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Abstract

In this thesis we investigate Schrödinger operators with unbounded complex potentials on unbounded domains. With the help of general representation theorems we find closed Neumann realisations with non-empty resolvent sets and construct quasi boundary triples. Both sectorial and accretive potentials are considered. In the process we prove a new version of Kato‘s inequality which applies to the Neumann case. Depending on the potential we introduce different notions of Dirichlet and Neumann traces.

Kurzfassung

Diese Arbeit beschäftigt sich mit Schrödinger Operatoren mit unbeschränkten komplexen Potentialen auf unbeschränkten Gebieten. Mithilfe allgemeiner Darstellungssätze finden wir abgeschlossene Neumann Realisierungen und konstruieren Quasi Boundary Triples. Sowohl sektorielle als auch akkretive Potentiale werden behandelt. Dabei beweisen wir eine neue Version von Kato's Ungleichung, welche auf den Neumann Fall angewendet werden kann. Je nach Art des Potentials werden verschiedene Definitionen von Dirichlet- und Neumann-Spuren eingeführt.

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Introduction

Motivation

Schrödinger operators $A = -\Delta + q$ with complex-valued and unbounded potentials $q : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ arise in various fields of physics and engineering. They occur in applications such as reversed-time lasers [9], active LRC circuits [28], magnetic resonance imaging [18, 17], superconductors [1], the damped wave equation [15] and hydrodynamics [16].

Providing a proper mathematical framework remains challenging though, since the complex potential leads to non-symmetric operators in $L^2(\Omega)$. A discussion of the theory of non-symmetric operators along with its difficulties is given in [19, 11, 29, 12].

In particular one might be interested in certain restriction of these operators which model Robin boundary conditions or δ-interactions at the boundary. Such boundary conditions occur naturally in problems such as the Block-Torry equation [17] or in the Ginzburg–Landau model [1, 2]. One method to analyse such restrictions of non-symmetric operators are quasi boundary triples for adjoint pairs of operators, as introduced in [5].

Generalised boundary triples were already constructed for Schrödinger operators with complex-valued L^p potentials on domains with compact boundary, see [4]. There this approach eventually leads to closed realisations of the operators under Robin boundary conditions with non-empty resolvent set, and even a explicit resolvent formula can be found.

Motivated by these results, the aim of this work is find quasi boundary triples for a more general setting, were we also consider unbounded complex-valued Schrödinger operators on possibly unbounded domains in dimension $d \geq 2$.

Representation Theorem and Forms

Chapter 1 gives an overview of preliminary results, such as basics of Sobolev spaces, traces, tubular mappings, elliptic regularity results and the definition of quasi boundary triples. Furthermore it contains Section 1.12 about a representation theorem that will be one of the main tools throughout the work.

Ultimately we are always interested in a closed realisation of the Schrödinger operator $A = -\Delta + q$ in $L^2(\Omega)$ with non-empty resolvent set and dense domain in $L^2(\Omega)$. We will start by viewing $-\Delta + q$ as a bounded operator $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with bounded inverse $\hat{A}^{-1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces such that there exist continuous embedding of $\mathcal{H}_1$ into $L^2(\Omega)$ and $L^2(\Omega)$ into $\mathcal{H}_2$ with dense range. The operator A will then be the restriction of $\hat{A}$ to its maximal domain in $L^2(\Omega)$, i.e.

$$\mathrm{Dom}(A) = \{f \in \mathcal{H}_1 : \hat{A}f \in L^2(\Omega)\}. \tag{0.1}$$

Theorem 1.12.1 states that in this setting A fulfils the desired properties mentioned above.

The operator $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ will be defined via a bounded sesquilinear form

$$\mathbf{a} : \mathcal{G} \times \mathcal{K} \to \mathbb{C}, \quad \mathbf{a}(u, v) = \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega qu\overline{v}\mathrm{d}x, \quad u \in \mathcal{G},\, v \in \mathcal{K}, \tag{0.2}$$

as

$$\hat{A} : \mathcal{G} \to \mathcal{K}^*, \quad \hat{A}f = \mathbf{a}(f, \cdot). \tag{0.3}$$

One of the challenges is to choose the form domains $\mathcal{G}$ and $\mathcal{K}$ in a way such that the corresponding operator $\hat{A}$ is bounded and boundedly invertible. Lemma 1.12.5 shows that this operator is well-defined and bounded if the sesquilinear form $\mathbf{a}$ is bounded. In this case the bijectivity of the operator $\hat{A}$ can be characterised by the conditions

(rf1)

$$\exists \alpha > 0 \quad \text{s.t.} \quad \inf_{0 \neq u \in \mathcal{G}} \sup_{0 \neq v \in \mathcal{K}} \frac{|\mathbf{a}(u, v)|}{\|u\|_\mathcal{G} \|v\|_\mathcal{K}} \geq \alpha, \tag{0.4}$$

(rf2)

$$(\mathbf{a}(u, v) = 0 \quad \forall u \in \mathcal{G}) \quad \Rightarrow \quad v = 0, \tag{0.5}$$

where (rf1) holds exaclty if $\hat{A}$ is injective and has closed range, and (rf2) holds if and only if $\hat{A}$ has dense range in $\mathcal{H}_2 = \mathcal{K}^*$, see Corollary 1.12.6. The boundedness of $\hat{A}^{-1}$ follows also from (rf1).

