An open problem: Accumulation of nonreal eigenvalues of indefinite Sturm-Liouville operators

Jussi Behrndt

Abstract. In this note we conjecture that the eigenvalues of singular indefinite Sturm-Liouville operators accumulate to the real axis whenever the eigenvalues of the corresponding definite Sturm-Liouville operator accumulate to the bottom of the essential spectrum from below.

Mathematics Subject Classification (2000). 34B25, 34L15, 47E05, 47B50.

Keywords. Sturm-Liouville operator, indefinite weight function, nonreal eigenvalues.

Let $q, p^{-1}, r \in L^1_{\text{loc}}(\mathbb{R})$ be real functions, assume that $p > 0$ and $r \neq 0$ a.e., and consider the singular Sturm-Liouville differential expressions

$$\tau = \frac{1}{r}\left(-\frac{d}{dx}\,p\,\frac{d}{dx} + q\right) \qquad \text{and} \qquad \ell = \frac{1}{|r|}\left(-\frac{d}{dx}\,p\,\frac{d}{dx} + q\right).$$

The pecularity here is that the weight function r is allowed to change its sign. More precisely, we shall assume that the following condition (I) is satisfied:

(I) There exist $a, b \in \mathbb{R}$, $a \leq b$, such that $r \restriction (-\infty, a) < 0$ and $r \restriction (b, \infty) > 0$ a.e.

Suppose that the differential expression ℓ is in the limit point case at both singular endpoints $+\infty$ and $-\infty$. It is well known that under this assumption ℓ gives rise to the selfadjoint operator

$$Tf = \ell(f),$$
$$\operatorname{dom} T = \bigl\{f \in L^2(\mathbb{R}, |r|) : f, pf' \text{ absolutely continuous}, \ \ell(f) \in L^2(\mathbb{R}, |r|)\bigr\},$$

in the weighted L^2-Hilbert space $L^2(\mathbb{R}, |r|)$; here $L^2(\mathbb{R}, |r|)$ denotes the space of (equivalence classes of) measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $f^2 r \in L^1(\mathbb{R})$ and is equipped with the scalar product $(f, g) = \int f\bar{g}\,|r|$.

Let us assume that the following condition (II) holds for the spectrum of the *definite* Sturm-Liouville operator T:

(II) $\sigma(T)$ is bounded from below and $\sigma(T) \cap (-\infty, 0)$ consists of eigenvalues which accumulate to 0.

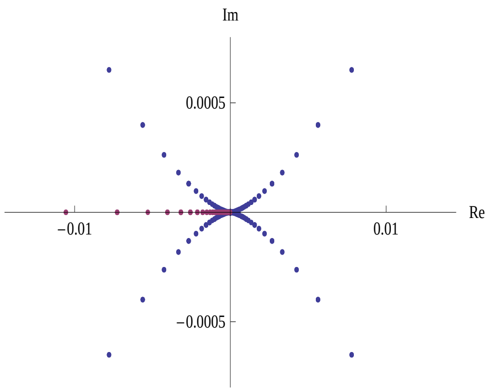

Figure 1. Numerical example for the accumulation of complex eigenvalues to zero of the indefinite differential operator A (blue points) and negative eigenvalues of T (red points) for the case $p(x) = 1$, $r(x) = \operatorname{sgn}(x)$ and $q(x) = -(1 + |x|)^{-1}$.

The open problem which we formulate below concerns the nonreal spectrum of the *indefinite* Sturm-Liouville operator

$$Af = \tau(f), \qquad \operatorname{dom} A = \operatorname{dom} T,$$

which arises from the selfadjoint operator T by multiplying it from the left with the operator $J = \operatorname{sgn}(r)$, i.e., $A = JT$. Note that $J = J^* = J^{-1}$ and that A can be viewed as an operator which is selfadjoint with respect to the indefinite inner product $[f, g] = \int f\bar{g}\, r$ in $L^2(\mathbb{R}, |r|)$. The following problem was originally formulated as a conjecture at a conference on the occasion of the retirement of A. Dijksma in Groningen, Netherlands, from 22–24 February 2006. It was posed recently as an open problem at the ICMS Workshop on *Mathematical aspects of the physics with non-self-adjoint operators* in Edinburgh, UK, 11–15 March 2013. The author is pleased to award solutions of the open problem with a bottle of finest single malt Scotch whisky.

Open Problem. *Show that under conditions* (I) *and* (II) *there exist nonreal eigenvalues of A which accumulate to* 0.

It is known that under conditions (I) and (II) the nonreal spectrum of A consists of eigenvalues only and that 0 is the only possible accumulation point of the nonreal eigenvalues. In fact, perturbation techniques for selfadjoint operators in Krein spaces imply that the operator A is definitizable over the set $\overline{\mathbb{C}} \setminus \{0\}$; cf. [1, 4]. We note that the existence of a potential q such that the nonreal eigenvalues of A accumulate to 0 was proved in [4].

The proposed open problem admits natural generalizations to the case that $\min \sigma_{\text{ess}}(T) < 0$ and can be formulated for higher order ordinary differential operators and partial differential operators with indefinite weights in the same form. The reader is referred to [3, 5] for more details on spectral theory of indefinite Sturm-Liouville operators. We also mention that a typical simple situation where conditions (I) and (II) are met is the case $p = 1$, $r = \text{sgn}$ and $q \in L^\infty(\mathbb{R})$ is such that

$$\lim_{x\to\pm\infty} q(x) = 0 \quad \text{and} \quad \limsup_{x\to\infty} x^2 q(x) < -\frac{1}{4}.$$

Figure 1 shows a numerical example in this situation with $q(x) = -(1 + |x|)^{-1}$ which was originally published in [2].

References

[1] J. Behrndt, On the spectral theory of singular indefinite Sturm-Liouville operators, J. Math. Anal. Appl. 334 (2007), 1439–1449.

[2] J. Behrndt, Q. Katatbeh, and C. Trunk, Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators, J. Phys. A: Math. Theor. 41 (2008), 244003.

[3] B. Ćurgus and H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differential Equations 79 (1989), 31–61.

[4] I.M. Karabash and C. Trunk, Spectral properties of singular Sturm-Liouville operators, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009), 483–503.

[5] A. Zettl, Sturm-Liouville theory, AMS, Providence, RI, 2005.

Jussi Behrndt
Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30
8010 Graz
Austria
e-mail: behrndt@tugraz.at