On the Negative Squares of Indefinite Sturm-Liouville Operators

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Abstract

The number of negative squares of all self-adjoint extensions of a simple symmetric operator of defect one with finitely many negative squares in a Krein space is characterized in terms of the behaviour of an abstract Titchmarsh-Weyl function near 0 and ∞. These results are applied to a large class of symmetric and self-adjoint indefinite Sturm-Liouville operators with indefinite weight functions.

Key words: Sturm-Liouville operators, Krein spaces, operators with finitely many negative squares, definitizable operators, symmetric and self-adjoint operators, boundary triplets, Weyl functions, generalized Nevanlinna functions, definitizable functions

1 Introduction

Let $p^{-1}, q, r \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be real functions such that $p > 0$, $r \neq 0$ almost everywhere and assume that the Sturm-Liouville differential expression

$$\ell = \frac{1}{|r|}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right) + q\right) \tag{1.1}$$

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is regular at 0 and in the limit point case at $+\infty$, that is, there exists a (up to scalar multiples) unique solution u of $\ell(y)=\lambda y$ which belongs to the Hilbert space $L^2_{|r|}(\mathbb{R}_+)$ of measurable functions f satisfying $|f|^2|r|\in L^1(\mathbb{R}_+)$. Then the minimal operator

$$Tf=\ell(f),\qquad \operatorname{dom} T=\Big\{f\in\mathcal{D}_{\max}: f(0)=f'(0)=0\Big\},$$

is symmetric in $L^2_{|r|}(\mathbb{R}_+)$ and has deficiency indices $(1,1)$. Here $\mathcal{D}_{\max}$ denotes the usual maximal domain consisting of all functions $f\in L^2_{|r|}(\mathbb{R}_+)$ such that f and pf' are absolutely continuous and $\ell(f)$ belongs to $L^2_{|r|}(\mathbb{R}_+)$. It is well known that the maximal operator is given by $T^*f=\ell(f)$, $\operatorname{dom} T^*=\mathcal{D}_{\max}$, and that all self-adjoint extensions of T in $L^2_{|r|}(\mathbb{R}_+)$ can be parametrized in the form

$$B_\tau=T^*\upharpoonright \operatorname{dom} B_\tau,\quad \operatorname{dom} B_\tau=\Big\{f\in\mathcal{D}_{\max}: f'(0)=\tau f(0)\Big\},\quad \tau\in\overline{\mathbb{R}},$$

where $\tau=\infty$ corresponds to the Dirichlet boundary condition $f(0)=0$. Let $\lambda_0\in\mathbb{C}\backslash\mathbb{R}$ and denote by φ the unique solution of $\ell(y)=\lambda_0 y$ which belongs to $L^2_{|r|}(\mathbb{R}_+)$ and satisfies $\varphi(0)=1$. Then the spectral properties of B_∞ can be completely described with the help of the usual Titchmarsh-Weyl function m, which admits the representation

$$m(\lambda)=\operatorname{Re}\varphi'(0)+(\lambda-\operatorname{Re}\lambda_0)(\varphi,\varphi)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)\Big((B_\infty-\lambda)^{-1}\varphi,\varphi\Big).$$

Similarly, the complete spectral information of B_τ, $\tau\in\mathbb{R}$, is contained in the function $\lambda\mapsto-(m(\lambda)-\tau)^{-1}$, which, as well as m, belongs to the Nevanlinna class and admits a similar representation with the help of the resolvent of B_τ.

In this paper we assume that the weight function r changes its sign and we consider indefinite Sturm-Liouville differential expressions of the type

$$\frac{1}{r}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right) \tag{1.2}$$

instead of the differential expression (1.1). The minimal operator S associated to (1.2) is defined in the same way as T with ℓ replaced by (1.2). Then S is a symmetric operator in the Krein space $L^2_r(\mathbb{R}_+)=(L^2_{|r|}(\mathbb{R}_+),[\cdot,\cdot])$, where the indefinite inner product is defined by

$$[f,g]:=\int_a^b f(x)\overline{g(x)}r(x)\,dx,\qquad f,g\in L^2_{|r|}(\mathbb{R}_+).$$

The Krein space adjoint S^+ coincides with the maximal operator and the self-adjoint extensions A_τ, $\tau\in\overline{\mathbb{R}}$, of S in $L^2_r(\mathbb{R}_+)$ can be parametrized in the same way as the extensions B_τ of T. We emphasize that the spectral properties of the self-adjoint extensions A_τ differ essentially from the spectral properties of the self-adjoint operators B_τ in the Hilbert space $L^2_{|r|}(\mathbb{R}_+)$. For example, the real spectrum of A_τ is not semibounded, nonreal spectrum can appear, and it is not even known if the resolvent set $\rho(A_\tau)$ is nonempty in general.

It was shown by B. Curgus and H. Langer in [6] that under some additional assumptions on the symmetric operator S and the weight function r in a neighborhood of the singular endpoint $+\infty$ (see also Proposition 4.1 in Section 4) all self-adjoint realizations A_τ, $\tau \in \overline{\mathbb{R}}$, in $L^2_r(\mathbb{R}_+)$ have a nonempty resolvent set and a finite number κ_τ of negative squares, that is, for some $\kappa_\tau \in \mathbb{N}_0$ there exists a κ_τ-dimensional subspace in $\operatorname{dom} A_\tau$, such that the hermitian form $[A_\tau \cdot, \cdot]$ is negative definite on this subspace, but there is no $\kappa_\tau + 1$-dimensional subspace with this property. The number of negative squares of indefinite Sturm-Liouville operators is intimately connected with the signature of $[\cdot,\cdot]$ in algebraic eigenspaces, and hence with sign properties of solutions of homogeneous differential equations, cf. Theorems 3.1 and 3.2 in Section 3. We mention that self-adjoint operators with finitely many negative squares appear in many applications (see e.g. [3,4,6–11,22–24]).

The main focus of the present paper is an exact description of the number of negative squares of self-adjoint indefinite Sturm-Liouville operators in terms of the local behaviour of an analogue of the Titchmarsh-Weyl function from classical Sturm-Liouville theory. The functions that come into play here belong to the classes D_κ, $\kappa = 0, 1, 2, \ldots$, which were introduced by the authors in [3] (see Definition 2.1 in Section 2) as subclasses of the definitizable functions, cf. [20,21]. In the case that A_∞ is the self-adjoint indefinite Sturm-Liouville operator in $L^2_r(\mathbb{R}_+)$ corresponding to the Dirichlet boundary condition $f(0) = 0$ and A_∞ has $\kappa_\infty \in \mathbb{N}_0$ negative squares, then for $\lambda_0 \in \rho(A_\infty)$ and $\psi \in \ker(S^+ - \lambda_0)$, $\psi(0) = 1$, the (abstract) Weyl function or Q-function

$$M(\lambda) = \operatorname{Re}\psi'(0) + (\lambda - \operatorname{Re}\lambda_0)[\psi,\psi] + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)\Big[(A_\infty - \lambda)^{-1}\psi, \psi\Big],$$

corresponding to the pair $\{S, A_\infty\}$ turns out to belong to the class D_{κ_∞}, see Proposition 4.8. As the function $-(M(\lambda) - \tau)^{-1}$, $\tau \in \mathbb{R}$, is represented with the help of the resolvent of A_τ in an analogous form it can be shown that the number of negative squares of A_τ coincides with the index of the D_{κ_τ}-class to which $-(M(\lambda) - \tau)^{-1}$ belongs.

Therefore we investigate the reciprocals of functions from the class D_κ in Section 2. It will be shown that the index $\widetilde{\kappa}$ of the reciprocal function can differ at most by one from κ and the dependence of $\widetilde{\kappa}$ will be exactly described in terms of the behaviour of the D_κ-function in 0 and ∞. With the help of this result we easily obtain a characterization of the number of negative squares of the self-adjoint extensions of a simple symmetric operator of defect one with finitely many negative squares in a Krein space, cf. Theorems 3.6 and 3.7 in Section 3. We note that V. Derkach has obtained more general results with different methods in [11]. In the special case that the symmetric operator is nonnegative P. Jonas and H. Langer characterized the canonical self-adjoint extensions in [22].

The main objective of Section 4 is to apply the general results from Sections 2 and 3 to a large class of symmetric and self-adjoint Sturm-Liouville operators with indefinite weight functions which correspond to differential expressions of the form (1.2) on finite and infinite intervals.

In Section 4.3 the indefinite Sturm-Liouville differential expression (1.2) is considered on an interval (a,b), where a is assumed to be regular and b is either limit point or regular. Then the minimal operator S associated to (1.2) is a symmetric operator in the Krein space $L^2_r((a,b))$ and it will be shown that S is simple, see Proposition 4.8. With the help of Theorems 3.6 and 3.7 the number of negative squares of the self-adjoint extensions A_τ, $\tau \in \mathbb{R}$, can be precisely described in terms of the number of negative squares of A_∞ and the behaviour of the Weyl function M in 0 and ∞. For two simple examples the Weyl function and the negative squares of all self-adjoint extensions are calculated explicitely.

Special attention is paid to the differential expression $\operatorname{sgn}(\cdot)(-\frac{d^2}{dx^2}+q)$ on $\mathbb{R}$, where it is assumed that $\pm\infty$ are limit point, see Section 4.2 and also [1,9,23,24] for similar problems. Then the minimal operator A_0 is self-adjoint in the Krein space $L^2_{\text{sgn}}(\mathbb{R})$ and can be regarded as a singular perturbation of the direct sum $A = A_+ \times A_-$ of the self-adjoint realizations A_+ and A_- of $-\frac{d^2}{dx^2}+q\restriction_{\mathbb{R}_+}$ and $\frac{d^2}{dx^2}-q\restriction_{\mathbb{R}_-}$ in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ corresponding to Dirichlet boundary conditions at 0. If $\sigma(A_+)\cap\mathbb{R}_-$ consists of κ_+ eigenvalues and $\sigma(A_-)\cap\mathbb{R}_-$ consists of κ_- eigenvalues, then A has $\kappa_+ + \kappa_-$ negative squares. Due to the special structure of the perturbation A and A_0 are both self-adjoint extensions of a symmetric differential operator S of defect one in the Krein space $L^2_{\text{sgn}}(\mathbb{R})$. Under the additional assumption $\sigma_p(A_+)\cap\sigma_p(A_-)=\emptyset$ it will be shown that S is simple and hence the negative squares of A can be characterized with the help of the general results from Section 3.

2 Functions from the class D_κ and their reciprocals

The class of all functions τ which are piecewise meromorphic in $\mathbb{C}\backslash\mathbb{R}$ and symmetric with respect to the real axis, that is $\tau(\overline{\lambda}) = \overline{\tau(\lambda)}$, is denoted by $M(\mathbb{C}\backslash\mathbb{R})$. By $\mathbb{C}^+$ ($\mathbb{C}^-$) we denote the open upper (resp. lower) half plane. For the extended real line and the extended complex plane we write $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively. For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ the union of all points of holomorphy of τ in $\mathbb{C}\backslash\mathbb{R}$ and all points $\lambda \in \overline{\mathbb{R}}$ such that τ can be analytically continued to λ and the continuations from $\mathbb{C}^+$ and $\mathbb{C}^-$ coincide is denoted by $\mathfrak{h}(\tau)$.

Let $\tau \in M(\mathbb{C}\backslash\mathbb{R})$. We shall say that the *growth of* τ *near* $\overline{\mathbb{R}}$ *is of finite order* if there exist constants $M, m > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$ such that $\mathcal{U}\backslash\overline{\mathbb{R}} \subset \mathfrak{h}(\tau)$ and

$$|\tau(\lambda)| \leq \frac{M(1+|\lambda|)^{2m}}{|\operatorname{Im}\lambda|^m}$$

holds for all $\lambda \in \mathcal{U}\backslash\overline{\mathbb{R}}$. An open subset $\Delta \subset \overline{\mathbb{R}}$ is said to be of *positive type with respect to* τ if for every sequence $(\lambda_n) \subset \mathfrak{h}(\tau) \cap \mathbb{C}^+$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{n\to\infty} \operatorname{Im}\tau(\lambda_n) \geq 0.$$

An open subset $\Delta \subset \overline{\mathbb{R}}$ is said to be of *negative type with respect to* τ if Δ is of positive type with respect to $-\tau$.