From Lemma 1.12.8 we see that a sesquilinear form with symmetric form domain $\mathcal{G} = \mathcal{K}$ satisfies (rf1) and (rf2) if it is coercive, i.e. if there exists $c > 0$ such that

$$|\mathbf{a}(u, u)| \geq c\|u\|_\mathcal{G}^2, \quad u \in \mathcal{G}. \tag{0.6}$$

Another result for symmetric form domains is Lemma 1.12.10 which states that (rf1) and (rf2) are satisfied if the form is Almog-Helffer coervice, i.e. if there exist bounded operators $\Theta_1, \Theta_2 : \mathcal{G} \to \mathcal{G}$ such that

(AH1)

$$\exists \alpha_1 > 0 \quad \text{s.t.}\ |\mathbf{a}(u, u)| + |\mathbf{a}(u, \Theta_1 u)| \geq \alpha_1 \|u\|_\mathcal{G}^2, \quad u \in \mathcal{G}, \tag{0.7}$$

(AH2)

$$\exists \alpha_2 > 0 \quad \text{s.t.}\ |\mathbf{a}(v, v)| + |\mathbf{a}(\Theta_2 v, v)| \geq \alpha_2 \|v\|_\mathcal{G}^2, \quad v \in \mathcal{G}. \tag{0.8}$$

Sectorial and Almog-Helffer coercive Operators

In Chapter 2 we consider Schrödinger operators on minimally smooth domains with potentials $q \in L^1_{\text{loc}}(\overline{\Omega})$ and $\operatorname{Re} q \geq 1$ a.e. which satisfy one of the following assumptions:

(SEC) q is sectorial, i.e. $\operatorname{Re} q \geq 1$ a.e. and there exists $\theta \in [0, \frac{\pi}{2})$ such that

$$|\operatorname{Im} q| \leq \tan(\theta) \operatorname{Re} q \text{ a.e.}. \tag{0.9}$$

(AHc2) $q \in W^{1,\infty}_{\text{loc}}(\Omega)$ and there exists $C > 0$ such that

$$|\nabla \operatorname{Im} q| \leq C(1 + |\operatorname{Im} q|^2)^{\frac{3}{2}}(1 + |q|)^{\frac{1}{2}}. \tag{0.10}$$

In Assumptions 2.1.1 there is also a third condition (AHc1) given, though it is more technical and holds true whenever (AHc2) does, see Lemma 2.1.2. While (SEC) requires that the imaginary part of the potential is controlled by its imaginary part, (AHc2) even allows for purely imaginary potentials and only states that the gradient of the potential has to be controlled.

In this setting we want to consider the space

$$\mathcal{G} = \mathcal{K} = \mathcal{V} = (H^1(\Omega) \cap \operatorname{Dom}|q|^{\frac{1}{2}}, \|\cdot\|^2_{\mathcal{V}} = \|\cdot\|^2_{H^1} + \||q|^{\frac{1}{2}} \cdot\|^2_{L^2}), \tag{0.11}$$

and work with the operator $\hat{A} : \mathcal{V} \to \mathcal{V}^*$ corresponding to the form

$$\mathbf{a}(u, v) = \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega q u \overline{v} \mathrm{d}x, \quad u, v \in \mathcal{V}. \tag{0.12}$$

If the potential q satifies (SEC) it can be easily shown that $\mathbf{a}$ is coercive, see Lemma 2.5.1. In Lemma 2.6.1 we show that $\mathbf{a}$ is Almog-Helffer coercive if it satisfies (AHc2).

Either way if the assumptions stated above are satisfied we see that $\hat{A}$ is bounded with bounded inverse. This allows us to use the representation theorem to obtain the L^2 realisation of $\hat{A}$, which is a closed operator with non-empty resolvent set and, after we defined a suitable notion of a Neumann trace, turns out to be the Neumann realisation of the Schrödinger operator, see Theorem 2.7.2.

Accretive Schrödinger Operators

In Chapter 4 we will focus on Schrödinger operators on $\mathcal{C}^\infty$-domains with a potential $q \in L^p_{\text{loc}}(\overline{\Omega})$, $q \geq 1$ a.e. and

$$p \begin{cases} > 1 & \text{if } d = 2, \\ > \frac{2d}{d+2} & \text{if } d \geq 3. \end{cases} \tag{0.13}$$

In order to obtain a Neumann realisation of the Schrödinger operator in L^2 we will again work with the representation theorem, but in this setting it is not that obvious to find a suitable form domain. We will eventually work with an operator $\hat{A}$ from $\mathcal{G}$ to $\mathcal{K}^* = (H^1(\Omega))^*$ with $\mathcal{K} = H^1(\Omega)$, where $\mathcal{G} \subset H^1(\Omega)$ is only implicitly given, for details see (4.40).

