to prove both items (i) and (ii) for $s = 1$ since the general case then follows by taking both sides to the power of $s \in [0, 2]$ and in the case of (i) applying the elementary inequality $|a+b|^s \leq 2(|a|^s + |b|^s)$.

(i) If $w < 1$, then (2.4) gives us

$$|K_0(w) + \ln w| \leq C\left(1 + |w|^2 |\ln w|\right) \leq C$$

and since $\ln \alpha = \ln w - \ln|x-y|$ this implies

$$|K_0(w) + \ln \alpha| = |K_0(w) + \ln w - \ln|x-y|| \leq C(1 + |\ln|x-y||). \tag{2.8}$$

If $w \geq 1$ we use that K_0 is monotonically decreasing and infer with $\alpha \in (0, \frac{1}{e})$

$$|K_0(w) + \ln \alpha| \leq |K_0(w)| + |\ln \alpha| \leq |K_0(1)| + |\ln \alpha| \leq C|\ln \alpha| \tag{2.9}$$

Since $e < \alpha^{-1} \leq |x-y|$ if $w \geq 1$ by (2.6), we conclude

$$|K_0(w) + \ln \alpha| \leq C|\ln \alpha| = C \ln(\alpha^{-1}) \leq C \ln |x-y|,$$

which together with (2.8) shows (i).

(ii) By the definition (2.6) of w we have that $\alpha|x-y| \geq 1$ is equivalent to $w \geq 1$ so the claim follows from (2.9).

(iii) By applying the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we find

$$\mathcal{G}(x,y;\alpha)^2 \leq 2\biggl(\mathcal{G}(x,y;\alpha) + \frac{\ln \alpha}{2\pi}\biggr)^2 + 2\biggl(\frac{\ln \alpha}{2\pi}\biggr)^2$$

so the claim follows from (i) with $s = 2$. □

As a consequence of Lemma 2.1 we obtain the compactness of the Birman-Schwinger operator Q as well as the decay of $\|Q(\alpha)\|$ as $\alpha \to +\infty$.

Proposition 2.2. *Assume that $V \in L^1(\mathbb{R}^2)$ is real-valued and satisfies* (1.6)*. Then for any $\alpha > 0$ the Birman-Schwinger operator $Q(\alpha)$ in* (2.1) *is a Hilbert-Schmidt operator and one has $\|Q(\alpha)\|_{\mathrm{HS}} \to 0$ as $\alpha \to +\infty$.*

Proof. We start by proving that $Q(\alpha)$ is a Hilbert-Schmidt operator for any $\alpha > 0$, that is, we verify

$$\|Q(\alpha)\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}^4} |V(x)|\mathcal{G}(x,y;\alpha)^2|V(y)| \,\mathrm{d}(x,y) < \infty. \tag{2.10}$$

Recall first that $K_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonically decreasing. Thus, for fixed $x, y \in \mathbb{R}^2$, $x \neq y$, we have by (2.3) that $\mathcal{G}(x,y;\alpha)^2$ is monotonically decreasing in α and hence it suffices to prove the claim for $\alpha \in (0, \frac{1}{e})$.

Fix $\alpha \in (0, \frac{1}{e})$; we show (2.10) by splitting the integral into the two regions $\alpha|x-y| < 1$ and $\alpha|x-y| \geq 1$. For the region $\alpha|x-y| < 1$ we apply Lemma 2.1(iii) and obtain for some $C > 0$ the inequality

$$\mathcal{G}(x,y;\alpha)^2 \leq C((\ln \alpha)^2 + (\ln|x-y|)^2), \quad \alpha|x-y| < 1. \tag{2.11}$$

For the region $\alpha|x-y| \geq 1$ we use (2.3) and the monotonicity of K_0 to infer

$$\mathcal{G}(x,y;\alpha)^2 \leq \left(\frac{K_0(1)}{2\pi}\right)^2, \quad \alpha|x-y| \geq 1. \tag{2.12}$$

By combining (2.11) and (2.12) we obtain for some $C > 0$

$$\|Q(\alpha)\|_{\mathrm{HS}}^2 \leq C\biggl(\|V\|_{L^1(\mathbb{R}^2)}^2 + \int_{\alpha|x-y|<1} |V(x)|(\ln|x-y|)^2|V(y)| \,\mathrm{d}(x,y)\biggr).$$

The latter integral is finite. This is easily seen after further splitting the region $\alpha|x-y| < 1$ into $|x-y| < e$ and $e \leq |x-y| < 1/\alpha$ (note that the latter region is non-empty since by assumption $\alpha \in (0, \frac{1}{e})$). For $|x-y| < e$

the finiteness now follows from Assumption (1.6) and for $e \le |x-y| < 1/\alpha$ it follows from the boundedness of $(\ln|x-y|)^2$. In summary, (2.10) holds.