If for some $\lambda_0 \in \overline{\mathbb{R}}$ the limit

$$\lim_{\lambda \widehat{\to} \lambda_0} \tau(\lambda)$$

exists and is real we set $\tau(\lambda_0) := \lim_{\lambda \widehat{\to} \lambda_0} \tau(\lambda)$. Here $\lambda \widehat{\to} \lambda_0$ denotes the nontangential limit from $\mathbb{C}^+$. In this case, by the symmetry of τ, the nontangential limit from $\mathbb{C}^-$ exists and has the same value.

Let in the following the growth of $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ near $\overline{\mathbb{R}}$ be of finite order. Let $\alpha \in \mathbb{R}$ and assume that there exists an open interval I_α, $\alpha \in I_\alpha$, such that $I_\alpha\backslash\{\alpha\}$ is of positive type with respect to τ. Let $\nu_\alpha \geq 0$ be the smallest integer such that

$$-\infty < \lim_{\lambda \widehat{\to} \alpha} (\lambda - \alpha)^{2\nu_\alpha+1}\tau(\lambda) \leq 0.$$

Due to the finite order growth of τ near $\overline{\mathbb{R}}$ such an integer ν_α always exists. If $\nu_\alpha > 0$, then α is said to be a *generalized pole of nonpositive type of* τ *with multiplicity* ν_α. Assume that there exists a number $k_\infty > 0$ such that (k_∞, ∞) and $(-\infty, -k_\infty)$ are of positive type with respect to τ and let $\nu_\infty \geq 0$ be the smallest integer such that

$$0 \leq \lim_{\lambda \widehat{\to} \infty} \frac{\tau(\lambda)}{\lambda^{2\nu_\infty+1}} < \infty.$$

Again, such an integer ν_∞ always exists. If $\nu_\infty > 0$, then ∞ is said to be a *generalized pole of nonpositive type of* τ *with multiplicity* ν_∞.

Let $\beta \in \mathbb{R}$ and assume that there exists an open interval I_β, $\beta \in I_\beta$, such that $I_\beta\backslash\{\beta\}$ is of positive type with respect to τ. Suppose that $\lim_{\lambda \widehat{\to} \beta} \frac{\tau(\lambda)}{(\lambda-\beta)^{2\gamma_\beta-1}}$

exists for some integer $\gamma_\beta \geq 0$ and let $\eta_\beta \geq 0$ be the largest integer such that

$$-\infty < \lim_{\lambda \hat{\to} \beta} \frac{\tau(\lambda)}{(\lambda - \beta)^{2\eta_\beta - 1}} \leq 0.$$

If $\eta_\beta > 0$, then $\beta \in \mathbb{R}$ is said to be a *generalized zero of nonpositive type of τ with multiplicity η_β*. Assume that there exists a number $l_\infty > 0$ such that (l_∞, ∞) and $(-\infty, -l_\infty)$ are of positive type with respect to τ, that $\lim_{\lambda \hat{\to} \infty} \lambda^{2\gamma_\infty - 1}\tau(\lambda)$ exists for some integer $\gamma_\infty \geq 0$ and let $\eta_\infty \geq 0$ be the largest integer such that

$$0 \leq \lim_{\lambda \hat{\to} \infty} \lambda^{2\eta_\infty - 1}\tau(\lambda) < \infty.$$

If $\eta_\infty > 0$, then ∞ is said to be a *generalized zero of nonpositive type of τ with multiplicity η_∞*.

The notions of generalized poles and generalized zeros of nonpositive type appear often in the investigation of the classes N_κ, $\kappa \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, of generalized Nevanlinna functions. Recall that a function $G \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to the class N_κ if the kernel N_G,

$$N_G(\lambda, \mu) := \frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}},$$

has κ negative squares (see [25]). It follows from [20, Corollary 2.6] that a function $G \in M(\mathbb{C}\backslash\mathbb{R})$ is a generalized Nevanlinna function if and only if the growth of G near $\overline{\mathbb{R}}$ is of finite order and there exists a finite set $e \subset \overline{\mathbb{R}}$ such that $\overline{\mathbb{R}}\backslash e$ is of positive type with respect to G. The class N_0 coincides with the class of Nevanlinna functions. This class consists of functions which are holomorphic in $\mathbb{C}^+ \cup \mathbb{C}^-$ and have a nonnegative imaginary part on $\mathbb{C}^+$.

Note that $G \in N_\kappa$ has poles in $\mathbb{C}^+$ and generalized poles of nonpositive type in $\mathbb{R} \cup \{\infty\}$ of total multiplicity κ. Moreover, if $G \in N_\kappa$ is not identically equal to zero, then G has zeros in $\mathbb{C}^+$ and generalized zeros of nonpositive type in $\mathbb{R} \cup \{\infty\}$ of total multiplicity κ (cf. [26]).

Next we recall the definition of the class D_κ from [3]. These function will play an important role throughout this paper.

Definition 2.1 *A function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to the class D_κ, $\kappa \in \mathbb{N}_0$, if there exists a point $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$, a function $G \in N_\kappa$ holomorphic in λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ such that*

$$\frac{\lambda}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds for all points λ where τ, G and g are holomorphic.

It was shown in [3] that the number κ in Definition 2.1 does not depend on the choice of $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$. We note that the classes D_κ, $\kappa \in \mathbb{N}_0$, are subclasses the class of definitizable functions, see [20,21].

Example 2.2 *The function*

$$\tau(\lambda) = \begin{cases} i\lambda, \text{ if } \lambda \in \mathbb{C}^+, \\ -i\lambda, \text{ if } \lambda \in \mathbb{C}^-, \end{cases}$$

is not a generalized Nevanlinna function. We have

$$\frac{\lambda}{\lambda^2+1}\tau(\lambda) = \frac{\lambda}{\lambda^2+1}(\tau(\lambda)+1) - \frac{\lambda}{\lambda^2+1}$$

and it is easy to see

$$\text{Im}\left(\frac{\lambda}{\lambda^2+1}(\tau(\lambda)+1)\right) > 0 \quad \text{for} \quad \lambda \in \mathbb{C}^+.$$

Therefore τ belongs to the class D_0. The set $(0,\infty)$ ($(-\infty,0)$) is of positive type (resp. negative type) with respect to τ. Moreover, $\mathfrak{h}(\tau)$ contains no real points and $\tau(0)$ exists.

In the next two theorems we show that for a function $\tau \in D_\kappa$, $\kappa \in \mathbb{N}_0$, not identically equal to zero it follows that $-\frac{1}{\tau}$ belongs to some class $D_{\tilde{\kappa}}$, where $\tilde{\kappa} \in \{\kappa-1, \kappa, \kappa+1\}$, $\tilde{\kappa} \in \mathbb{N}_0$, and we describe the number $\tilde{\kappa}$ in dependence of the behaviour of the functions $\lambda \mapsto \lambda^{-1}\tau(\lambda)$ and $\lambda \mapsto \lambda\tau(\lambda)$ at the points 0 and ∞, respectively.

Theorem 2.3 *Let $\tau \in D_\kappa$, $\kappa \geq 1$, be not identically equal to zero. Then*

$$-\frac{1}{\tau} \in D_{\tilde{\kappa}}, \quad \textit{where} \quad \tilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty,$$

$$\Delta_0 = \begin{cases} 0, \text{ if 0 is not a generalized zero of nonpositive type of } \lambda \mapsto \lambda\tau(\lambda), \\ -1, \text{ if 0 is a generalized zero of nonpositive type of } \lambda \mapsto \lambda\tau(\lambda), \end{cases}$$

and

$$\Delta_\infty = \begin{cases} 1, \text{ if } \infty \text{ is not a generalized zero of nonpositive type of } \lambda \mapsto \lambda^{-1}\tau(\lambda), \\ 0, \text{ if } \infty \text{ is a generalized zero of nonpositive type of } \lambda \mapsto \lambda^{-1}\tau(\lambda). \end{cases}$$

Proof. 1. According to Definition 2.1 we may choose $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, $\lambda_0 \in \mathfrak{h}(\tau)$, such that $\tau(\lambda_0) \neq 0$ and a function $G \in N_\kappa$ holomorphic in λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ such that

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds for all points λ where τ, G and g are holomorphic. It follows that

$$g(\lambda) = \frac{a\lambda^2 + b\lambda + c}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)},$$

$a, b, c \in \mathbb{R}$, has simple poles in λ_0 and $\overline{\lambda}_0$, hence $a\lambda_0^2 + b\lambda_0 + c \neq 0$ and $a\overline{\lambda}_0^2 + b\overline{\lambda}_0 + c \neq 0$. Therefore g belongs to the generalized Nevanlinna class N_1 and, as G is holomorphic in λ_0, it follows that $G + g \in N_{\kappa+1}$. Denote by α_j (β_i), $j = 1, \ldots, r$ ($i = 1, \ldots, s$) the poles (zeros) in $\mathbb{C}^+$ and the generalized poles (generalized zeros) of nonpositive type in $\mathbb{R}$ with multiplicities ν_j (η_i) of $G + g$ (cf. [26,29]). We set $\alpha_1 = \lambda_0$ and $\nu_1 = 1$. Then we have $\alpha_j \neq \lambda_0$ for $j = 2, \ldots, r$. By [16] (see also [13]) there exists a Nevanlinna function G_0 such that

$$G(\lambda) + g(\lambda) = \frac{\prod_{i=1}^{s}(\lambda-\beta_i)^{\eta_i}(\lambda-\overline{\beta}_i)^{\eta_i}}{\prod_{j=1}^{r}(\lambda-\alpha_j)^{\nu_j}(\lambda-\overline{\alpha}_j)^{\nu_j}} G_0(\lambda),$$

where

$$\max\left\{\sum_{i=1}^{s}\eta_i, \sum_{j=1}^{r}\nu_j\right\} = \kappa + 1.$$

Moreover, the difference

$$\kappa + 1 - \sum_{i=1}^{s}\eta_i \quad \left(\kappa + 1 - \sum_{j=1}^{r}\nu_j\right), \tag{2.1}$$

if positive, is the order of ∞ as a generalized zero (pole, resp.) of nonpositive type of $G + g$. We define

$$Z(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\left(\frac{-1}{\tau(\lambda)}\right) \tag{2.2}$$

and we have

$$Z(\lambda) = \frac{\lambda^2}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)} \frac{\prod_{j=2}^{r}(\lambda-\alpha_j)^{\nu_j}(\lambda-\overline{\alpha}_j)^{\nu_j}}{\prod_{i=1}^{s}(\lambda-\beta_i)^{\eta_i}(\lambda-\overline{\beta}_i)^{\eta_i}}\left(\frac{-1}{G_0(\lambda)}\right). \tag{2.3}$$

Furthermore,

$$\frac{\tau(\lambda)}{\lambda} = \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda^2}(G(\lambda) + g(\lambda)) \tag{2.4}$$

and

$$\lambda\tau(\lambda) = (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(G(\lambda) + g(\lambda)) \tag{2.5}$$

holds. Observe that $\lambda \mapsto \lambda^{-1}\tau(\lambda)$ and $\lambda \mapsto \lambda\tau(\lambda)$ have growth of finite order near $\overline{\mathbb{R}}$.