Once we have defined a Neumann trace and shown that this operator is boundedly invertible, we see again that the operator from the representation theorem is in fact the Neumann realisation of the Schrödinger operator.

However, showing that the operator $\hat{A}$ is indeed boundedly invertible is challenging. It is straightforward to show that the corresponding form satisfies (rf1), but in order to show that also (rf2) we need a suitable version of Kato's inequality, which we discuss in Chapter 3.

Kato's inequality for Neumann boundary conditions

The standard version of Kato's inequality as introduced in [21] states that for an open, bounded, non-empty domain Ω and $u \in L^1_{\text{loc}}(\Omega)$ with $\Delta u \in L^1_{\text{loc}}(\Omega)$ it holds true that

$$\int_\Omega |u|\Delta\phi \mathrm{d}x \geq \int_\Omega \mathrm{Re}\left(\mathrm{sgn}(\overline{u})\Delta u\right)\phi \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega),\ \phi \geq 0, \tag{0.14}$$

where

$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{0.15}$$

Note that this version only tests with functions from $\mathcal{D}(\Omega)$, which vanish near the boundary. Thus it is not suitable if we want to use it in Chapter 4 for functions with non-zero Dirichlet trace. Consequently we need Theorem 3.2.2, which states, to the best of the author's knowledge, a new inequality which applies also to the Neumann case. In Chapter 3 we proof this theorem, which is one of the main results of this thesis.

Theorem A. *Let Ω be a $\mathcal{C}^\infty$-domain, $u \in H^1(\Omega)$ with $\Delta u \in L^p_{\text{loc}}(\overline{\Omega})$ for some $p > 1$, and assume that*

$$\langle \nabla u, \nabla \phi \rangle_{H^1} + \int_\Omega \Delta u \overline{\phi} \mathrm{d}x = 0, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega. \tag{0.16}$$

Then it holds true that

$$-\int_\Omega \nabla|u|\nabla\phi \mathrm{d}x \geq \int_\Omega \mathrm{Re}\left[\mathrm{sgn}(\overline{u})\Delta u\right]\phi \mathrm{d}x, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi \geq 0. \tag{0.17}$$

Note that in contrast to the standard version of the Kato inequality we need a C^∞-boundary, local integrability up to the boundary and $p > 1$. The former arises due to the use of tubular mappings in the C^∞-setting of differential geometry, the later is a consequence of the used elliptic regularity results.

Most steps of the proof are rather similar to a proof of the standard version. However, the main technical difficulty is to find for any given bounded subset G of Ω a sequence of smooth functions with vanishing normal derivatives such that they converge to u in $H^1(G)$ and their Laplacians converge to Δu in $L^p(G)$. This was obvious if we needed Kato's inequality only for test functions from $\mathcal{D}(\Omega)$ since their support is compact in Ω and thus mollification in a neighbourhood of this support gives the desired functions. In our setting we first have to extend the function u locally to a small tubular neighbourhood of the boundary. Here we require the previously mentioned elliptic regularity result in order to guarantee that the Laplacian of the extension is still in $L^p_{\mathrm{loc}}(\overline{\Omega})$. Next one flattens the boundary locally via the tubular mapping, which requires the C^∞-regularity of the boundary, and mollifies, which leads to a vanishing normal derivative and smooth local approximations of u. In the end we can put all the local pieces together with a suitably constructed partition of unity and find the desired sequence.

Traces

Another challenge for the construction of quasi boundary triples is to find suitable traces. For accretive operators such as in Chapter 4 it is sufficient to consider the standard definition of the Dirichlet trace γ_D with its bounded right-inverse $\mathcal{E}$, and define the Neumann trace $\gamma_N : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ via

$$(\gamma_N u, \psi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} = (\hat{A}u, \mathcal{E}\psi)_{(H^1)^* \times H^1} - \langle(-\Delta + q)u, \mathcal{E}\psi\rangle_{L^2}, \quad \psi \in H^{\frac{1}{2}}(\Omega), \tag{0.18}$$

on the appropriate domain. For $q = 0$ this reduces to the standard definition of the Neumann trace, and in any case this notion of a Neumann trace is consistent with the standard definition on the overlap of their domains, see Lemma 4.6.4. Furthermore in Lemma 4.7.1 we show that this Neumann trace is surjective.

For operators as in Chapter 2 the situation is a bit more complicated since the operator $\hat{A}$ maps $\mathcal{V}$ into $\mathcal{V}^*$. Consequently we want a Dirichlet trace which is surjective on $\mathcal{V}$ and has a bounded right-inverse. This motivates the construction of a boundary space

$$\mathcal{W} = \mathrm{Ran}\gamma_D \restriction_{\mathcal{V}}, \tag{0.19}$$

with an appropriate norm which makes it a Hilbert space and the right-inverse $\mathcal{E}^{\mathcal{V}}$ bounded. The corresponding Dirichlet trace will be called $\gamma_D^{\mathcal{V}}$.