For the second claim, we justify below that dominated convergence yields

$$\int_{\mathbb{R}^2} |V(x)||\mathcal{G}(x,y;\alpha)^2|V(y)|\,\mathrm{d}(x,y) \to 0, \quad \alpha \to +\infty. \tag{2.13}$$

To this end, (2.5) implies $K_0(\alpha|x-y|) \to 0$ and by (2.3) also $\mathcal{G}(x,y;\alpha) \to 0$ as $\alpha \to +\infty$ for every $x \neq y$ and hence for almost every $(x,y) \in \mathbb{R}^4$. Furthermore, by the monotonicity of K_0 on $\mathbb{R}^+$ we have for any $\alpha_0 \in (0, \frac{1}{e})$ and all $\alpha \ge \alpha_0$

$$|V(x)|\mathcal{G}(x,y;\alpha)^2|V(y)| \le |V(x)|\mathcal{G}(x,y;\alpha_0)^2|V(y)|, \quad x,y \in \mathbb{R}^2, \quad x \neq y,$$

which by the first part of this proof is an integrable upper bound. □

It turns out that the Birman-Schwinger operator being Hilbert-Schmidt also ensures that V is a relatively form compact perturbation of $-\Delta$ in $L^2(\mathbb{R}^2)$, that is,

$$\operatorname{dom}(-\Delta+1)^{\frac{1}{2}} = H^1(\mathbb{R}^2) \subset \operatorname{dom}|V|^{\frac{1}{2}} \quad \text{and} \quad |V|^{\frac{1}{2}}(-\Delta+1)^{-\frac{1}{2}} \tag{2.14}$$

is a compact operator in $L^2(\mathbb{R}^2)$.

Corollary 2.3. *Assume that $V \in L^1(\mathbb{R}^2)$ is real-valued and satisfies* (1.6). *Then V is relatively form compact with respect to $-\Delta$ in $L^2(\mathbb{R}^2)$.*

Proof. We show first that $A \coloneqq |V|^{\frac{1}{2}}(-\Delta+1)^{-\frac{1}{2}}$ is bounded and everywhere defined. Consider for $n \in \mathbb{N}$ the operators A_n in $L^2(\mathbb{R}^2)$ defined by

$$A_n \coloneqq V_n(-\Delta+1)^{-\frac{1}{2}}, \quad V_n(x) \coloneqq \min\{|V(x)|^{\frac{1}{2}}, n\}.$$

Clearly, $V_n \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and hence each A_n, $n \in \mathbb{N}$, is bounded and everywhere defined. A short computation taking adjoints shows

$$\|A_n^* f\|^2_{L^2(\mathbb{R}^2)} = \|(-\Delta+1)^{-\frac{1}{2}} V_n f\|^2_{L^2(\mathbb{R}^2)} \le \|V_n(-\Delta+1)^{-1}V_n\|_{\mathrm{HS}} \|f\|^2_{L^2(\mathbb{R}^2)}$$

for any $f \in L^2(\mathbb{R}^2)$. Using $|V_n(x)|^2 \le |V(x)|$ for all $x \in \mathbb{R}^2$ we obtain by the monotonicity of the Hilbert-Schmidt norms and Proposition 2.2 that

$$\|A_n\|^2 = \|A_n^*\|^2 \le \|Q(1)\|_{\mathrm{HS}} < \infty, \quad n \in \mathbb{N}.$$

Since $V_n(x)^2 \uparrow |V(x)|$ as $n \to \infty$ monotone convergence implies

$$\begin{aligned}\int_{\mathbb{R}^2} |V(x)||(-\Delta+1)^{-\frac{1}{2}} f(x)|^2\,\mathrm{d}x &= \lim_{n\to\infty} \int_{\mathbb{R}^2} V_n(x)^2 |(-\Delta+1)^{-\frac{1}{2}} f(x)|^2\,\mathrm{d}x \\ &\le \limsup_{n\to\infty} \|A_n\|^2 \|f\|^2_{L^2(\mathbb{R}^2)} \\ &\le \|Q(1)\|_{\mathrm{HS}} \|f\|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

for any $f \in L^2(\mathbb{R}^2)$. Thus we conclude the first inclusion in (2.14) and that A is bounded and everywhere defined in $L^2(\mathbb{R}^2)$. This also implies

$$AA^* = |V|^{\frac{1}{2}}(-\Delta+1)^{-\frac{1}{2}}\overline{(-\Delta+1)^{-\frac{1}{2}}|V|^{\frac{1}{2}}} = Q(1);$$

since $Q(1)$ is compact (see Proposition 2.2) it follows that A and A^* are compact as well so the second claim in (2.14) follows. □

We now employ the compactness of the Birman-Schwinger operator to show that $R(\alpha)$ given by (2.2) defines the resolvent of a self-adjoint operator.

Proposition 2.4. *Assume that* $V \in L^1(\mathbb{R}^2)$ *is real-valued and satisfies* (1.6). *Then the operator* $R(\alpha)$ *given by* (2.2) *for* $\alpha \in \{\beta > 0 : 1 \in \rho(Q(\beta))\}$ *defines a self-adjoint operator* H *in* $L^2(\mathbb{R}^2)$ *by*

$$R(\alpha) = (H + \alpha^2)^{-1}, \tag{2.15}$$

which has the more explicit form

$$H = -\Delta - V, \quad \operatorname{dom} H = \bigl\{ f \in H^1(\mathbb{R}^2) : (-\Delta - V)f \in L^2(\mathbb{R}^2) \bigr\}. \tag{2.16}$$

Moreover, H *is bounded from below,* $\sigma_{\mathrm{ess}}(H) = [0, \infty)$*, and one has*

$$\dim \ker(H + \alpha^2) = \dim \ker(I - Q(\alpha)), \quad \alpha > 0. \tag{2.17}$$

Finally, H *is a self-adjoint extension of the symmetric operator* $-\Delta - V$ *defined on* $\operatorname{dom}(-\Delta - V) = H^2(\mathbb{R}^2) \cap \operatorname{dom} V$ *in* $L^2(\mathbb{R}^2)$.