2. Let us assume that ∞ is not a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda^{-1}\tau(\lambda)$. Then, by (2.4), it follows that ∞ is not a generalized zero of nonpositive type of the function $G + g$, that is, see (2.1),

$$\sum_{j=2}^{r} \nu_j < \sum_{i=1}^{s} \eta_i = \kappa + 1.$$

If 0 is not a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$, then, by (2.5), we have that 0 is not a generalized zero of nonpositive type of the function $G + g$, that is $\beta_i \neq 0$, $i = 1, \dots, s$, and $Z \in N_{\kappa+2}$ follows. On the other hand, if 0 is a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$, then, by (2.5), 0 is a generalized zero of nonpositive type of the function $G + g$ and $Z \in N_{\kappa+1}$ follows.

3. We assume now that ∞ is a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda^{-1}\tau(\lambda)$. Then, by (2.4), ∞ is a generalized zero of nonpositive type of the function $G + g$, that is, see (2.1),

$$\sum_{i=1}^{s} \eta_i < \sum_{j=1}^{r} \nu_j = \kappa + 1.$$

As $\nu_1 = 1$, we conclude

$$\sum_{i=1}^{s} \eta_i \leq \sum_{j=2}^{r} \nu_j = \kappa.$$

And, similarly as in step 2, if 0 is not a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$, then $Z \in N_{\kappa+1}$. If 0 is a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$, then $Z \in N_\kappa$.

4. We define a function r by

$$r(\lambda) = \frac{1}{\lambda_0 - \overline{\lambda}_0}\left(\frac{-\lambda_0^2}{(\lambda - \lambda_0)(a\lambda_0^2 + b\lambda_0 + c)} + \frac{\overline{\lambda}_0^2}{(\lambda - \overline{\lambda}_0)(a\overline{\lambda}_0^2 + b\overline{\lambda}_0 + c)}\right) \tag{2.6}$$

Then the function $Z - r$ is holomorphic at λ_0 and $\overline{\lambda}_0$. Obviously the multiplicity of the poles in $\mathbb{C}^+ \backslash \{\lambda_0, \overline{\lambda}_0\}$ of Z and $Z - r$ coincide. Moreover $\alpha \in \overline{\mathbb{R}}$ is a generalized pole of multiplicity ν_α with respect to Z if and only if α is a generalized pole of multiplicity ν_α with respect to $Z - r$. Therefore the function

$Z-r$ belongs to the class $N_{\kappa+1}$ (N_κ or $N_{\kappa-1}$) if and only if Z belongs to $N_{\kappa+2}$ ($N_{\kappa+1}$ or N_κ, respectively). As

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\left(\frac{-1}{\tau(\lambda)}\right) = Z(\lambda) = Z(\lambda) - r(\lambda) + r(\lambda) \tag{2.7}$$

holds, Theorem 2.3 is proved. □

Theorem 2.4 *Let $\tau \in D_0$ be not identically equal to zero. Then*

$$-\frac{1}{\tau} \in D_1$$

if and only if ∞ is not a generalized zero of nonpositive type of $\lambda \mapsto \lambda^{-1}\tau(\lambda)$ and 0 is not a generalized zero of nonpositive type of $\lambda \mapsto \lambda\tau(\lambda)$, and

$$-\frac{1}{\tau} \in D_0$$

otherwise.

Proof. We choose λ_0, G and g as in Theorem 2.3. Then $G+g \in N_1$ and we define Z as in (2.2).

1. Let us assume that ∞ is not a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda^{-1}\tau(\lambda)$. Then there exists a Nevanlinna function G_0 and $\beta \in \mathbb{C}^+ \cup \mathbb{R}$ such that, by (2.3),

$$Z(\lambda) = \frac{\lambda^2}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)} \cdot \frac{1}{(\lambda-\beta)(\lambda-\overline{\beta})}\left(\frac{-1}{G_0(\lambda)}\right).$$

As in the proof of Theorem 2.3 we conclude that $Z \in N_2$ if $\beta \neq 0$, i.e., if 0 is not a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$ and that $Z \in N_1$ if $\beta = 0$, i.e. if 0 is a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$.

2. Let us assume that ∞ is a generalized zero of nonpositive type of the function $\lambda \mapsto \lambda^{-1}\tau(\lambda)$. Then there exists a Nevanlinna function G_0 such that

$$Z(\lambda) = \frac{\lambda^2}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\left(\frac{-1}{G_0(\lambda)}\right),$$

hence $Z \in N_1$.

3. Choose r as in (2.6). Then $Z-r$ belongs to N_1 (N_0) if and only if Z belongs to N_2 (N_1, respectively). Together with (2.7) Theorem 2.4 is proved. □

The fact that for a function $\tau \in D_\kappa$ not identically equal to zero the function $-\frac{1}{\tau}$ belongs to $D_{\kappa-1} \cup D_\kappa \cup D_{\kappa+1}$ (or to $D_0 \cup D_1$ if $\kappa = 0$) was already shown in

[3, Theorem 9] with the help of a perturbation argument applied to a so-called minimal self-adjoint representing operator or relation of the function τ.

Example 2.5 *We consider the function*

$$\tau(\lambda) = \lambda^2 + \alpha,$$

where $\alpha \in \mathbb{R}$. *Then* $\tau, -\frac{1}{\tau} \in N_1$ *and*

$$\lambda^{-1}\tau(\lambda) = \lambda + \alpha\lambda^{-1}, \tag{2.8}$$

so that $\tau \in D_0$ *follows. Moreover, equation* (2.8) *implies that* ∞ *is not a generalized zero of nonpositive type of the function* $\lambda \mapsto \lambda^{-1}\tau(\lambda)$. *As* 0 *is a generalized zero of nonpositive type of the function* $\lambda \mapsto \lambda\tau(\lambda)$ *if and only if* $\alpha \leq 0$, *we obtain*

$$-\frac{1}{\tau} \in \begin{cases} D_1, \text{ if } \alpha > 0, \\ D_0, \text{ if } \alpha \leq 0. \end{cases}$$

3 Symmetric and self-adjoint operators with finitely many negative squares

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space, let S be a densely defined linear operator in $\mathcal{K}$ and denote the adjoint of S with respect to the Krein space inner product $[\cdot,\cdot]$ by S^+. We shall say that S is *symmetric* (*self-adjoint*) if $S \subset S^+$ (resp. $S = S^+$). In the following we are in particular interested in symmetric and self-adjoint operators with finitely many negative squares. Recall that a densely defined closed symmetric operator S has κ *negative squares*, $\kappa \in \mathbb{N}_0$, if the hermitian form $\langle\cdot,\cdot\rangle$ on $\operatorname{dom} S$, defined by

$$\langle f, g\rangle := [Sf, g], \qquad f, g \in \operatorname{dom} S,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M}$ in $\operatorname{dom} S$ such that $\langle v, v\rangle < 0$ if $v \in \mathcal{M}$, $v \neq 0$, but no $\kappa+1$ dimensional subspace with this property. S is called *nonnegative* if S has $\kappa = 0$ negative squares.

Self-adjoint operators with finitely many negative squares and a nonempty resolvent set belong to the class of definitizable operators introduced and comprehensively studied by H. Langer in [27,28]. Recall that a self-adjoint operator A in $\mathcal{K}$ is said to be *definitizable* if $\rho(A)$ is nonempty and there exists a polynomial p, $p \neq 0$, such that

$$[p(A)x, x] \geq 0, \qquad x \in \operatorname{dom} p(A),$$

holds. A definitizable operator possesses a spectral function defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not belong to some finite set of so-called critical points (see [27,28]).

In the following theorem we recall some spectral properties of self-adjoint operators with finitely many negative squares. The statements are well known and are consequences of the general results in [27,28]. However, for the reader not familiar with the spectral function of a definitizable operator we give a short sketch of the proof.

Theorem 3.1 *Let A be a self-adjoint operator in the Krein space $(\mathcal{K}, [\cdot,\cdot])$, assume that $\rho(A)$ is nonempty and that A has κ negative squares. Then the following holds.*

(i) *The nonreal spectrum of A consists of at most κ pairs $\{\mu_i, \overline{\mu}_i\}$, $\mu_i \in \mathbb{C}^+$, of eigenvalues with finite dimensional algebraic eigenspaces. Denote for an eigenvalue λ of A the signature of the inner product $[\cdot,\cdot]$ on the algebraic eigenspace by $\{\kappa_-(\lambda), \kappa_0(\lambda), \kappa_+(\lambda)\}$. Then*

$$\sum_{\lambda\in\sigma_p(A)\cap(-\infty,0)} (\kappa_+(\lambda)+\kappa_0(\lambda)) + \sum_{\lambda\in\sigma_p(A)\cap(0,\infty)} (\kappa_-(\lambda)+\kappa_0(\lambda)) + \sum_i \kappa_0(\mu_i) \le \kappa, \tag{3.1}$$

and, if $0 \notin \sigma_p(A)$, then equality holds in (3.1)*.*

(ii) *There are at most κ different real nonzero eigenvalues of A with corresponding Jordan chains of length greater than one. The length of each of these chains is at most $2\kappa+1$.*

(iii) *Let B be a self-adjoint operator in $(\mathcal{K}, [\cdot,\cdot])$ with $\rho(A)\cap\rho(B) \neq \emptyset$ and assume*

$$\dim\Big(\mathrm{ran}\,\big((A-\lambda)^{-1} - (B-\lambda)^{-1}\big)\Big) = n_0 < \infty$$

for some (and hence for every) $\lambda \in \rho(A)\cap\rho(B)$. Then B has $\widetilde{\kappa} \ge 0$ negative squares, where $|\widetilde{\kappa} - \kappa| \le n_0$.

Proof. By [28] there exists a definitizing polynomial p for A which is nonnegative on $(0,\infty)$, nonpositive on $(-\infty,0)$ and each $\lambda \in \sigma_p(A)\cap(0,\infty)$ ($\lambda \in \sigma_p(A)\cap(-\infty,0)$) with $\kappa_-(\lambda)+\kappa_0(\lambda) > 0$ ($\kappa_0(\lambda)+\kappa_+(\lambda) > 0$, respectively) is a zero of p. Let E_A be the spectral function of A (cf. [28]) and choose $[a,b] \subset (0,\infty)$ such that $[a,b]$ contains exactly one zero λ of the definitizing polynomial p. Then $(E_A([a,b])\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space and the rank of negativity is $\kappa_-(\lambda)+\kappa_0(\lambda)$, see [28]; an analogous statement holds for the negative zeros of p. Moreover, the algebraic eigenspace corresponding to a nonreal eigenvalue μ_i is neutral with respect to $[\cdot,\cdot]$ and the rank of negativity of $(E_A(\{\mu_i,\overline{\mu}_i\})\mathcal{K}, [\cdot,\cdot])$ is $\kappa_0(\mu_i)$. Using the Riesz-Dunford calculus, we define a square root of the operator

$$A \upharpoonright (E_A([a,b])\mathcal{K}).$$

Using this square root, it follows easily that the forms $[\cdot,\cdot]$ and $[A\,\cdot,\cdot]$ restricted to the spectral subspace $E_A([a,b])\mathcal{K}$ have the same number of negative squares. A similar argument shows that the number of negative squares of the forms $[\cdot,\cdot]$ and $[A\,\cdot,\cdot]$ restricted to the spectral subspace $E_A(\{\mu_i,\overline{\mu}_i\})\mathcal{K}$ coincide. This implies (i).

The assertions of (ii) follow from the reasoning above and the fact that the first $\kappa+1$ elements of a Jordan chain of length $2\kappa+2$ span a $(\kappa+1)$-dimensional neutral subspace, $\kappa\geq 0$.