Using this, we define the Neumann trace $\gamma_N^q : \mathcal{W} \to \mathcal{W}^*$ in a similar way as in (0.18) via

$$(\gamma_N^q u, \psi)_{\mathcal{W}^* \times \mathcal{W}} = (\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^* \times \mathcal{V}} - \langle(-\Delta + q)u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}, \quad \psi \in \mathcal{W}. \tag{0.20}$$

This notion is again in a suitable sense consistent with the standard definition and the definition above, see Lemma 2.4.3 and Lemma 4.6.5. Additionally we show in Lemma 2.8.1 that this Neumann trace is also surjective.

Final Results

Once we have obtained all the results described above, we find that in all considered cases we have closed Neumann realisations of the Schrödinger operators in L^2 with non-empty resolvent sets, this is stated in Theorem 2.7.2 for the sectorial and Almog-Helffer coercive potentials and in Theorem 4.8.1 for the accretive potentials.

We use them to construct the desired quasi boundary triples, see Section 2.9 for the sectorial and Almog-Helffer coercive potentials and Section 4.9 for the accretive potentials.

1 Preliminaries

1.1 Notations

In this first section we introduce some notations used throughout this thesis.

For $\epsilon > 0$ and $x \in \mathbb{R}^d$ we denote the open ϵ-ball around x as

$$\mathbb{B}_\epsilon(x) = \{y \in \mathbb{R}^d : |x - y| < \epsilon\}. \tag{1.1}$$

For a multi-index $\alpha \in \mathbb{N}_0^d$ we denote its length by

$$|\alpha| = \sum_{i=1}^{d} \alpha_i. \tag{1.2}$$

For $\alpha, \beta \in \mathbb{N}_0^d$ we write $\beta \leq \alpha$, if

$$\beta_i \leq \alpha_i, \quad i \in \{1, ..., d\}. \tag{1.3}$$

Furthermore we define the generalised binomial coefficient as

$$\binom{\alpha}{\beta} = \prod_{i=1}^{d} \binom{\alpha_i}{\beta_i}, \quad \alpha, \beta \in \mathbb{N}_0^d,\ \beta \leq \alpha. \tag{1.4}$$

Let $\Omega \subset \mathbb{R}^d$ be open and non-empty. If $u \in C^k(\Omega)$, i.e. k-times continuously differentiable, and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, then we denote the partial derivative of order α as

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}}. \tag{1.5}$$

Let $K \subset \mathbb{R}^d, \Omega \subset \mathbb{R}^d$ be open and non-empty. Then we write $K \Subset \Omega$ if $\overline{K}$ is compact and $\overline{K} \subset \Omega$.

For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we write its orthogonal sum $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = \{(u, v) : u \in \mathcal{H}_1,\ v \in \mathcal{H}_2\}, \tag{1.6}$$

equipped with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{H}} = \langle u_1, u_2, \rangle_{\mathcal{H}_1} + \langle v_1, v_2 \rangle_{\mathcal{H}_2}, \quad u_1, u_2 \in \mathcal{H}_1,\ v_1, v_2 \in \mathcal{H}_2. \tag{1.7}$$

For a mapping T between non-empty sets X and Y we denote its domain by $\mathrm{Dom}T$ and its range by

$$\mathrm{Ran}T = \{y \in Y : \exists x \in \mathrm{Dom}T \text{ s.t. } Tx = y\}. \tag{1.8}$$

Restrictions of a mapping to a subdomain $W \subset \mathrm{Dom}T$ are denoted by $T\restriction_W$.

If T is a linear mapping between normed spaces X and Y we define the kernel of T as

$$\ker T = \{x \in \mathrm{Dom}T : Tx = 0\}. \tag{1.9}$$

If T is a closed and densely defined linear operator in a Hilbert space $\mathcal{H}$ we denote its resolvent set as

$$\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda) \text{ is bijective and boundedly invertible}\}. \tag{1.10}$$

1.2 Weak Derivatives and Sobolev spaces

The aim of this subsection is to define weak derivatives and Sobolev spaces. To this end we introduce the space of test functions and its dual space, the distributions.

Definition 1.2.1. *Let $\Omega \in \mathbb{R}^d$ be an open and non-empty set. The test functions on Ω are the set of infinitely often differentiable functions with compact support in Ω, i.e.*

$$C_0^\infty(\Omega) = \{\phi \in C^\infty(\Omega) : \operatorname{supp}\phi \Subset \Omega\}. \tag{1.11}$$

If we view $C_0^\infty(\Omega)$ as a vector space we will use the notation $\mathcal{D}(\Omega)$ and say that a sequence $(\phi_n)_n \subset \mathcal{D}(\Omega)$ is a zero sequence in $\mathcal{D}(\Omega)$ if there exists $K \Subset \Omega$ such that

$$\operatorname{supp}\phi_n \subset K, \quad n \in \mathbb{N}, \tag{1.12}$$

and

$$\sup_{x\in\Omega} |\partial^\alpha \phi_n(x)| \xrightarrow{n\to\infty} 0, \quad \alpha \in \mathbb{N}_0^d. \tag{1.13}$$

Sequentially continuous and anti-linear functionals on $\mathcal{D}(\Omega)$ are called distributions, and we denote the space of distributions as $\mathcal{D}'(\Omega)$.