Proof of Proposition 2.4. The statement essentially follows from well-known results by Kato [17] and Konno and Kuroda [23]. However, we follow the presentation in [14] and apply the results therein with $\mathcal{H} = \mathcal{K} = L^2(\mathbb{R}^2)$, $H_0 = -\Delta$, $\operatorname{dom} H_0 = H^2(\mathbb{R}^2)$, $A = -|V|^{\frac{1}{2}}$ and $B = V^{\frac{1}{2}}$, where A and B are understood as maximal multiplication operators in $L^2(\mathbb{R}^2)$.

More precisely, since by assumption $V \in L^1(\mathbb{R}^2)$, we have $|V|^{\frac{1}{2}} \in L^2(\mathbb{R}^2)$ and as a consequence, the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ implies $\operatorname{dom} H_0 \subset \operatorname{dom} |V|^{\frac{1}{2}}$ (actually, by (2.14) we even know $H^1(\mathbb{R}^2) \subset \operatorname{dom} |V|^{\frac{1}{2}}$). Next, Proposition 2.2 yields the compactness of the Birman-Schwinger operator $Q(\alpha)$ and $\{\beta > 0 : 1 \in \rho(Q(\beta))\} \neq \emptyset$. Hence, [14, Thm. 2.3] implies that H is a densely defined and closed extension of the symmetric operator $-\Delta - V$ defined on $\operatorname{dom}(-\Delta - V) = H^2(\mathbb{R}^2) \cap \operatorname{dom} V$ in $L^2(\mathbb{R}^2)$ and [14, Thm. 3.2] yields the Birman-Schwinger principle (2.17). Moreover, V being real-valued and $\|Q(\alpha)\| \to 0$ as $\alpha \to 0+$ imply that H is bounded from below, and, in particular, self-adjoint in $L^2(\mathbb{R}^2)$. The stability of the essential spectrum follows from [14, Thm. 4.5].

Finally, H having the explicit form (2.16) follows from the relative form compactness of V in Corollary 2.3. Indeed, the operator H defined by (2.15) coincides with the self-adjoint realization (2.16) of $-\Delta - V$ in $L^2(\mathbb{R}^2)$ constructed via standard form methods, see, e.g., [18, Chap. VI.2 and VI.3], [28, Chap. VIII.6], and [2, Lemma 3.8]. □

3. Weak coupling regime and the proof of Theorem 1.2

In the following we are interested in the negative eigenvalues of the operators H_ε formally given by $-\Delta - \varepsilon V$ in $L^2(\mathbb{R}^2)$ and defined rigorously below using the construction from the previous section (with V replaced by εV), and we prove our main result Theorem 1.2.

Assume that $\varepsilon > 0$ is small and that $V \in L^1(\mathbb{R}^2)$ is real-valued and satisfies (1.6). Following the construction in Section 2, set $R_0(\alpha) := (-\Delta + \alpha^2)^{-1}$ and define

$$\begin{aligned} R_\varepsilon(\alpha) = R_0(\alpha) + \overline{\varepsilon R_0(\alpha) V^{\frac{1}{2}}}[I - \varepsilon Q(\alpha)]^{-1} |V|^{\frac{1}{2}} R_0(\alpha), \\ \alpha \in \{\beta > 0 : 1 \in \rho(\varepsilon Q(\beta))\}, \end{aligned} \tag{3.1}$$

where $Q(\alpha)$ is the compact Birman-Schwinger operator in $L^2(\mathbb{R}^2)$ given by (2.1). It follows as in Proposition 2.4 that the operator $R_\varepsilon(\alpha)$ for $\alpha \in \{\beta > 0 : 1 \in \rho(\varepsilon Q(\beta))\}$ defines a self-adjoint operator H_ε in $L^2(\mathbb{R}^2)$ by

$$R_\varepsilon(\alpha) = (H_\varepsilon + \alpha^2)^{-1}, \tag{3.2}$$

which has the more explicit form

$$H_\varepsilon = -\Delta - \varepsilon V, \quad \operatorname{dom} H_\varepsilon = \bigl\{ f \in H^1(\mathbb{R}^2) : (-\Delta - \varepsilon V) f \in L^2(\mathbb{R}^2) \bigr\}. \tag{3.3}$$

Moreover, H_ε is bounded from below, $\sigma_{\mathrm{ess}}(H_\varepsilon) = [0, \infty)$, one has

$$\dim \ker(H_\varepsilon + \alpha^2) = \dim \ker(I - \varepsilon Q(\alpha)), \quad \alpha > 0, \tag{3.4}$$

and H_ε is a self-adjoint extension of the symmetric operator $-\Delta - \varepsilon V$ defined on $\operatorname{dom}(-\Delta - \varepsilon V) = H^2(\mathbb{R}^2) \cap \operatorname{dom} V$ in $L^2(\mathbb{R}^2)$.