In order to verify (iii) note that

$$T := A \upharpoonright \Big\{x\in \operatorname{dom} A\cap \operatorname{dom} B : Ax = Bx\Big\}$$

is a (in general nondensely defined) closed symmetric operator in $\mathcal{K}$ with the property $\dim(\operatorname{graph} A/\operatorname{graph} T) = n_0$. This implies that the hermitian form $[T\cdot,\cdot]$ on $\operatorname{dom} T$ has $\kappa' \in \{\kappa-n_0,\dots,\kappa\}$ negative squares and as B is an n_0-dimensional self-adjoint extension of T the assertion follows. □

The next theorem on invariant subspaces and similarity to self-adjoint operators in Hilbert spaces follows from the general results in [5] and [28].

Theorem 3.2 *Let A be a self-adjoint operator in $(\mathcal{K},[\cdot,\cdot])$ with κ negative squares and*

$$(0,\alpha)\subset\rho(A) \quad \text{or} \quad (-\alpha,0)\subset\rho(A)$$

for some $\alpha>0$. Assume $\dim\ker A<\infty$ and assume that there exists a bounded and boundedly invertible operator W with $[Wx,x]\geq 0$, $x\in\mathcal{K}$, and $W\operatorname{dom} A\subset\operatorname{dom} A$.

Then $\mathcal{K}$ decomposes into the direct sum of two A-invariant closed subspaces $\mathcal{K}'$ and $\mathcal{K}''$ which are orthogonal with respect to $[\cdot,\cdot]$ and $(\mathcal{K}',[\cdot,\cdot])$, $(\mathcal{K}'',-[\cdot,\cdot])$ are Pontryagin spaces with finite rank of negativity. If, in addition, A has no Jordan chains of length greater than one and $\sigma_p(A)\cap\sigma_{ess}(A)=\emptyset$ holds, then A is similar to a self-adjoint operator in a Hilbert space.

We assume in the following that S is a densely defined closed symmetric operator in the Krein space $\mathcal{K}$ which is of *defect one*, that is, there exists a self-adjoint extension A' in $\mathcal{K}$ such that $\dim(\operatorname{graph} A'/\operatorname{graph} S)=1$. If, in addition S has κ negative squares, then it is obvious that each self-adjoint extension A of S in $\mathcal{K}$ has κ or $\kappa+1$ negative squares. Our aim is to describe the number of negative squares of the self-adjoint extensions of S in terms of an abstract boundary condition and an abstract analogue of the Titchmarsh-Weyl function from Sturm-Liouville theory, see Theorem 3.6 below. For this we first briefly recall the notions of boundary triplets and associated Weyl functions.

Definition 3.3 *Let S be a densely defined closed symmetric operator of defect one in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a* boundary triplet *for S^+ if there exist linear mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} S^+ \to \mathbb{C}$ such that*

$$[S^+ f, g] - [f, S^+ g] = \Gamma_1 f\, \overline{\Gamma_0 g} - \Gamma_0 f\, \overline{\Gamma_1 g}$$

holds for all $f, g \in \operatorname{dom} S^+$ and the mapping $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^+ \to \mathbb{C}^2$ is surjective.

For basic facts on boundary triplets and further references, see, e.g., [11,12,14,15]. We recall only a few important facts. Let S be a densely defined closed symmetric operator of defect one in the Krein space $\mathcal{K}$. Then a boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for S^+ always exists, but is not unique. All self-adjoint extensions A_τ of S in $\mathcal{K}$ can be characterized by

$$A_\tau := \begin{cases} S^+ \upharpoonright \ker(\Gamma_1 - \tau\Gamma_0), & \text{if } \tau \in \mathbb{R}, \\ S^+ \upharpoonright \ker \Gamma_0, & \text{if } \tau = \infty. \end{cases} \tag{3.2}$$

For brevity we shall sometimes write $A_\tau = S^+ \upharpoonright \ker(\Gamma_1 - \tau\Gamma_0)$, $\tau \in \overline{\mathbb{R}}$, instead of relation (3.2). Moreover, we will usually write A instead of A_∞, that is, $A = A_\infty = S^+ \upharpoonright \ker \Gamma_0$.

For a point λ of regular type of S we set $\mathcal{N}_\lambda := \ker(S^+ - \lambda)$. In the following we will assume that the self-adjoint operator $A = S^+ \upharpoonright \ker \Gamma_0$ has a nonempty resolvent set. Then the functions

$$\lambda \mapsto \gamma(\lambda) := \left(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\right)^{-1} \quad \text{and} \quad \lambda \mapsto M(\lambda) := \Gamma_1 \left(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\right)^{-1}, \quad \lambda \in \rho(A),$$

are well defined and holomorphic on $\rho(A)$, they are called the *γ-field* and the *Weyl function* corresponding to S and the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, see [11]. The γ-field and Weyl function satisfy

$$\gamma(\lambda) = \left(1 + (\lambda - \mu)(A - \lambda)^{-1}\right)\gamma(\mu) \qquad \lambda, \mu \in \rho(A), \tag{3.3}$$

and

$$M(\lambda) - M(\overline{\mu}) = (\lambda - \overline{\mu})\gamma(\mu)^+\gamma(\lambda), \qquad \lambda, \mu \in \rho(A),$$

and it follows that

$$M(\lambda) = \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+\left((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A - \lambda)^{-1}\right)\gamma(\lambda_0)$$

holds for any fixed $\lambda_0 \in \rho(A)$ and all $\lambda \in \rho(A)$. The Weyl function can be used to describe the spectral properties of the extensions A_τ, $\tau \in \overline{\mathbb{R}}$. We mention only that a point $\lambda \in \rho(A)$ belongs to $\rho(A_\tau)$, $\tau \in \mathbb{R}$, if and only if $M(\lambda) \neq \tau$, and that

$$(A_\tau - \lambda)^{-1} = (A - \lambda)^{-1} + \gamma(\lambda)\left(\tau - M(\lambda)\right)^{-1}\gamma(\overline{\lambda})^+$$

holds for all $\lambda \in \rho(A) \cap \rho(A_\tau)$.

Now we focus on the special case of symmetric and self-adjoint operators with finitely many negative squares. In the following a densely defined closed symmetric operator S of defect one with finitely many negative squares is said to be *simple* if there exists a self-adjoint extension A' of S with a nonempty resolvent set such that the condition

$$\mathcal{K} = \operatorname{clsp}\left\{\mathcal{N}_\lambda : \lambda \in \rho(A')\right\} \tag{3.4}$$

holds. The following proposition (cf. [6, Proposition 1.1]) together with Runge's theorem shows that relation (3.4) does not depend on the choice of A'.

Proposition 3.4 *Let S be a densely defined closed symmetric operator of defect one in $\mathcal{K}$ with finitely many negative squares and assume that there exists a self-adjoint extension A' of S with a nonempty resolvent set. Then every self-adjoint extension of S has a nonempty resolvent set and finitely many negative squares.*

The statements in the next proposition can be found in [3, Lemma 7] and [21, Theorem 1.12 and § 3].

Proposition 3.5 *Let S be a densely defined closed symmetric operator of defect one in $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+ with Weyl function M. Assume that $A = S^+ \upharpoonright \ker \Gamma_0$ has finitely many negative squares and a nonempty resolvent set and that S is simple. Then the following holds.*

(i) *A has κ negative squares if and only if M belongs to the class D_κ,*
(ii) *$\rho(A) = \mathfrak{h}(M) \backslash \{\infty\}$,*
(iii) *λ is a pole of multiplicity ν of M if and only if λ is an isolated eigenvalue of A with $\dim(\operatorname{ran} E_A(\{\lambda\})) = \nu$.*

Now we use Theorem 2.3 in order to give a characterization of the number of negative squares of the self-adjoint extensions of a simple symmetric operator of defect one with finitely many negative squares. We note that V. Derkach has obtained more general results in [11] with different methods and that the statements in Theorem 3.6 and Theorem 3.7 can be deduced from [11, Corollary 5.1]. Recall, that for a function τ from some class D_κ we write $\tau(\lambda_0)$, $\lambda_0 \in \overline{\mathbb{R}}$, if the nontangential limit $\lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)$ from the upper halfplane exists and is real (see Section 2). If $\operatorname{Im}(\lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)) \neq 0$ or $\lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)$ does not exist we shall say that $\tau(\lambda_0)$ does not exist.

Theorem 3.6 *Let S be a densely defined closed simple symmetric operator of defect one in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Assume that $A = S^+ \upharpoonright \ker \Gamma_0$ has $\kappa \geq 1$ negative squares and a nonempty* *resolvent set, let*

$$A_\tau = S^+ \upharpoonright \ker\Big(\Gamma_1 - \tau\Gamma_0\Big), \qquad \tau \in \mathbb{R},$$

and denote the Weyl function corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ *by* M. *If* $M(0)$ *or* $M(\infty)$ *does not exist, we set* $M(0) := \infty$ *and* $M(\infty) := -\infty$, *respectively. Then*

$$A_\tau \text{ has } \widetilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty \text{ negative squares,}$$

where

$$\Delta_0 = \begin{cases} 0, \text{ if } \tau < M(0), \\ -1, \text{ otherwise,} \end{cases} \quad \text{and} \quad \Delta_\infty = \begin{cases} 1, \text{ if } M(\infty) < \tau, \\ 0, \text{ otherwise.} \end{cases}$$

Proof. The operator S has defect one, hence all self-adjoint extensions A_τ, $\tau \in \mathbb{R}$, of S in $\mathcal{K}$ have $\kappa' \in \{\kappa-1, \kappa, \kappa+1\}$ negative squares. By Proposition 3.4 we have $\rho(A_\tau) \neq \emptyset$ for all $\tau \in \mathbb{R}$ and the nonreal spectrum of A_τ consists only of finitely many eigenvalues, see Theorem 3.1(i).

It is easy to see that $\{\mathbb{C}, \Gamma_1 - \tau\Gamma_0, -\Gamma_0\}$, $\tau \in \mathbb{R}$, is a boundary triplet for S^+ with corresponding γ-field γ_τ and Weyl function M_τ given by

$$\gamma_\tau(\lambda) = \gamma(\lambda)(M(\lambda) - \tau)^{-1}, \qquad \lambda \in \rho(A) \cap \rho(A_\tau),$$

and

$$M_\tau(\lambda) = -(M(\lambda) - \tau)^{-1}, \qquad \lambda \in \rho(A) \cap \rho(A_\tau),$$

respectively. Note that $\rho(A) \cap \rho(A_\tau) = (\mathfrak{h}(M) \cap \mathfrak{h}((M - \tau)^{-1})) \backslash \{\infty\}$ holds. Since

$$\mathcal{K} = \operatorname{clsp}\Big\{\mathcal{N}_\lambda : \lambda \in \rho(A)\Big\} = \operatorname{clsp}\Big\{\mathcal{N}_\lambda : \lambda \in \rho(A) \cap \rho(A_\tau)\Big\}$$

we can apply Proposition 3.5(i), so that, M_τ belongs to the class $D_{\widetilde{\kappa}}$ if and only if $A_\tau = S^+ \upharpoonright \ker(\Gamma_1 - \tau\Gamma_0)$ has $\widetilde{\kappa}$ negative squares.

In order to determine the class $D_{\widetilde{\kappa}}$ to which M_τ belongs we use Theorem 2.3. By our assumptions on A and Proposition 3.5(i) the functions M and $M - \tau$ belong to the class D_κ. Assume first that the function $\lambda \mapsto M(\lambda)$ admits a continuation into the points 0 and ∞ such that

$$M(0) = \lim_{\lambda \widehat{\rightarrow} 0} M(\lambda) \quad \text{and} \quad M(\infty) = \lim_{\lambda \widehat{\rightarrow} \infty} M(\lambda)$$

are real. Then 0 is a generalized zero of nonpositive type of the function

$$\lambda \mapsto \lambda(M(\lambda) - \tau) \tag{3.5}$$

if and only if $M(0) \leq \tau$. Moreover, ∞ is a generalized zero of nonpositive type of the function

$$\lambda \mapsto \frac{1}{\lambda}(M(\lambda) - \tau) \tag{3.6}$$

if and only if $\tau \leq M(\infty)$. Hence we conclude from Theorem 2.3 that

$$M_\tau \in D_{\tilde{\kappa}}, \quad \text{where} \quad \tilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty,$$

$$\Delta_0 = \begin{cases} \;\;0, \text{ if } \tau < M(0), \\ -1, \text{ otherwise,} \end{cases} \quad \text{and} \quad \Delta_\infty = \begin{cases} 1, \text{ if } M(\infty) < \tau, \\ 0, \text{ otherwise,} \end{cases}$$

and Proposition 3.5(i) implies the statements of Theorem 3.6.