A special type of distributions are locally integrable functions.

Definition 1.2.2. *Let $\Omega \in \mathbb{R}^d$ be an open, non-empty set and $u : \Omega \to \mathbb{C}$ be measurable. We say that $u \in L^p_{\mathrm{loc}}(\Omega)$ for $p \in [1, \infty]$ if for every $K \Subset \Omega$ it holds that*

$$\|u\|_{L^p(K)} < \infty. \tag{1.14}$$

We say that $u \in L^p_{\mathrm{loc}}(\overline{\Omega})$ if the same holds for every $K \Subset \overline{\Omega}$.

Lemma 1.2.3. *Let $\Omega \in \mathbb{R}^d$ be an open, non-empty set and $u \in L^1_{\text{loc}}(\Omega)$. Then the functional $T_u : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by*

$$T_u(\phi) = \int_\Omega u\overline{\phi}\mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega), \tag{1.15}$$

is a distribution.

Proof. The anti-linearity is clear from the linearity of the integral.

Now let $(\phi_n)_n \subset \mathcal{D}(\Omega)$ be a zero-sequence. Then there exists $K \Subset \Omega$ such that $\operatorname{supp}\phi_n \subset K$ for all $n \in \mathbb{N}$, and consequently

$$T_u(\phi_n) = \int_K u\overline{\phi_n}\mathrm{d}x \leq \sup_{x\in\Omega} |\phi_n(x)| \int_K \mathrm{d}x \xrightarrow{n\to\infty} 0, \tag{1.16}$$

since K is by assumption bounded. This shows that T_u is sequentially continuous and thus $T_u \in \mathcal{D}'(\Omega)$. □

Remark 1.2.4. *In the setting of Lemma 1.2.3 we will just write $u \in \mathcal{D}'(\Omega)$ instead of T_u for the distribution defined in (1.15).*

We will now define distributional and weak derivatives.

Definition 1.2.5. *Let $\Omega \in \mathbb{R}^d$ be an open, non-empty set, $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. We say that $S \in \mathcal{D}'(\Omega)$ is the distributional derivative of order α of T if*

$$(S, \phi)_{\mathcal{D}'\times\mathcal{D}} = (-1)^{|\alpha|}(T, \partial^\alpha\phi)_{\mathcal{D}'\times\mathcal{D}}, \quad \phi \in \mathcal{D}(\Omega). \tag{1.17}$$

In this case we write $S = \partial^\alpha T$.

If $u \in L^1_{\text{loc}}(\Omega)$ is a regular distribution and it holds that also $\partial^\alpha u \in L^1_{\text{loc}}(\Omega)$, then we call $\partial^\alpha u$ the weak derivative of order α of u.

The following Lemma shows that distributions are infinitely often differentiable in the sense of Definition 1.2.5.

Lemma 1.2.6. *Let $\Omega \in \mathbb{R}^d$ be an open, non-empty set, $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. Then there exists a unique distribution $S \in \mathcal{D}'(\Omega)$ such that $S = \partial^\alpha T$.*

Proof. For the existence, lets define

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|}(T, \partial^\alpha\phi)_{\mathcal{D}'\times\mathcal{D}}, \quad \phi \in \mathcal{D}(\Omega). \tag{1.18}$$

It is clear that this is a anti-linear functional, what remains is to show the sequential continuity.

To this end, let $(\phi_n)_n \subset \mathcal{D}(\Omega)$ be a zero-sequence. Then there exists a $K \Subset \Omega$ such that

$$\operatorname{supp}\phi_n \subset K, \quad n \in \mathbb{N}, \tag{1.19}$$

and consequently we see that

$$\operatorname{supp} \partial^\alpha \phi_n \subset \operatorname{supp} \phi_n \subset K, \quad n \in \mathbb{N}. \tag{1.20}$$

Additionally we can use Schwarz's theorem for smooth functions to see that for any $\beta \in \mathbb{N}_0^d$

$$\sup_{x\in\Omega} |\partial^\beta(\partial^\alpha \phi_n(x))| = \sup_{x\in\Omega} |\partial^{\beta+\alpha}\phi_n(x)| \xrightarrow{n\to\infty} 0. \tag{1.21}$$

So $(\partial^\alpha \phi_n)_n \subset \mathcal{D}(\Omega)$ is also a zero-sequence and consequently the sequential continuity of $T \in \mathcal{D}'(\Omega)$ gives us that

$$\lim_{n\to\infty} \partial^\alpha T(\phi_n) = (-1)^{|\alpha|} \lim_{n\to\infty} (T, \partial^\alpha \phi_n)_{\mathcal{D}'\times\mathcal{D}} = 0, \tag{1.22}$$

which shows that $\partial^\alpha T \in \mathcal{D}'(\Omega)$.