3.1. **Preparatory estimates.**

To analyze the eigenvalues of H_ε in the weak coupling limit Simon [30] decomposed the Birman-Schwinger operator $Q(\alpha)$ as

$$Q(\alpha) = L(\alpha) + M(\alpha), \quad \alpha > 0; \tag{3.5}$$

with integral operators $L(\alpha)$ and $M(\alpha)$ in $L^2(\mathbb{R}^2)$ that have the kernels

$$\mathcal{L}(x, y; \alpha) := |V(x)|^{\frac{1}{2}} g(\alpha) V(y)^{\frac{1}{2}}, \tag{3.6}$$

$$\mathcal{M}(x, y; \alpha) := |V(x)|^{\frac{1}{2}} \left(\mathcal{G}(x, y; \alpha) - g(\alpha) \right) V(y)^{\frac{1}{2}}, \tag{3.7}$$

respectively, and the function g is given by

$$g(\alpha) = -(2\pi)^{-1} \ln \alpha. \tag{3.8}$$

It is useful to notice that L is of rank one and that its norm is given by

$$\|L(\alpha)\| = |g(\alpha)| \|V\|_{L^1(\mathbb{R}^2)},$$

and hence $\|L(\alpha)\|$ has a logarithmic singularity at $\alpha = 0$. Meanwhile, the singularity of $\|M(\alpha)\|$ is weaker, which is the content of the next lemma.

Lemma 3.1. *Assume that* $V \in L^1(\mathbb{R}^2)$ *is real-valued and satisfies* (1.6) *and* (1.7) *for some* $s \in [0, 1)$*. Then*

$$\int_{\mathbb{R}^4} |V(x)| |\ln |x - y||^s |V(y)| \, \mathrm{d}(x, y) < \infty \tag{3.9}$$

and for the integral operator $M(\alpha)$ *in* $L^2(\mathbb{R}^2)$ *for* $\alpha \to 0+$ *one has*

(i) $\|M(\alpha)\|^2 = o(|g(\alpha)|^{2-s})$,

(ii) $|(M(\alpha)|V|^{\frac{1}{2}}, V^{\frac{1}{2}})| = o(|g(\alpha)|^{1-s})$.

Proof. We start with some preliminary observations and verify (3.9). Note that the inequality

$$|\ln|x-y||^r \leq 1 + (\ln|x-y|)^2, \quad x,y \in \mathbb{R}^2, \quad x \neq y, \quad r \in [0,2], \tag{3.10}$$

holds. To check (3.9), observe that the existence of the integral over the region $|x-y| < e$ follows from (3.10) applied with $r = s \in [0,1)$ and our assumptions $V \in L^1(\mathbb{R}^2)$ and (1.6). For the region $|x-y| \geq e$ we make use of the elementary inequality

$$\ln(1+|x|) \leq \ln 2 + \mathbb{1}_{\{|x|\geq 1\}}(x) \ln|x|. \tag{3.11}$$

Since $e \leq |x-y| \leq |x| + |y| \leq (1+|x|)(1+|y|)$ we have by (3.11)

$$\begin{aligned}(\ln|x-y|)^s &\leq (\ln(1+|x|) + \ln(1+|y|))^s \\ &\leq \big(2\ln 2 + \mathbb{1}_{\{|x|\geq 1\}}(x)\ln|x| + \mathbb{1}_{\{|y|\geq 1\}}(y)\ln|y|\big)^s \\ &\leq C\big(1 + \mathbb{1}_{\{|x|\geq 1\}}(x)|\ln|x||^s + \mathbb{1}_{\{|y|\geq 1\}}(y)|\ln|y||^s\big);\end{aligned}$$

this together with $V \in L^1(\mathbb{R}^2)$ and (1.7) shows the finiteness of the integral over the region $|x-y| \geq e$. Summing up, we have shown (3.9).

For the proofs of assertions (i) and (ii) we again use the convention that C denotes a positive constant that may change between estimates. By the definition (3.7) of the kernel of M we have

$$\begin{aligned}\|M(\alpha)\|^2 &\leq \int_{\mathbb{R}^4} |V(x)||\mathcal{G}(x,y;\alpha) - g(\alpha)|^2 |V(y)| \,\mathrm{d}(x,y), \\ |(M(\alpha)|V|^{\frac{1}{2}}, V^{\frac{1}{2}})| &\leq \int_{\mathbb{R}^4} |V(x)||\mathcal{G}(x,y;\alpha) - g(\alpha)||V(y)| \,\mathrm{d}(x,y).\end{aligned}$$

Hence, employing (3.8), both assertions follow if we show for $t \in \{1,2\}$ that

$$\int_{\mathbb{R}^4} |V(x)||\mathcal{G}(x,y;\alpha) - g(\alpha)|^t |V(y)| \,\mathrm{d}(x,y) = o(|\ln\alpha|^{t-s}) \tag{3.12}$$

as $\alpha \to 0+$. In the following we assume $\alpha \in (0, \frac{1}{e})$ so that the inequalities in Lemma 2.1 are valid. In particular, by Lemma 2.1(i), there exists $C > 0$ such that for all $\alpha \in (0, \frac{1}{e})$, we have

$$|\mathcal{G}(x,y;\alpha) - g(\alpha)|^t \leq C\left(1 + |\ln|x-y||^t\right), \quad (x,y) \in \mathbb{R}^4, \quad x \neq y. \tag{3.13}$$