In the case that $\lim_{\lambda \hat{\to} \infty} M(\lambda)$ does not exist, or, if it exists, it is nonreal, it is clear that the function in (3.6) has no generalized zero of nonpositive type in ∞. Thus, if $M(0)$ exists and is real, we have $M_\tau \in D_{\kappa+1}$ if and only if $\tau < M(0)$ and $M_\tau \in D_\kappa$ otherwise. If $\lim_{\lambda \hat{\to} 0} M(\lambda)$ does not exist, or, if it exists, it is nonreal, it is clear that the function in (3.5) has no generalized zero of nonpositive type in 0. Thus, if $M(\infty)$ exists and is real, we have $M_\tau \in D_{\kappa+1}$ if and only if $M(\infty) < \tau$ and $M_\tau \in D_\kappa$ otherwise. Finally, if $\lim_{\lambda \hat{\to} \infty} M(\lambda)$ and $\lim_{\lambda \hat{\to} 0} M(\lambda)$ do not exist or are nonreal Theorem 2.3 implies $M_\tau \in D_{\kappa+1}$ for all $\tau \in \mathbb{R}$. Together with Proposition 3.5(i) this proves Theorem 3.6. □

For the special case that A is nonnegative P. Jonas and H. Langer characterized the negative squares of the canonical self-adjoint extensions of A in [22]. As a consequence of Theorem 2.4 we obtain the following result.

Theorem 3.7 *Let S be a densely defined closed simple symmetric operator of defect one in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Assume that $A = S^+ \upharpoonright \ker \Gamma_0$ is nonnegative and that $\rho(A)$ is nonempty, let*

$$A_\tau = S^+ \upharpoonright \ker\Big(\Gamma_1 - \tau\Gamma_0\Big), \qquad \tau \in \mathbb{R},$$

and denote the Weyl function corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ by M. If $M(0)$ or $M(\infty)$ does not exist, we set $M(0) := \infty$ and $M(\infty) := -\infty$, respectively. Then A_τ has one negative square if and only if

$$M(\infty) < \tau < M(0)$$

and A_τ is a nonnegative operator otherwise.

4 Sturm-Liouville operators with an indefinite weight

In this section we show that the general results from the previous sections can be applied to a large class of symmetric and self-adjoint Sturm-Liouville operators with indefinite weight functions and we discuss some explicit examples.

4.1 Indefinite Sturm-Liouville differential expressions

Let $-\infty \leq a < b \leq \infty$ and assume that $r \in L^1_{\text{loc}}((a,b))$ is a real valued function on (a,b) such that $r \neq 0$ almost everywhere and that the sets

$$\Delta_+ := \{x \in (a,b) : r(x) > 0\} \qquad \text{and} \qquad \Delta_- := \{x \in (a,b) : r(x) < 0\}$$

have positive Lebesgue measure. By $L^2_{|r|}((a,b))$ we denote the space of all equivalence classes of measurable functions f defined on (a,b) for which

$$\int_a^b |f(x)|^2 |r(x)| dx$$

is finite. We equip $L^2_{|r|}((a,b))$ with the indefinite inner product

$$[f,g] := \int_a^b f(x)\overline{g(x)} r(x) dx, \qquad f,g \in L^2_{|r|}((a,b)),$$

and denote the corresponding Krein space $(L^2_{|r|}((a,b)), [\cdot,\cdot])$ by $L^2_r((a,b))$. A fundamental symmetry in $L^2_r((a,b))$ is given by $(Jf)(x) := (\operatorname{sgn} r(x)) f(x)$, $f \in L^2_r((a,b))$, and the corresponding fundamental decomposition is

$$L^2_r((a,b)) = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-, \quad \mathcal{K}_+ := L^2_{|r|}(\Delta_+), \quad \mathcal{K}_- := L^2_{|r|}(\Delta_-). \tag{4.1}$$

Note that $[J\cdot,\cdot]$ coincides with the usual Hilbert scalar product

$$(f,g) := \int_a^b f(x)\overline{g(x)} |r(x)|\, dx, \quad f,g \in L^2_{|r|}((a,b)),$$

on $L^2_{|r|}((a,b))$.

Let $p^{-1}, q \in L^1_{\text{loc}}((a,b))$ be real valued functions and assume $p > 0$ almost everywhere. In the following we consider the indefinite Sturm-Liouville differential expression

$$\frac{1}{r}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right) + q\right) \tag{4.2}$$

and we define different operators in the Krein space $L^2_r((a,b))$ associated with this differential expression. We shall say that the boundary point a (b) is *regular* if $-\infty < a$ ($b < \infty$, respectively) and for one (and hence for all) $c \in (a,b)$ the functions p^{-1}, q, r belong to $L^1((a,c))$ ($L^1((c,b))$, respectively). If the endpoint a (b) is not regular, then we say that a (b, respectively) is *singular*.

The maximal operator associated to (4.2) is denoted by $T_{\max}$,

$$T_{\max} f = \frac{1}{r}\Big(-(pf')' + qf\Big), \qquad \operatorname{dom} T_{\max} = \mathcal{D}_{\max}, \tag{4.3}$$

where $\mathcal{D}_{\max}$ is the maximal domain in $L^2_r((a,b))$ defined by

$$\mathcal{D}_{\max} := \Big\{ f \in L^2_{|r|}((a,b)) : f, pf' \in AC((a,b)), \tfrac{1}{|r|}(-(pf')' + qf) \in L^2_{|r|}((a,b)) \Big\}.$$

Here $AC((a,b))$ is the linear space of absolutely continuous functions on (a,b).

Later we will consider several situations where the self-adjoint operators associated to (4.2) turn out to have a finite number of negative squares. We remark that this is in general not true, see e.g. [1,2]. The next proposition is an immediate consequence of Propositions 2.2-2.5 from [6] for the case of second order differential operators. It states that under suitable assumptions all self-adjoint restrictions of $T_{\max}$ have finitely many negative squares.

Proposition 4.1 *Assume that $A \subseteq T_{\max}$ is a self-adjoint operator in the Krein space $L^2_r((a,b))$. Then the following holds.*

- (i) *If a and b are regular, then A has a finite number of negative squares and $\rho(A)$ is nonempty.*
- (ii) *If a (or b) is singular and there exists $a' \in (a,b)$ (or $b' \in (a,b)$) such that the hermitian form $[A\cdot,\cdot]$ is positive on all $f \in \operatorname{dom} A$ which vanish outside of (a,a') (or (b',b)) and r is of constant sign a.e. on (a,a') (or (b',b)), then A has a finite number of negative squares and $\rho(A)$ is nonempty.*

4.2 A singular indefinite Sturm-Liouville operator on $\mathbb{R}$

In this subsection we consider the special case $(a,b) = (-\infty,\infty)$ and $r(x) = \operatorname{sgn}(x)$. Moreover we assume for simplicity $p(x) = 1$ although this is not essential in the following investigations. In other words, we study the differential expression

$$\operatorname{sgn}(\cdot)\left(-\frac{d^2}{dx^2} + q\right) \tag{4.4}$$

in the Krein space $L^2_{\mathrm{sgn}}(\mathbb{R})$ with $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ real. We choose $J = \mathrm{sgn}\,(\cdot)$ as a fundamental symmetry. Then, by (4.1), $\mathcal{K}_\pm = L^2(\mathbb{R}_\pm)$, where $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_- = (-\infty, 0)$. In the following it will be assumed that the differential expression

$$-\frac{d^2}{dx^2} + q \tag{4.5}$$

is in the limit point case at $+\infty$ and $-\infty$, that is, for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the homogeneous equation $-f'' + qf = \lambda f$ has (up to scalar multiples) unique solutions $\varphi_{\lambda,\infty}$, $\varphi_{\lambda,-\infty}$ such that $\varphi_{\lambda,\infty} \in L^2((c_+,\infty))$ and $\varphi_{\lambda,-\infty} \in L^2((-\infty, c_-))$ for some (and hence for all) $c_+, c_- \in \mathbb{R}$.

In the following we write A_0 for the maximal operator $T_{\max}$ from (4.3) and (4.4), this notation will become clear later in Proposition 4.4. Obviously JA_0, $\mathrm{dom}\, A_0 = \mathcal{D}_{\max}$, coincides with the maximal operator associated to the differential expression (4.5) in the Hilbert space $L^2(\mathbb{R})$, which is self-adjoint in $L^2(\mathbb{R})$, see, e.g., [17,30,31]. This implies that A_0 is self-adjoint in the Krein space $L^2_{\mathrm{sgn}}(\mathbb{R})$ and hence we have proved the following proposition.

Proposition 4.2 *Assume that the differential expression* (4.5) *is in the limit point case at* $\pm\infty$. *Then the maximal operator*

$$(A_0 f)(x) = \mathrm{sgn}\,(x)\Big(-f''(x) + (qf)(x)\Big), \qquad \mathrm{dom}\, A_0 = \mathcal{D}_{\max},$$

is self-adjoint in the Krein space $L^2_{\mathrm{sgn}}(\mathbb{R})$.

In the following we identify functions $f \in L^2(\mathbb{R})$ with elements $\{f_+, f_-\}$, where $f_\pm := f \restriction_{\mathbb{R}_\pm} \in L^2(\mathbb{R}_\pm)$. Similarly we write $q = \{q_+, q_-\}$, $q_\pm \in L^1_{\mathrm{loc}}(\mathbb{R}_\pm)$. Note that the differential expressions

$$-\frac{d^2}{dx^2} + q_+ \qquad \text{and} \qquad \frac{d^2}{dx^2} - q_-$$

in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ are both regular at the endpoint 0 and in the limit point case at the singular endpoint $+\infty$ and $-\infty$, respectively. Therefore the operators

$$A_+ f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A_- f_- = f_-'' - q_- f_- \tag{4.6}$$

defined on

$$\mathrm{dom}\, A_\pm = \Big\{ f_\pm \in \mathcal{D}_{\max,\pm} : f_\pm(0) = 0 \Big\}, \tag{4.7}$$

with

$$\mathcal{D}_{\max,+} = \Big\{ f_+ \in L^2(\mathbb{R}_+) : f_+, f_+' \in AC(\mathbb{R}_+),\ -f_+'' + q_+ f_+ \in L^2(\mathbb{R}_+) \Big\},$$
$$\mathcal{D}_{\max,-} = \Big\{ f_- \in L^2(\mathbb{R}_-) : f_-, f_-' \in AC(\mathbb{R}_-),\ f_-'' - q_- f_- \in L^2(\mathbb{R}_-) \Big\},$$

are self-adjoint in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [17,30,31]. We agree to denote the spectral function of $A_\pm$ by $E_{A_\pm}$. It is clear that

$$\begin{aligned}(Af)(x) &:= \operatorname{sgn}(x)\Big(-f''(x)+(qf)(x)\Big)\\ &= \Big\{-f_+''(x)+(q_+f_+)(x), f_-''(x)-(q_-f_-)(x)\Big\},\\ \operatorname{dom} A &:= \Big\{f=\{f_+,f_-\} : f_\pm\in\operatorname{dom} A_\pm\Big\},\end{aligned}\tag{4.8}$$

is self-adjoint in the Krein space $L^2_{\rm sgn}(\mathbb{R})$. Here A is the diagonal block operator matrix with entries A_+ and A_- with respect to the fundamental decomposition $L^2(\mathbb{R}_+)[\dot{+}]L^2(\mathbb{R}_-)$ of $L^2_{\rm sgn}(\mathbb{R})$ and, hence, A is a fundamentally reducible operator in $L^2_{\rm sgn}(\mathbb{R})$, cf. e.g. [19, Section 3].