The uniqueness is clear, since Definition 1.2.5 gives an explicit formula for distributional derivatives. □

Remark 1.2.7. *Since we can interpret every function $u \in L^1_{\text{loc}}(\Omega)$ as a distribution in the sense of (1.15), we will from now on assume that derivatives $\partial^\alpha u$ are meant in the distributional sense, which always exists due to Lemma 1.2.6, and explicitly state whenever they have higher regularity.*

For sufficiently regular functions the distributional derivatives coincide with the classical derivative, which is a simple consequence of integration by parts.

After these preparations we are now able to define Sobolev spaces of integer order.

Definition 1.2.8. *Let $\Omega \subset \mathbb{R}^d$ be open and non-empty, $m \in \mathbb{N}$ and $p \in [1,\infty]$. The Sobolev space $W^{m,p}(\Omega)$ is given by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega),\, |\alpha| \le m\}, \tag{1.23}$$

equipped with the norm

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}}, \quad u \in W^{m,p}(\Omega). \tag{1.24}$$

If $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$, and define a inner product via

$$\langle u, v\rangle_{H^m} = \sum_{|\alpha|\le m} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2}, \quad u, v \in H^m(\Omega). \tag{1.25}$$

Furthermore we can also define the local Sobolev spaces

$$W^{m,p}_{\text{loc}}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) : u\restriction_K \in W^{m,p}(K)\, \forall K \Subset \Omega\}. \tag{1.26}$$

For the definition of Sobolev spaces of fractional order we will consider the Slobodeckij-seminorm for $r \in (0,1)$ and $p \in [1,\infty)$, which is given by

$$|u|_{r,p,\Omega} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+pr}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}. \tag{1.27}$$

Definition 1.2.9. *Let $\Omega \subset \mathbb{R}^d$ be open and non-empty, $m \in \mathbb{N}$, $r \in (0,1), s = m + r$ and $p \in [1,\infty)$. The Sobolev space $W^{s,p}(\Omega)$ is given by*

$$W^{s,p}(\Omega) = \{u \in W^{m,p}(\Omega) : |\partial^\alpha u|_{r,p,\Omega} < \infty, \, |\alpha| = m\}, \tag{1.28}$$

with the norm

$$\|u\|_{W^{s,p}} = \left(\|u\|^p_{W^{m,p}} + \sum_{|\alpha|=m} |\partial^\alpha u|^p_{r,p,\Omega} \right)^{\frac{1}{p}}. \tag{1.29}$$

For $p = 2$ we again write $W^{s,p}(\Omega) = H^s(\Omega)$, and the norm is induced by

$$\langle u, v\rangle_{H^s} = \langle u, v\rangle_{H^m} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{\overline{[\partial^\alpha u(x) - \partial^\alpha u(y)]}\,[\partial^\alpha v(x) - \partial^\alpha v(y)]}{|x-y|^{d+2r}} \mathrm{d}x\mathrm{d}y. \tag{1.30}$$

The next Lemma states that multiplication with test functions does not change the regularity of a Sobolev function, which will be a very useful result later on.

Lemma 1.2.10. *Let $\Omega \subset \mathbb{R}^d$ be open and non-empty, $\phi \in C_0^\infty(\mathbb{R}^d)$ and $u \in H^s(\Omega)$. Then it holds true that $\phi u \in H^s(\Omega)$, and for every $r \in \mathbb{N}$ with $|s| \le r$ there exists $C_r > 0$ such that*

$$\|\phi u\|_{H^s} \le C_r \|\phi\|_{W^{r,\infty}} \|u\|_{H^s}, \tag{1.31}$$

and it holds true that

$$\partial^\alpha(\phi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta \phi \partial^{\alpha-\beta} u, \quad |\alpha| \le s. \tag{1.32}$$

Proof. See [25, Theorem 3.20] and [13, Chapter 5, Theorem 1]. □

The following result gives an insight of how weak and classical derivatives are connected.

Theorem 1.2.11. *Let $\Omega \subset \mathbb{R}^d$ be open and non-empty and let $u : \Omega \to \mathbb{C}$ be Lipschitz continuous. Then u is almost everywhere differentiable.*

If u is additionally bounded, then $u \in W^{1,\infty}(\Omega)$, and the weak gradient coincides with the classical gradient almost everywhere, in particular we have that for the weak gradient

$$\|\nabla u\|_{L^\infty} \le L, \tag{1.33}$$

where $L > 0$ denotes the Lipschitz constant of u.

Proof. See [14, Theorem 3.2 and Theorem 4.5]. □

1.3 Mollification

Mollification is a straightforward way to approximate functions in Sobolev spaces by smooth functions. We will give a definition and summarise properties in this section. Let us first define the family of standard mollifiers.