Next, introduce the sets

$$\begin{aligned}\Omega_1 = \Omega_1(\alpha) &= \big\{(x,y) \in \mathbb{R}^4 : \ln|x-y| < 1\big\}, \\ \Omega_2(\alpha) &= \big\{(x,y) \in \mathbb{R}^4 : 1 \leq \ln|x-y| < |\ln\alpha|^{\frac{1}{2}}\big\}, \\ \Omega_3(\alpha) &= \big\{(x,y) \in \mathbb{R}^4 : |\ln\alpha|^{\frac{1}{2}} \leq \ln|x-y| < |\ln\alpha|\big\}, \\ \Omega_4(\alpha) &= \big\{(x,y) \in \mathbb{R}^4 : |\ln\alpha| \leq \ln|x-y|\big\}.\end{aligned} \tag{3.14}$$

Note that $\mathbb{R}^4 = \bigcup_{k=1}^4 \Omega_k(\alpha)$ as a disjoint union and that $\Omega_k(\alpha) \neq \emptyset$ for each $k \in \{1,\dots,4\}$ since by assumption $\alpha \in (0, \frac{1}{e})$, and thus $1 < |\ln\alpha|^{\frac{1}{2}} < |\ln\alpha|$.

Consequently, (3.12) follows if we consider for $t \in \{1, 2\}$ and $k \in \{1, \dots, 4\}$ the integrals

$$I_k(\alpha) := \int_{\Omega_k(\alpha)} |V(x)||\mathcal{G}(x, y; \alpha) - g(\alpha)|^t |V(y)| \, \mathrm{d}(x, y), \tag{3.15}$$

and show for each $k \in \{1, \dots, 4\}$ that

$$I_k(\alpha) = o(|\ln \alpha|^{t-s}), \quad \alpha \to 0+. \tag{3.16}$$

It is clear that (3.16) holds for $I_1(\alpha)$ since by (3.13)

$$I_1(\alpha) \leq C\bigg(\|V\|^2_{L^1(\mathbb{R}^2)} + \int_{|x-y|<e} |V(x)||\ln|x-y||^t |V(y)| \, \mathrm{d}(x, y) \bigg) < \infty;$$

the last integral is finite by (3.10) applied with $r = t$ and our assumptions $V \in L^1(\mathbb{R}^2)$ and (1.6). In particular, $I_1(\alpha)$ is bounded for $\alpha \in (0, \frac{1}{e})$ and since $|\ln \alpha|^{t-s} \to \infty$ as $\alpha \to 0+$ the claim (3.16) follows for $k = 1$.

We continue with the integral $I_2(\alpha)$. Note that for any $(x, y) \in \Omega_2(\alpha)$

$$\begin{aligned} 1 \leq (\ln|x-y|)^t &= (\ln|x-y|)^{t-s} (\ln|x-y|)^s \\ &< |\ln \alpha|^{\frac{t-s}{2}} (\ln|x-y|)^s. \end{aligned} \tag{3.17}$$

By employing this inequality in (3.13) we infer

$$|\mathcal{G}(x, y; \alpha) - g(\alpha)|^t \leq C(1 + |\ln \alpha|^{\frac{t-s}{2}} (\ln|x-y|)^s) \leq C|\ln \alpha|^{\frac{t-s}{2}} (\ln|x-y|)^s$$

and hence by (3.9)

$$I_2(\alpha) \leq C|\ln \alpha|^{\frac{t-s}{2}}.$$

In particular, since by assumption $s < 1$ and $t \in \{1, 2\}$, we conclude

$$\frac{I_2(\alpha)}{|\ln \alpha|^{t-s}} \leq C|\ln \alpha|^{\frac{s-t}{2}} \to 0, \quad \alpha \to 0+,$$

i.e., (3.16) holds for $k = 2$.

Next we show (3.16) for $I_3(\alpha)$ and $I_4(\alpha)$. A similar estimate as in (3.17) implies for any $(x, y) \in \Omega_3(\alpha)$

$$(\ln|x-y|)^t \leq |\ln \alpha|^{t-s} (\ln|x-y|)^s.$$

Using the inequality in (3.13) implies that

$$\begin{aligned} |\mathcal{G}(x, y; \alpha) - g(\alpha)|^t &\leq C(1 + |\ln \alpha|^{t-s} (\ln|x-y|)^s) \\ &\leq C|\ln \alpha|^{t-s} (\ln|x-y|)^s, \quad (x, y) \in \Omega_3(\alpha). \end{aligned} \tag{3.18}$$