Proposition 4.3 *Assume that the differential expression* (4.5) *is in the limit point case at $\pm\infty$ and let $\kappa\in\mathbb{N}_0$. Then the operator A in* (4.8) *has κ negative squares if and only if*

$$\kappa=\dim\Big(\operatorname{ran} E_{A_+}((-\infty,0))\Big)+\dim\Big(\operatorname{ran} E_{A_-}((0,\infty))\Big).\tag{4.9}$$

In this case, the operator A_0 has $\kappa'\geq 0$ negative squares, where $|\kappa'-\kappa|\leq 1$.

Proof. It follows from the definition of the operator A and from

$$[Af,f]=(A_+f_+,f_+)-(A_-f_-,f_-),\qquad f=\{f_+,f_-\}\in\operatorname{dom} A,$$

that A has κ negative squares if and only if κ satisfies (4.9). The operator

$$\begin{aligned}(Sf)(x) &:= \operatorname{sgn}(x)\Big(-f''(x)+(qf)(x)\Big)\\ &= \Big\{-f_+''(x)+(q_+f_+)(x), f_-''(x)-(q_-f_-)(x)\Big\},\\ \operatorname{dom} S &:= \Big\{f=\{f_+,f_-\} : f_\pm\in\mathcal{D}_{\max,\pm},\ f_\pm(0)=0,\ f_+'(0)=f_-'(0)\Big\},\end{aligned}\tag{4.10}$$

is a closed densely defined symmetric operator in $L^2_{\rm sgn}(\mathbb{R})$ which has defect one and A is a self-adjoint extensions of S with a nonempty resolvent set. Furthermore, since

$$\operatorname{dom} A_0=\Big\{f=\{f_+,f_-\} : f_\pm\in\mathcal{D}_{\max,\pm},\ f_+(0)=f_-(0),\ f_+'(0)=f_-'(0)\Big\}$$

also A_0 is a self-adjoint extensions of S and $\rho(A_0)$ is nonempty by Proposition 3.4. Hence, for $\lambda\in\rho(A)\cap\rho(A_0)$,

$$\dim\Big(\operatorname{ran}\Big((A_0-\lambda)^{-1}-(A-\lambda)^{-1}\Big)\Big)=1$$

together with Proposition 3.1(iii) implies that A_0 has $\kappa'\in\{\kappa-1,\kappa,\kappa+1\}$, $\kappa'\geq 0$, negative squares. □

In the following we will assume that the condition

$$\dim\Big(\operatorname{ran} E_{A_+}((-\infty,0))\Big) + \dim\Big(\operatorname{ran} E_{A_-}((0,\infty))\Big) < \infty \tag{4.11}$$

is satisfied, i.e., the self-adjoint operator A_+ is semibounded from below and the self-adjoint operator A_- is semibounded from above, and $\sigma(A_+)\cap(-\infty,0)$ and $\sigma(A_-)\cap(0,\infty)$ consist of finitely many eigenvalues (which here necessarily have multiplicity one). Note that in particular the eigenvalues do not accumulate to 0 from the negative half-axis (positive half-axis, respectively). Condition (4.11) is satisfied if e.g. $\lim_{x\to\pm\infty} q_\pm(x) > 0$ or $q_\pm$ vanish in a neighbourhood of $\pm\infty$, see, e.g., [30]. Moreover we remark that condition (4.11) is independent of the choice of the self-adjoint boundary condition in $\operatorname{dom} A_\pm$ in the sense that $f_\pm(0) = 0$ could be replaced by $f_\pm(0) = \alpha_\pm f_\pm'(0)$ for any real constant $\alpha_\pm$.

In the next proposition we choose a boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the adjoint

$$\begin{aligned}(S^+ f)(x) &= \operatorname{sgn}(x)\Big(-f''(x) + (qf)(x)\Big)\\ &= \Big\{-f_+''(x) + (q_+f_+)(x), f_-''(x) - (q_-f_-)(x)\Big\},\\ \operatorname{dom} S^+ &= \Big\{f = \{f_+, f_-\} : f_\pm \in \mathcal{D}_{\max,\pm},\ f_+(0) = f_-(0)\Big\},\end{aligned} \tag{4.12}$$

of the symmetric operator S in (4.10) such that A and A_0 are the self-adjoint extensions with domain $\ker\Gamma_0$ and $\ker\Gamma_1$, respectively. The proof is straightforward and will be omitted.

Proposition 4.4 *Let S be the symmetric operator from* (4.10) *and let A and A_0 be as above. Then $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 f := f_+(0) \qquad \text{and} \qquad \Gamma_1 f := f_+'(0) - f_-'(0), \tag{4.13}$$

$f = \{f_+, f_-\} \in \operatorname{dom} S^+$, is a boundary triplet for S^+ such that $A = S^+ \upharpoonright \ker\Gamma_0$ and $A_0 = S^+ \upharpoonright \ker\Gamma_1$ holds.

In the following we will express the Weyl function M corresponding to S and the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ in Proposition 4.4 in terms of Weyl functions of the symmetric operators

$$T_+f_+ = -f_+'' + q_+f_+ \quad \text{and} \quad T_-f_- = f_-'' - q_-f_- \tag{4.14}$$

in $L^2(\mathbb{R}_\pm)$ defined on

$$\operatorname{dom} T_\pm = \Big\{f_\pm \in \mathcal{D}_{\max,\pm} : f_\pm(0) = f_\pm'(0) = 0\Big\},$$

corresponding to the boundary triplets

$$\{\mathbb{C}, f_+ \mapsto f_+(0), f_+ \mapsto f_+'(0)\} \quad \text{and} \quad \{\mathbb{C}, f_- \mapsto f_-(0), f_- \mapsto f_-'(0)\}, \tag{4.15}$$

$f_\pm \in \mathcal{D}_{\max,\pm}$, respectively. Here the adjoint operators $T_\pm^*$ in $L^2(\mathbb{R}_\pm)$ are the usual maximal operators defined on $\mathcal{D}_{\max,\pm}$. Note that the operators $A_\pm$ are self-adjoint extensions of $T_\pm$ in $L^2(\mathbb{R}_\pm)$ corresponding to the first boundary mappings $\mathcal{D}_{\max,\pm} \ni f_\pm \mapsto f_\pm(0)$. The Weyl functions m_+ and m_- corrsponding to T_+ and T_- and the boundary triplets in (4.15) are scalar Nevanlinna functions and it follows from (4.11) that m_+ and m_- have at most finitely many poles in $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively.

Remark 4.5 *If $\varphi_{\lambda,+}$, $\psi_{\lambda,+}$ and $\varphi_{\lambda,-}$, $\psi_{\lambda,-}$ denote the fundamental solutions of the differential equations $-f_+'' + q_+ f_+ = \lambda f_+$ and $f_-'' - q_- f_- = \lambda f_-$, $\lambda \in \mathbb{C}$, satisfying*

$$\varphi_{\lambda,\pm}(0) = \psi'_{\lambda,\pm}(0) = 1 \quad \text{and} \quad \varphi'_{\lambda,\pm}(0) = \psi_{\lambda,\pm}(0) = 0,$$

then for each $\lambda \in \mathbb{C}\backslash\mathbb{R}$

$$x \mapsto \varphi_{\lambda,\pm}(x) + m_\pm(\lambda)\psi_{\lambda,\pm}(x) \in L^2(\mathbb{R}_\pm)$$

holds, i.e. the functions $m_\pm$ coincide with the classical Titchmarsh-Weyl functions or Titchmarsh-Weyl coefficients of the differential expressions $-\frac{d^2}{dx^2} + q_+$ and $\frac{d^2}{dx^2} - q_-$.

In order to ensure that the symmetric operator S in (4.10) is simple we assume in the following that A_+ and A_- have no common eigenvalues.

Proposition 4.6 *Let S and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be as in* (4.10) *and Proposition* 4.4*, and let $m_\pm$ be the Weyl functions of the boundary triplets in* (4.15)*. Assume that the operators A_+ and A_- in* (4.6)*-*(4.7) *satisfy $\sigma_p(A_+) \cap \sigma_p(A_-) = \emptyset$, and that* (4.11) *is true. Then the following holds.*

(i) *The operator S is simple.*
(ii) *The Weyl function corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is $M = m_+ - m_-$.*
(iii) *M belongs to the class D_κ, where κ is given by* (4.9)*.*

Proof. (i) The operators T_+ and T_- have deficiency indices $(1,1)$. Denote by $f_{\lambda,+}$ and $f_{\lambda,-}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, a nonzero vector in $\ker(T_+^* - \lambda)$ and $\ker(T_-^* - \lambda)$, respectively. By [18, Theorem 3] the operators $T_\pm$ are simple, so that

$$L^2(\mathbb{R}_+) = \operatorname{clsp}\Big\{f_{\lambda,+} : \lambda \in \mathbb{C}\backslash\mathbb{R}\Big\} \quad \text{and} \quad L^2(\mathbb{R}_-) = \operatorname{clsp}\Big\{f_{\lambda,-} : \lambda \in \mathbb{C}\backslash\mathbb{R}\Big\}$$

hold and we have

$$L^2_{\text{sgn}}(\mathbb{R}) = \operatorname{clsp}\Big\{\{f_{\lambda,+}, 0\}, \{0, f_{\lambda,-}\} : \lambda \in \mathbb{C}\backslash\mathbb{R}\Big\}. \tag{4.16}$$

Let $\mu \in \mathbb{C}\backslash\mathbb{R}$ and set, for simplicity,

$$\mathcal{H} := \operatorname{clsp}\Big\{\ker(S^+ - \lambda) : \lambda \in \mathbb{C}\backslash\mathbb{R}\Big\}.$$

We will show

$$\{f_{\mu,+}, 0\} \in \mathcal{H}.$$

Denote by $g_\mu = \{g_{\mu,+}, g_{\mu,-}\}$ a nonzero vector in $\ker(S^+ - \mu)$, $\mu \in \mathbb{C}\backslash\mathbb{R}$. Then, by (3.3),

$$\Big(1 + (\lambda - \mu)(A - \lambda)^{-1}\Big)g_\mu \in \ker(S^+ - \lambda), \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

where A is defined as in (4.8). Hence

$$\begin{aligned}\Big(1+&(\lambda - \mu)(A - \lambda)^{-1}\Big)g_\mu = \\ =&\Big\{\Big(1 + (\lambda - \mu)(A_+ - \lambda)^{-1}\Big)g_{\mu,+}, \Big(1 + (\lambda - \mu)(A_- - \lambda)^{-1}\Big)g_{\mu,-}\Big\}.\end{aligned}$$

This gives

$$\begin{aligned}\mathcal{H} &= \operatorname{clsp}\Big\{\Big(1 + (\lambda - \mu)(A - \lambda)^{-1}\Big)g_\mu : \lambda \in \mathbb{C}\backslash\mathbb{R}\Big\} \\ &= \operatorname{clsp}\Big\{\{g_{\mu,+}, g_{\mu,-}\}, \{(A_+ - \lambda)^{-1}g_{\mu,+}, (A_- - \lambda)^{-1}g_{\mu,-}\} : \lambda \in \mathbb{C}\backslash\mathbb{R}\Big\}.\end{aligned} \tag{4.17}$$

We consider only the special case $(-\infty, 0) \subset \rho(A_+)$, $0 \notin \sigma_p(A_+)$, $\lambda_0 \in \sigma_p(A_-)\backslash\sigma_p(A_+)$ for some $\lambda_0 > 0$ and $(0, \infty)\backslash\{\lambda_0\} \subset \rho(A_-)$. The slightly more general case $\sigma_p(A_+) \cap \sigma_p(A_-) = \emptyset$ can be treated very similarly, and we leave this to the reader.