Definition 1.3.1. *The standard mollifier is a function* $\omega : \mathbb{R}^d \to \mathbb{R}$ *given by*

$$\omega(x) = \begin{cases} Ce^{-\frac{1}{1-|x|^2}} & \textit{for } |x| < 1, \\ 0 & \textit{else,} \end{cases} \tag{1.34}$$

where $C > 0$ *is chosen such that*

$$\int_{\mathbb{R}^d} \omega(x) = 1. \tag{1.35}$$

The family of standard mollifiers are the functions $(\omega_\rho)_{\rho>0}$*, where*

$$\omega_\rho(x) = \tfrac{1}{\rho^d}\omega(\tfrac{x}{\rho}). \tag{1.36}$$

Lemma 1.3.2. *The standard mollifiers* $(\omega_\rho)_{\rho>0}$ *satisfy the following:*

i) $\omega_\rho \in C_0^\infty(\mathbb{R}^d)$,

ii) $\int_{\mathbb{R}^d} \omega_\rho(x)\mathrm{d}x = 1$,

iii) $\operatorname{supp}(\omega_\rho) = \overline{\mathbb{B}}_\rho(0)$.

Proof. See [12, Equation (1.1) in Chapter V and remarks afterwards] for property i). Property ii) follows simply from change of variables, and property iii) is trivial. □

Theorem 1.3.3. *Let* $\Omega \subset \mathbb{R}^d$ *be open and non-empty,* $u \in L^p(\Omega)$ *for* $p \in [1, \infty)$ *and* $\tilde{u}$ *its zero extension to all of* $\mathbb{R}^d$*. Then it holds that* $\omega_\rho * \tilde{u} \in C^\infty(\mathbb{R}^d)$ *and*

$$\|\omega_\rho * \tilde{u} - u\|_{L^p(\Omega)} \xrightarrow{\rho\to 0} 0. \tag{1.37}$$

Proof. See [12, Theorem V.1.5 and remarks before that]. □

Remark 1.3.4. *Note that Theorem 1.3.3 does not hold for* $p = \infty$*, as continuous functions can only converge uniformly towards a continuous function, but we do get convergence on compact subsets of* Ω *if* u *is also continuous, see [12, Lemma V.1.4].*

Theorem 1.3.5. *Let* $\Omega \subset \mathbb{R}^d$ *be open and non-empty,* $u \in L^1_{\mathrm{loc}}(\Omega)$ *and* $\partial^\alpha u L^1_{\mathrm{loc}}(\Omega)$ *for some multi-index* $\alpha \in \mathbb{N}_0^d$*. Then it holds that*

$$\omega_\rho * (\partial^\alpha u)(x) = \partial^\alpha(\omega_\rho * u)(x), \quad x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \rho. \tag{1.38}$$

Proof. See [12, Theorem V.2.2]. □

1.4 Sobolev spaces on the boundary

In this section we define Sobolev spaces on the boundary of a domain. To this end we will need to assume that the domain satisfies regularity conditions known as minimally smooth.

Definition 1.4.1. *We say that $\kappa : \mathbb{R}^d \to \mathbb{R}^d$ is a rigid body motion if there exists an orthogonal matrix $R \in \mathbb{R}^{d\times d}$ (i.e. $R^T R = RR^T = I$), and a vector $t \in \mathbb{R}^d$ such that*

$$\kappa(x) = Rx + t, \quad x \in \mathbb{R}^d. \tag{1.39}$$

Definition 1.4.2. *An open, non-empty set $\Omega \subset \mathbb{R}^d$ is called a special Lipschitz domain if there exists a rigid body motion $\kappa : \mathbb{R}^d \to \mathbb{R}^d$ and a Lipschitz continuous function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that*

$$\Omega = \{x \in \mathbb{R}^d : \, (\kappa(x))_d > \xi\left((\kappa(x))_1, ..., (\kappa(x))_{d-1}\right)\}. \tag{1.40}$$

We say that an open, non-empty set $\Omega \subset \mathbb{R}^d$ is minimally smooth if there exist $\epsilon > 0, M > 0, N \in \mathbb{N}$ and a countable family of open sets $(U_j)_{j\in J}$ such that the following assertions holds true:

i) If $x \in \partial\Omega$, then $\mathbb{B}_\epsilon(x) \in U_j$ for some index $j \in J$.

ii) At most N sets in $(U_j)_{j\in J}$ have non-empty intersection.

iii) For each index $j \in J$ there exists a special Lipschitz domain Ω_j defined by a function ξ_j and rigid body motion κ_j such that

$$U_j \cap \Omega = U_j \cap \Omega_j \tag{1.41}$$

and

$$\|\nabla \xi_j\|_{L^\infty(\mathbb{R}^{d-1})} \le M. \tag{1.42}$$

If all the functions ξ_j in this definition are in $C^k(\mathbb{R}^{d-1})$, we say that Ω is a C^k- domain.

We call the set $((U_j, \xi_j, \kappa_j))_{j\in J}$ a Lipschitz atlas for Ω.

Lemma 1.4.3. *Let $\Omega \subset \mathbb{R}^d$ be a minimally smooth domain. Then there exists a countable family of open sets $(U_j)_{j\in J}$ which satisfy i)-iii) in Definition 1.4.2 such that for every $j \in J$ there exist $x_{j,1}, x_{j,2} \in \mathbb{R}^d$ with*

$$\mathbb{B}_\epsilon(x_{j,1}) \subset U_j \subset \mathbb{B}_{2\epsilon}(x_{j,2}), \tag{1.43}$$

where $\epsilon > 0$ is as in Definition 1.4.2.