For $(x, y) \in \Omega_4(\alpha)$ we see (recall that $\alpha \in (0, \frac{1}{e})$)

$$|x-y| \geq e^{|\ln \alpha|} = e^{-\ln \alpha} = \frac{1}{\alpha}$$

so we can employ Lemma 2.1(ii) and (i), respectively, to obtain

$$\begin{aligned}|\mathcal{G}(x,y;\alpha)-g(\alpha)|^t &= |\mathcal{G}(x,y;\alpha)-g(\alpha)|^{t-s}|\mathcal{G}(x,y;\alpha)-g(\alpha)|^s\\ &\le C|\ln\alpha|^{t-s}(1+(\ln|x-y|)^s)\\ &\le C|\ln\alpha|^{t-s}(\ln|x-y|)^s, \quad (x,y)\in\Omega_4(\alpha).\end{aligned} \tag{3.19}$$

Finally, by combining (3.18) with (3.19) we obtain

$$I_3(\alpha)+I_4(\alpha)\le C|\ln\alpha|^{t-s}\int_{\Omega_3(\alpha)\cup\Omega_4(\alpha)}|V(x)|(\ln|x-y|)^s|V(y)|\,\mathrm{d}(x,y).$$

By (3.9) and the definition of $\Omega_3(\alpha)$ and $\Omega_4(\alpha)$ dominated convergence implies that the last integral tends to zero as $\alpha\to 0+$ so (3.16) for $k\in\{3,4\}$ follows. □

3.2. **Negative eigenvalues of H_ε.** In the following we use the decomposition (3.5) to characterize the negative eigenvalues of H_ε in certain subsets as zeros of a function Λ_ε.

Proposition 3.2. *Let $\varepsilon>0$, assume that $V\in L^1(\mathbb{R}^2)$ is real-valued and satisfies* (1.6)*, let $H_\varepsilon=-\Delta-\varepsilon V$ be defined as in* (3.3)*, and let g be given by* (3.8)*. Then for any $\alpha>0$ such that $\|\varepsilon M(\alpha)\|<1$ we have that $-\alpha^2$ is an eigenvalue of H_ε if and only if*

$$\Lambda_\varepsilon(\alpha)\coloneqq 1-\varepsilon g(\alpha)([I-\varepsilon M(\alpha)]^{-1}|V|^{\frac12},V^{\frac12})=0. \tag{3.20}$$

Moreover, any eigenvalue of H_ε of this form is simple.

Proof. If $\alpha>0$ is such that $\|\varepsilon M(\alpha)\|<1$ we make use of (3.5) and the bounded invertibility of $I-\varepsilon M(\alpha)$ and obtain

$$I-\varepsilon Q(\alpha)=(I-\varepsilon M(\alpha))\left(I-[I-\varepsilon M(\alpha)]^{-1}\varepsilon L(\alpha)\right). \tag{3.21}$$

With the help of the Birman-Schwinger principle (3.4) we then conclude

$$\begin{aligned}\dim\ker(H_\varepsilon+\alpha^2) &= \dim\ker(I-\varepsilon Q(\alpha))\\ &= \dim\ker\left(I-[I-\varepsilon M(\alpha)]^{-1}\varepsilon L(\alpha)\right).\end{aligned}$$

Next, note that (3.6) implies $L(\alpha)=g(\alpha)L_2L_1$ with

$$L_1:\begin{cases}L^2(\mathbb{R}^2)\to\mathbb{C},\\ f\mapsto (f,V^{\frac12}),\end{cases}\qquad L_2:\begin{cases}\mathbb{C}\to L^2(\mathbb{R}^2),\\ \varphi\mapsto\varphi|V|^{\frac12}.\end{cases} \tag{3.22}$$

Recall that for two bounded and everywhere defined operators $A:\mathcal{H}\to\mathcal{G}$ and $B:\mathcal{G}\to\mathcal{H}$ one has $\sigma_p(AB)\cup\{0\}=\sigma_p(BA)\cup\{0\}$ and

$$\dim\ker(I_{\mathcal{G}}-AB)=\dim\ker(I_{\mathcal{H}}-BA),$$

see, e.g., [7, Chap. VII.4, Prob. 7]. In particular, using the factorization (3.22) of L we conclude that

$$\dim\ker\left(I-[I-\varepsilon M(\alpha)]^{-1}\varepsilon L(\alpha)\right)=\dim\ker\left(1-\varepsilon g(\alpha)L_1[I-\varepsilon M(\alpha)]^{-1}L_2\right).$$

Since the definition of L_1 and L_2 implies

$$1 - \varepsilon g(\alpha) L_1 [I - \varepsilon M(\alpha)]^{-1} L_2 = 1 - \varepsilon g(\alpha)([I - \varepsilon M(\alpha)]^{-1} |V|^{\frac{1}{2}}, V^{\frac{1}{2}}) = \Lambda_\varepsilon(\alpha)$$

we have shown (3.20).