For $f_{\mu,+}$ there exits $\alpha \in \mathbb{C}$ with $f_{\mu,+}(0) = \alpha f_{\mu,-}(0)$, that is, by (4.12), $\{f_{\mu,+}, \alpha f_{\mu,-}\} \in \ker(S^+ - \mu)$. For arbitrary $M > \lambda_0$, $M \notin \sigma_p(A_+)$, and $\epsilon, \delta > 0$ the vector

$$\begin{aligned}\frac{1}{2\pi i}\int_0^{\lambda_0-\delta} &\Big\{\Big((A_+ - \lambda - i\epsilon)^{-1} - (A_+ - \lambda + i\epsilon)^{-1}\Big)f_{\mu,+}, \\ &\qquad \Big((A_- - \lambda - i\epsilon)^{-1} - (A_- - \lambda + i\epsilon)^{-1}\Big)\alpha f_{\mu,-}\Big\}\, d\lambda\end{aligned}$$

and the vector

$$\begin{aligned}\frac{1}{2\pi i}\int_{\lambda_0+\delta}^{M} &\Big\{\Big((A_+ - \lambda - i\epsilon)^{-1} - (A_+ - \lambda + i\epsilon)^{-1}\Big)f_{\mu,+}, \\ &\qquad \Big((A_- - \lambda - i\epsilon)^{-1} - (A_- - \lambda + i\epsilon)^{-1}\Big)\alpha f_{\mu,-}\Big\}\, d\lambda\end{aligned}$$

belong to $\mathcal{H}$, see (4.17). Therefore, if ϵ and δ tend to zero, we find

$$\Big\{E_{A_+}((0, M))f_{\mu,+}, 0\Big\} \in \mathcal{H},$$

and hence, for $M \to \infty$, $\{f_{\mu,+}, 0\} \in \mathcal{H}$. A similar argument shows that $\{0, f_{\mu,-}\}$ belongs to $\mathcal{H}$ for arbitrary $\mu \in \mathbb{C}\backslash\mathbb{R}$ and by (4.16) we have

$$\mathcal{H} = L^2_{\text{sgn}}(\mathbb{R}).$$

Therefore, S is simple.

(ii) Let $g_\lambda = \{g_{\lambda,+}, g_{\lambda,-}\} \in \ker(S^+ - \lambda)$, $\lambda \in \rho(A)$, so that, in particular, $g_{\lambda,\pm} \in \ker(T_\pm^* - \lambda)$. If $m_\pm$ denote the Weyl functions corresponding to the boundary triplets (4.15), then

$$m_\pm(\lambda) = \frac{g'_{\lambda,\pm}(0)}{g_{\lambda,\pm}(0)}, \qquad \lambda \in \rho(A_\pm),$$

holds and therefore (4.12) and (4.13) imply

$$M(\lambda) = \frac{\Gamma_1 g_\lambda}{\Gamma_0 g_\lambda} = \frac{g'_{\lambda,+}(0) - g'_{\lambda,-}(0)}{g_{\lambda,+}(0)} = m_+(\lambda) - m_-(\lambda), \quad \lambda \in \rho(A).$$

(iii) This is a consequence of (i) and (ii), see Proposition 3.5. □

As a consequence of Theorem 3.6 we find (in the case where κ in (4.9) is not zero) that the number of negative squares of the operator A_0 can be characterized with the help of the Weyl functions m_+ and m_- as follows. The case $\kappa = 0$ can be treated analogously with Theorem 3.7.

Theorem 4.7 *Let A_0 be the self-adjoint operator in $L^2_{\mathrm{sgn}}(\mathbb{R})$ from Proposition 4.2, assume that condition (4.11) and $\sigma_p(A_+) \cap \sigma_p(A_-) = \emptyset$ hold, and let κ in (4.9) be nonzero. Let $m_\pm$ be the Weyl functions of (4.15) and $M = m_+ - m_-$. If $M(0)$ or $M(\infty)$ does not exist, we set $M(0) := \infty$ and $M(\infty) := -\infty$, respectively. Then*

$$A_0 \text{ has } \widetilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty \text{ negative squares,}$$

where

$$\Delta_0 := \begin{cases} 0, \text{ if } 0 < M(0), \\ -1, \text{ otherwise,} \end{cases} \quad \text{and} \quad \Delta_\infty := \begin{cases} 1, \text{ if } M(\infty) < 0, \\ 0, \text{ otherwise.} \end{cases}$$

4.3 Indefinite Sturm-Liouville operators regular at one endpoint

In this section we consider indefinite Sturm-Liouville differential expressions of the form (4.2) in $L^2_r((a,b))$ which are regular at the left endpoint a and either singular (and in the limit point case) or regular at the right endpoint b. The first case that b is singular means that $-\infty < a < \infty$ and $p^{-1}, q, r \in L^1((a,c))$ for some $c \in (a,b)$ and $b = \infty$, or at least one of the functions p^{-1}, q, r does not belong to $L^1((c',b))$ for some (and hence for every) $c' \in (a,b)$. In addition, it is assumed that in the singular case the differential expression $|r|^{-1}(-\frac{d}{dx}(p\frac{d}{dx})+q)$

is limit point at b, that is, the homogeneous equation

$$-(pf')' + qf = \lambda |r| f, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

has a unique solution φ_λ (up to scalar multiples) in $L^2_{|r|}((a,b))$. The second case that (4.2) is regular at b means $p^{-1}, q, r \in L^1((a,b))$ and $b < \infty$. For brevity we will treat both cases simultaneously.

Define a symmetric operator S in the Krein space $L^2_r((a,b))$ by

$$\begin{aligned} Sf &:= \frac{1}{r}\Big(-(pf')' + qf\Big), \\ \operatorname{dom} S &:= \Big\{f \in \mathcal{D}_{\max} : f(a) = (pf')(a) = 0,\, [\alpha f(b) = (pf')(b)]_{\text{reg}}\Big\}, \end{aligned} \tag{4.18}$$

where $\alpha \in \overline{\mathbb{R}}$ is fixed and $[\cdot]_{\text{reg}}$ indicates that the boundary condition $\alpha f(b) = (pf')(b)$ is imposed in the regular case only. In this case $\alpha = \infty$ in (4.18) means $f(b) = 0$. The next proposition collects some properties of the operator S and its self-adjoint extensions A_τ in $L^2_r((a,b))$. We will again use the notation $[\cdot]_{\text{reg}}$ in (4.18) for the additional boundary condition at b in the regular case.

Proposition 4.8 *Let S be the indefinite Sturm-Liouville operator from* (4.18). *In the case that b is singular it is assumed that there exists $b' \in (a,b)$ as in Proposition* 4.1(ii). *Then S is a densely defined closed simple symmetric operator of defect one in the Krein space $L^2_r((a,b))$. The adjoint operator S^+ is given by*

$$\begin{aligned} S^+ f &= \frac{1}{r}\Big(-(pf')' + qf\Big), \\ \operatorname{dom} S^+ &= \Big\{f \in \mathcal{D}_{\max} : [\alpha f(b) = (pf')(b)]_{\text{reg}}\Big\}, \end{aligned}$$

and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, $\Gamma_0 f = f(a)$, $\Gamma_1 f = (pf')(a)$, $f \in \operatorname{dom} S^+$, is a boundary triplet for S^+. All self-adjoint extensions A_τ, $\tau \in \overline{\mathbb{R}}$, are given by

$$\begin{aligned} A_\tau f &= \frac{1}{r}\Big(-(pf')' + qf\Big), \\ \operatorname{dom} A_\tau &= \begin{cases} \{f \in \operatorname{dom} S^+ : \tau f(a) = (pf')(a)\}, & \text{if } \tau \in \mathbb{R}, \\ \{f \in \operatorname{dom} S^+ : f(a) = 0\}, & \text{if } \tau = \infty, \end{cases} \end{aligned}$$

and each A_τ, $\tau \in \overline{\mathbb{R}}$, has a nonempty resolvent set and a finite number of negative squares.

Proof. Besides the assertion that S is simple all statements of the proposition follow from the properties of the densely defined closed symmetric operator

$$JSf = \frac{1}{|r|}\Big(-(pf')' + qf\Big), \qquad \operatorname{dom} JS = \operatorname{dom} S,$$

in the Hilbert space $(L^2_{|r|}((a,b)),[J\cdot,\cdot])$ and Proposition 4.1. We leave the details to the reader.

It remains to verify that S is simple. Here we follow the lines of the proof of [18, Theorem 3] and make use of the spectral function of definitizable operators, cf. [28]. Let $A = S^+ \upharpoonright \ker \Gamma_0$ and suppose that there exists an element $k \in L^2_r((a,b))$ with the property

$$k\,[\perp]\,\mathrm{clsp}\Big\{\ker(S^+ - \lambda) : \lambda \in \rho(A)\Big\}.$$

Write the resolvent of A with the help of the Green's function G in the form

$$\Big((A-\lambda)^{-1}k\Big)(x) = \int_a^b G(x,y,\lambda)k(y)r(y)dy$$

and decompose G as in [18, Proof of Theorem 3],

$$G(x,y,\lambda) = G_0(x,y,\lambda) + G_1(x,y,\lambda),$$

where $y \mapsto G_1(x,y,\lambda)$ and $y \mapsto \frac{\partial}{\partial y}G_1(x,y,\lambda)$ are continuous also for $x = y$, so that $y \mapsto G_1(x,y,\lambda) \in \ker(S^+ - \lambda)$ and

$$\Big((A-\lambda)^{-1}k\Big)(x) = \int_a^b G_0(x,y,\lambda)k(y)r(y)dy$$

can be continued to a continuous function of $\{x,\lambda\} \in (a,b) \times \mathbb{C}$, cf. [18, Lemma 2]. Let $g \in L^2_{|r|}((a,b))$ be a function with compact support in (a,b). Then also

$$\lambda \mapsto R_{k,g}(\lambda) := \Big((A-\lambda)^{-1}k, g\Big) \tag{4.19}$$

defines a continuous function on $\mathbb{C}$.

As A is a self-adjoint operator with finitely many negative squares and a nonempty resolvent set, A is definitizable and the non-real spectrum of A consists only of finitely many nonreal eigenvalues which are symmetric with respect to the real axis, cf. Theorem 3.1 (i). Let E_A be the spectral function of A (cf. [28]) and denote by $e \subset \overline{\mathbb{R}}$ the set of critical points of A. Then for all $t_1 < t_2$, $t_1, t_2 \notin e$,

$$\Big(E_A((t_1,t_2))k, g\Big) = \lim_{\delta \searrow 0}\lim_{\varepsilon \searrow 0} \frac{1}{2\pi i}\int_{t_1+\delta}^{t_2-\delta} \Big(R_{k,g}(\lambda + i\varepsilon) - R_{k,g}(\lambda - i\varepsilon)\Big)\, d\lambda$$

holds (cf. [28, Proof of Theorem I.3.1]). Now the continuity of the function (4.19) implies

$$\Big(E_A((t_1,t_2))k, g\Big) = 0 \tag{4.20}$$

for all $t_1 < t_2$, $t_1, t_2 \notin e$. Similarly, if $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ is a nonreal eigenvalue of the operator A, $E_A(\{\lambda_0\})$ denotes the corresponding Riesz-Dunford projection and $\mathcal{C}_\varepsilon(\lambda_0) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \varepsilon\}$ with $\varepsilon > 0$ sufficiently small, then

$$\Big(E_A(\{\lambda_0\})k, g\Big) = -\frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon(\lambda_0)} R_{k,g}(\lambda)\, d\lambda \tag{4.21}$$

tends to zero for $\varepsilon \to 0$.