Finally, $\dim \mathbb{C} = 1$ so either $\dim \ker \Lambda_\varepsilon(\alpha) = 1$ if $\Lambda_\varepsilon(\alpha) = 0$ or alternatively $\dim \ker \Lambda_\varepsilon(\alpha) = 0$ if $\Lambda_\varepsilon(\alpha) \neq 0$. Hence any eigenvalue corresponding to a zero of Λ_ε is simple. □

3.3. **Proof of Theorem 1.2.** From the previous considerations it is clear that the operator $H_\varepsilon = -\Delta - \varepsilon V$ in (3.3) is self-adjoint in $L^2(\mathbb{R}^2)$, bounded from below, and one has $\sigma_{\mathrm{ess}}(H_\varepsilon) = [0, \infty)$. Thus it remains to verify (ii). By Proposition 3.2 it suffices to prove that the function Λ_ε given in (3.20) has a zero as $\varepsilon \to 0+$. For ease of notation we set

$$U := \int_{\mathbb{R}^2} V(x)\, \mathrm{d}x = (|V|^{\frac{1}{2}}, V^{\frac{1}{2}})$$

and note that by assumption $U > 0$. Let $\alpha > 0$ such that $\|\varepsilon M(\alpha)\| < 1$. By employing the expansion

$$[I - \varepsilon M(\alpha)]^{-1} = I + \varepsilon M(\alpha) + \varepsilon^2 [I - \varepsilon M(\alpha)]^{-1} M(\alpha)^2$$

in (3.20) we find

$$\Lambda_\varepsilon(\alpha) = 1 - g(\alpha)\big(U\varepsilon + r_\varepsilon(\alpha)\big), \tag{3.23}$$

where we have defined

$$r_\varepsilon(\alpha) := \varepsilon^2 (M(\alpha)|V|^{\frac{1}{2}}, V^{\frac{1}{2}}) + \varepsilon^3([I - \varepsilon M(\alpha)]^{-1} M(\alpha)^2 |V|^{\frac{1}{2}}, V^{\frac{1}{2}}). \tag{3.24}$$

Next, we make the substitution

$$\alpha(t) := g^{-1}\left(\frac{1}{U\varepsilon}(1+t)\right) = \exp\left(-\frac{2\pi}{U\varepsilon}(1+t)\right), \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{3.25}$$

which is well-defined for any $\varepsilon > 0$ since $g : (0,1) \to (0, \infty)$ is bijective and by assumption $U > 0$. Clearly, $\sup_{t \in [-\frac{1}{2}, \frac{1}{2}]} |\alpha(t)| \to 0$ as $\varepsilon \to 0+$ and

$$\sup_{t \in [-\frac{1}{2}, \frac{1}{2}]} |g(\alpha(t))| \leq C\varepsilon^{-1} \tag{3.26}$$

for some $C > 0$ and all $\varepsilon > 0$. Hence, all the following asymptotics as $\varepsilon \to 0+$ hold uniformly for all $t \in [-\frac{1}{2}, \frac{1}{2}]$. By Lemma 3.1(i) we have

$$\|\varepsilon M(\alpha(t))\|^2 = \varepsilon^2 \|M(\alpha(t))\|^2 = o(\varepsilon^s), \quad \varepsilon \to 0+, \tag{3.27}$$

i.e., $\|\varepsilon M(\alpha(t))\| \leq \frac{1}{2}$ for any $t \in [-\frac{1}{2}, \frac{1}{2}]$ and all $\varepsilon \in (0, \varepsilon_0)$ if $\varepsilon_0 > 0$ is chosen small enough. In particular, $\Lambda_\varepsilon \circ \alpha : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ is well-defined and even continuous for any $\varepsilon \in (0, \varepsilon_0)$, which is seen by expanding $[I - \varepsilon M(\alpha)]^{-1}$ in (3.23) into a Neumann series and using that the family M is real-analytic on $(0, \infty)$ and hence also on $\alpha([-\frac{1}{2}, \frac{1}{2}]) \subset (0, \infty)$. Note also that each zero t of $\Lambda_\varepsilon \circ \alpha$ corresponds to a simple eigenvalue $-\alpha(t)^2$ of H_ε, see Proposition 3.2.

We continue by proving the existence of a zero of $\Lambda_\varepsilon \circ \alpha$. First we derive an upper bound for $r_\varepsilon \circ \alpha$. By (3.27) we have

$$\varepsilon^3 \big|([I - \varepsilon M(\alpha(t))]^{-1} M(\alpha(t))^2 |V|^{\frac{1}{2}}, V^{\frac{1}{2}})\big| \leq \varepsilon^3 \frac{\|M(\alpha(t))\|^2 \|V\|_{L^1(\mathbb{R}^2)}}{1 - \|\varepsilon M(\alpha(t))\|}$$
$$= o(\varepsilon^{1+s}), \quad \varepsilon \to 0+,$$

and with Lemma 3.1(ii) and (3.26) we find

$$\varepsilon^2 \big|(M(\alpha(t))|V|^{\frac{1}{2}}, V^{\frac{1}{2}})\big| = o(\varepsilon^{1+s}), \quad \varepsilon \to 0+.$$

By combining the above estimates we see that the remainder r_ε in (3.24) satisfies

$$|r_\varepsilon(\alpha(t))| = o(\varepsilon^{1+s}), \quad \varepsilon \to 0+.$$

Hence, we conclude with (3.23) and (3.25) that

$$(\Lambda_\varepsilon \circ \alpha)(t) = 1 - g(\alpha(t)) \left(U\varepsilon + o(\varepsilon^{1+s})\right) = -t + o(\varepsilon^s), \quad \varepsilon \to 0+,$$

which implies that $\Lambda_\varepsilon \circ \alpha$ has a zero $t_\varepsilon = o(\varepsilon^s)$ as $\varepsilon \to 0+$.

By Proposition 3.2 it follows that $\lambda_\varepsilon = -\alpha(t_\varepsilon)^2$ is an eigenvalue of H_ε so by employing (3.25) we find

$$\ln(-\lambda_\varepsilon) = 2\ln(\alpha(t_\varepsilon)) = -\frac{4\pi}{U\varepsilon}(1 + t_\varepsilon) = -\frac{4\pi}{U\varepsilon}(1 + o(\varepsilon^s)), \quad \varepsilon \to 0+,$$

and all claims are shown. □

4. Details on Example 1.3(ii)

Here we check that the potential V_0 given by (1.11) satisfies (1.6), i.e.,

$$\int_{|x-y|<e} V_0(x)(\ln|x-y|)^2 V_0(y)\, \mathrm{d}(x,y) < \infty. \tag{4.1}$$

Recall, that $V_0 \in L^1(\mathbb{R}^2)$ and that we have

$$V_0(x)(\ln|x-y|)^2 V_0(y) = \frac{\mathbb{1}_{\{|x|<\frac{1}{3}\}}(x)}{|x|^2(\ln|x|)^4}(\ln|x-y|)^2 \frac{\mathbb{1}_{\{|y|<\frac{1}{3}\}}(y)}{|y|^2(\ln|y|)^4}. \tag{4.2}$$

Since $(\ln|x-y|)^2$ is bounded for $1 \leq |x-y| < e$ it is clear that the integral (4.1) is finite over the region $1 \leq |x-y| < e$. Hence, it suffices to consider the regions

$$\Omega_1 := \left\{(x,y) \in \mathbb{R}^4 : |x-y| < 1,\ |x-y| \geq \frac{|x|}{2}\right\},$$
$$\Omega_2 := \left\{(x,y) \in \mathbb{R}^4 : |x-y| < 1,\ |x-y| < \frac{|x|}{2}\right\}.$$

For $(x,y) \in \Omega_1$ we have

$$\mathbb{1}_{\{|x|<\frac{1}{3}\}}(x)(\ln|x-y|)^2 \leq \mathbb{1}_{\{|x|<\frac{1}{3}\}}(x)\left(\ln\left|\frac{x}{2}\right|\right)^2 \leq C\mathbb{1}_{\{|x|<\frac{1}{3}\}}(x)(\ln|x|)^2$$

and hence by (4.2)

$$\begin{aligned}\int_{\Omega_1} & V_0(x)(\ln|x-y|)^2 V_0(y)\,\mathrm{d}(x,y)\\ &\le C\int_{\Omega_1}\frac{\mathbb{1}_{\{|x|<\frac13\}}(x)}{|x|^2(\ln|x|)^2}\frac{\mathbb{1}_{\{|y|<\frac13\}}(y)}{|y|^2(\ln|y|)^4}\,\mathrm{d}(x,y)\\ &\le C\Big(\int_{|x|<\frac13}\frac{\mathrm{d}x}{|x|^2(\ln|x|)^2}\Big)\Big(\int_{|y|<\frac13}\frac{\mathrm{d}y}{|y|^2(\ln|y|)^4}\Big)\\ &<\infty.\end{aligned}$$

For $(x,y)\in\Omega_2$ the triangle inequality implies $|x|/2<|y|<3|x|/2$, and hence

$$\frac{\mathbb{1}_{\{|x|<\frac13\}}(x)\mathbb{1}_{\{|y|<\frac13\}}(y)}{|y|^2(\ln|y|)^4}\le C\frac{\mathbb{1}_{\{|x|<\frac13\}}(x)\mathbb{1}_{\{|y|<\frac13\}}(y)}{|x|^2(\ln|x|)^4}.$$

In particular, we find with (4.2)

$$V_0(x)(\ln|x-y|)^2V_0(y)\le C\frac{\mathbb{1}_{\{|x|<\frac13\}}(x)\mathbb{1}_{\{|y|<\frac13\}}(y)}{|x|^4(\ln|x|)^8}(\ln|x-y|)^2.$$

Using the substitution $u(y)=x-y$ and Fubini's theorem we conclude

$$\int_{\Omega_2}V_0(x)(\ln|x-y|)^2V_0(y)\,\mathrm{d}(x,y)\le C\int_{|x|<\frac13}\int_{|u|<\frac{|x|}{2}}\frac{(\ln|u|)^2\,\mathrm{d}u\,\mathrm{d}x}{|x|^4(\ln|x|)^8}.$$

For the inner integral a simple estimate after integrating by parts twice shows

$$\int_{|u|<\frac{|x|}{2}}(\ln|u|)^2\,\mathrm{d}u=2\pi\int_0^{\frac{|x|}{2}}r(\ln r)^2\,\mathrm{d}r\le C|x|^2(\ln|x|)^2$$

so we finally obtain

$$\int_{|x|<\frac13}\int_{|u|<\frac{|x|}{2}}\frac{(\ln|u|)^2\,\mathrm{d}u\,\mathrm{d}x}{|x|^4(\ln|x|)^8}\le C\int_{|x|<\frac13}\frac{\mathrm{d}x}{|x|^2(\ln|x|)^6}<\infty.$$

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