Let Δ, $0 \in \Delta$, be an open interval such that $E_A(\Delta)$ is defined. Then the self-adjoint operator $A \upharpoonright (I - E_A(\Delta))L^2_r((a,b))$ is a boundedly invertible operator in the Krein space $\mathcal{K}' := (I - E_A(\Delta))L^2_r((a,b))$ and the inverse

$$B := \Big(A \upharpoonright \mathcal{K}'\Big)^{-1} \in \mathcal{L}(\mathcal{K}')$$

is a definitizable operator (cf. [28, Lemma II.2.2] with

$$0 \notin \sigma_p(B). \tag{4.22}$$

Denote by E_B the spectral function of B. Then [28, Propositions II.5.1, II.5.2] and (4.22) imply

$$\mathcal{K}' = \operatorname{clsp}\Big\{E_B(\delta)\mathcal{K}' : \delta \text{ open interval}, 0 \notin \delta, E_B(\delta) \text{ exists}\Big\},$$

therefore

$$\mathcal{K}' = \operatorname{clsp}\Big\{E_A(\delta)\mathcal{K}' : \delta \text{ bounded open interval, } E_A(\delta) \text{ exists}\Big\}.$$

Together with (4.20) and (4.21) we conclude $(k, g) = 0$ for every $g \in L^2_{|r|}((a,b))$ with compact support in (a,b). This gives $k = 0$, that is, S is simple. □

If $f_\lambda \in L^2_r((a,b))$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, spans the defect subspace of S, $\ker(S^+ - \lambda) = \operatorname{sp}\{f_\lambda\}$, then the Weyl function M corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ from Proposition 4.8 is given by

$$M(\lambda) = \frac{\Gamma_1 f_\lambda}{\Gamma_0 f_\lambda} = \frac{(pf'_\lambda)(a)}{f_\lambda(a)}, \qquad \lambda \in \rho(A),$$

and belongs to the class D_κ, where the number κ coincides with the number of negative squares of the self-adjoint extension $A = S^+ \upharpoonright \ker\Gamma_0$ of S, see Proposition 3.5. As a consequence of Theorem 3.6 and Proposition 4.8 we obtain the following theorem.

Theorem 4.9 *Let $A = S^+ \upharpoonright \ker\Gamma_0$ be as above and assume that b is regular or that b is singular and there exists $b' \in (a,b)$ as in Proposition* 4.1 (ii)*. Then A has a nonempty resolvent set and κ negative squares, $\kappa \in \mathbb{N}_0$. Let M be the Weyl function corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ from*

Proposition 4.8. *If $M(0)$ or $M(\infty)$ does not exist, we set $M(0) := \infty$ and $M(\infty) := -\infty$, respectively. If $\kappa \geq 1$, then for $\tau \in \mathbb{R}$ the operator*

$$A_\tau = S^+ \restriction \ker\Big(\Gamma_1 - \tau\Gamma_0\Big) \text{ has } \widetilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty \text{ negative squares,}$$

where

$$\Delta_0 := \begin{cases} 0, \text{ if } \tau < M(0), \\ -1, \text{ otherwise,} \end{cases} \quad \text{and} \quad \Delta_\infty := \begin{cases} 1, \text{ if } M(\infty) < \tau, \\ 0, \text{ otherwise.} \end{cases}$$

The case $\kappa = 0$ can be treated analogously with Theorem 3.7.

4.3.1 A singular example

Let $-\infty < a < 0$, $b = \infty$, $p = 1$ and $q \in (0, \infty)$ be a positive real constant. As the indefinite weight function we choose $r(x) = \operatorname{sgn}(x)$, $x \in (a, \infty)$. Here the symmetric operator S in $L^2_{\text{sgn}}((a, \infty))$ from (4.18) has the form

$$\begin{aligned} (Sf)(x) &= \operatorname{sgn}(x)\Big(-f''(x) + qf(x)\Big), \\ \operatorname{dom} S &= \Big\{f \in W^2_2((a, \infty)) : f(a) = f'(a) = 0\Big\}, \end{aligned}$$

since the maximal domain $\mathcal{D}_{\max}$ coincides with the Sobolev space $W^2_2((a, \infty))$. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, $\Gamma_0 f = f(a)$, $\Gamma_1 f = f'(a)$, $f \in \operatorname{dom} S^+$, be the boundary triplet for the adjoint operator S^+, $\operatorname{dom} S^+ = W^2_2((a, \infty))$, from Proposition 4.8. We will calculate the Weyl function corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$. For $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the defect subspace $\ker(S^+ - \lambda)$ is spanned by

$$f_\lambda(x) := \begin{cases} \exp\Big(i(\sqrt{\lambda - q})x\Big), & \text{if } x \geq 0, \\ \eta(\lambda) \exp\Big((\sqrt{\lambda + q})x\Big) + \nu(\lambda) \exp\Big(-(\sqrt{\lambda + q})x\Big), & \text{if } a < x < 0, \end{cases}$$

where

$$\eta(\lambda) = \left(\frac{1}{2} + \frac{i}{2}\sqrt{\frac{\lambda - q}{\lambda + q}}\right) \quad \text{and} \quad \nu(\lambda) = \left(\frac{1}{2} - \frac{i}{2}\sqrt{\frac{\lambda - q}{\lambda + q}}\right),$$

and $\sqrt{\cdot}$ denotes the branch of the square root defined in $\mathbb{C}$ with a cut along $[0, \infty)$ and fixed by $\operatorname{Im} \sqrt{\lambda} > 0$ if $\lambda \notin [0, \infty)$. Moreover, $\sqrt{\cdot}$ is continued to $[0, \infty)$ via $\lambda \mapsto \sqrt{\lambda} \geq 0$ for $\lambda \in [0, \infty)$. The Weyl function M corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$M(\lambda) = \sqrt{\lambda + q}\, \frac{\eta(\lambda) \exp\Big(a\sqrt{\lambda + q}\Big) - \nu(\lambda) \exp\Big(-a\sqrt{\lambda + q}\Big)}{\eta(\lambda) \exp\Big(a\sqrt{\lambda + q}\Big) + \nu(\lambda) \exp\Big(-a\sqrt{\lambda + q}\Big)}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

As $\lim_{\lambda \hat{\to} 0} \eta(\lambda) = 0$ and $\lim_{\lambda \hat{\to} 0} \nu(\lambda) = 1$ we obtain

$$M(0) = \lim_{\lambda \hat{\to} 0} M(\lambda) = -\sqrt{q}.$$

Moreover it is not difficult to verify that the limit $\lim_{\lambda \hat{\to} \infty} M(\lambda)$ does not exist. Since $q > 0$ the operator $A = S^+ \upharpoonright \ker \Gamma_0$ in $L^2_{\mathrm{sgn}}((a, \infty))$ is nonnegative and we conclude from Theorem 3.7 that the self-adjoint operator

$$\begin{aligned} (A_\tau f)(x) &= \mathrm{sgn}\,(x)\Big(-f''(x) + qf(x)\Big), \\ \mathrm{dom}\, A_\tau &= \Big\{f \in W_2^2((a, \infty)) : \tau f(a) = f'(a)\Big\}, \end{aligned} \tag{4.23}$$

$\tau \in \mathbb{R}$, is nonnegative if and only if $\tau \geq -\sqrt{q}$ and has A_τ has one negative square if and only if $\tau < -\sqrt{q}$.

We note that $\sigma_{\mathrm{ess}}(A_\tau) = \sigma_{\mathrm{ess}}(A) = [q, \infty)$ for all $\tau \in \mathbb{R}$ and that the Weyl function M can be used to describe the spectra of the operators A_τ in more detail. E.g. it is straightforward to check that the poles of M on $(-\infty, q)$ do not accumulate to q, that is, the eigenvalues of A in $(-\infty, q)$ do not accumulate to $\sigma_{\mathrm{ess}}(A)$, see Proposition 3.5.

4.3.2 A regular example

Let $(a, b) = (-1, 1)$, $p = 1$, $q = 0$ and as indefinite weight we choose the function $r(x) = \mathrm{sgn}\,(x)$, $x \in (-1, 1)$. For $\alpha = \infty$ the operator S from (4.18) has the form

$$\begin{aligned} (Sf)(x) &= -\mathrm{sgn}\,(x) f''(x), \\ \mathrm{dom}\, S &= \Big\{f \in W_2^2((-1, 1)) : f(-1) = f'(-1) = f(1) = 0\Big\}, \end{aligned}$$

and is symmetric in the Krein space $L^2_{\mathrm{sgn}}((-1, 1))$. By Proposition 4.8 the adjoint operator $S^+ = -\mathrm{sgn}\,(\cdot)\frac{d^2}{dx^2}$ is defined on $\{f \in W_2^2((-1, 1)) : f(1) = 0\}$.

A simple calculation shows that $\ker(S^+ - \lambda)$ is spanned by the function

$$f_\lambda(x) := \begin{cases} \Big(\sin\sqrt{\lambda}\Big) \cosh(\sqrt{\lambda}x) - \Big(\cos\sqrt{\lambda}\Big) \sinh(\sqrt{\lambda}x) & x \in (-1, 0) \\ \sin(\sqrt{\lambda}(1 - x)) & x \in [0, 1) \end{cases}$$

if $\lambda \neq 0$, and by $f_0(x) = 1 - x$ if $\lambda = 0$. Again, the function $\lambda \mapsto \sqrt{\lambda}$ is defined as in Example 4.3.1.

We choose the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, where $\Gamma_0 f = f(-1)$ and $\Gamma_1 = f'(-1)$, $f \in \operatorname{dom} S^+$, according to Proposition 4.8. Then the self-adjoint extension $A = S^+ \upharpoonright \ker \Gamma_0$ corresponds to Dirichlet boundary conditions and it is easy to see that A is nonnegative in the Krein space $L^2_{\text{sgn}}((-1,1))$, its spectrum $\sigma(A)$ is discrete and accumulates to ∞ and $-\infty$. The Weyl function M corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$M(\lambda) = -\sqrt{\lambda}\,\frac{\sin\sqrt{\lambda}\sinh\sqrt{\lambda} + \cos\sqrt{\lambda}\cosh\sqrt{\lambda}}{\sin\sqrt{\lambda}\cosh\sqrt{\lambda} + \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}, \qquad \lambda \in \rho(A)\backslash\{0\}.$$

A point $\lambda \in \mathbb{R}\backslash\{0\}$ is an eigenvalue of A if and only if $\tan\sqrt{\lambda} = -\tanh\sqrt{\lambda}$ holds, see Proposition 3.5. Note that 0 belongs to $\rho(A)$.

The function M is holomorphic in a neighbourhood of 0, we have $M(0) = -\frac{1}{2}$ and the limit $\lim_{y\to+\infty} M(iy)$ does not exist. It follows from Theorem 3.7 that the self-adjoint operator

$$A_\tau := S^+ \upharpoonright \ker\Big(\Gamma_1 - \tau\Gamma_0\Big), \qquad \tau \in \mathbb{R},$$

that is,

$$\begin{aligned}(A_\tau f)(x) &= -\operatorname{sgn}(x)\, f''(x),\\ \operatorname{dom} A_\tau &= \Big\{ f \in W_2^2((-1,1)) : \tau f(-1) = f'(-1),\ f(1) = 0 \Big\},\end{aligned}$$

is nonnegative if and only if $\tau \in [-\frac{1}{2}, \infty)$ and A_τ has one negative square if and only if $\tau \in (-\infty, -\frac{1}{2})$. We remark, that this can also be shown by computing $[A_\tau f, f]$, $f \in \operatorname{dom} A_\tau$, and applying the Hölder inequality.

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