Proof. Since Ω is minimally smooth there exists countable family $(V_i)_{i\in I}$ which satisfies i)-iii) in Definition 1.4.2. We define balls

$$B_{i_1,...,i_d} = \mathbb{B}_{2\epsilon}\left(\left(i_1\frac{\epsilon}{\sqrt{d}}, ..., i_d\frac{\epsilon}{\sqrt{d}}\right)\right) \subset \mathbb{R}^d, \quad i_1, ..., i_d \in \mathbb{Z}, \tag{1.44}$$

and the sets

$$U_{i,i_1,...,i_d} = V_i \cap B_{i_1,...,i_d}, \quad i \in I, i_1, ..., i_d \in \mathbb{Z}. \tag{1.45}$$

$(U_j)_{j\in I\times\mathbb{Z}^d}$ is again a countable family of open sets. We want to show that these sets satisfy the desired properties.

Since $(V_i)_{i\in I}$ satisfies condition iii) in Definition 1.4.2 we see that for each $(i, i_1, ..., i_d) \in I \times \mathbb{Z}^d$ it holds true that

$$\begin{aligned} U_{i,i_1,...,i_d} \cap \Omega = (V_i \cap B_{i_1,...,i_d}) \cap \Omega = (V_i \cap \Omega) \cap B_{i_1,...,i_d} = (V_i \cap \Omega_i) \cap B_{i_1,...,i_d} \\ = (V_i \cap B_{i_1,...,i_d}) \cap \Omega_i = U_{i,i_1,...,i_d} \cap \Omega_i, \end{aligned} \tag{1.46}$$

where Ω_i is a special Lipschitz domain defined by a rigid body motion κ_i and a function ξ_i which satisfies

$$\|\nabla\xi_j\|_{L^\infty(\mathbb{R}^{d-1})} < M, \tag{1.47}$$

so we see that $(U_j)_{j\in I\times\mathbb{Z}^d}$ satisfies property iii) in Definition 1.4.2.

The balls $(B_{i_1,...,i_d})_{i_1,...,i_d\in\mathbb{Z}}$ all have the same size and the centres are uniformly distributed in $\mathbb{R}^d$, therefore there exists a maximum number $K \in \mathbb{N}$ of balls which can have non-empty intersection.

Now let $x \in \mathbb{R}^d$. Since $(V_i)_{i\in I}$ satisfies condition ii) in Definition 1.4.2 $x \in V_i$ can hold for at most N indices $i \in I$, w.l.o.g. assume there are N such indices and call them $b_1, .., b_N \in I$.

By the same argument, x is contained in at most K balls, w.l.o.g. assume that there are K such sets and we call their indices $c_1, ..., c_K \in \mathbb{Z}^d$, and so we can conclude that

$$x \in U_{i,i_1,...,i_d} = V_i \cap B_{i_1,...,i_d} \quad \Leftrightarrow \quad i \in \{b_1, ..., b_N\}, (i_1, ..., i_d) \in \{c_1, ..., c_K\}, \tag{1.48}$$

which shows that x is contained in NK sets from $(U_j)_{j\in I\times\mathbb{Z}^d}$.

Since $x \in \mathbb{R}^d$ was arbitrary we see that at most NK of the sets $(U_j)_{j\in I\times\mathbb{Z}^d}$ can have non-empty intersection, so they also satisfy ii) in Definition 1.4.2.

Next we want to show that for an arbitrary point $x \in \mathbb{R}^d$, the distance to the nearest center of one of the balls $(B_{i_1,...,i_d})_{i_1,...,i_d\in\mathbb{Z}}$ is less or equal than $\frac{\epsilon}{2}$.

Let $x \in \mathbb{R}^d$. Clearly there exist numbers $i_1, ..., i_d \in \mathbb{Z}$ such that

$$\left|i_k\frac{\epsilon}{\sqrt{d}} - x_k\right| \leq \frac{\epsilon}{2\sqrt{d}}, \quad k \in \{1, ..., d\}, \tag{1.49}$$

and thus

$$\left|\frac{\epsilon}{\sqrt{d}}(i_1, ..., i_d) - x\right|^2 \leq d\frac{\epsilon^2}{4d} = \frac{\epsilon^2}{4}, \tag{1.50}$$

which shows that the distance of x to the center of $B_{i_1,\ldots,i_d}$ is at most $\frac{\epsilon}{2}$.

Consequently, since all the balls $(B_{i_1,\ldots,i_d})_{i_1,\ldots,i_d\in\mathbb{Z}}$ have radius 2ϵ, we see that for every $x \in \mathbb{R}^d$ we find at least one ball $B_{i_1,\ldots,i_d}$ such that $\mathbb{B}_\epsilon(x) \subset B_{i_1,\ldots,i_d}$. Now if $x \in \partial\Omega$, one can choose V_i such that $\mathbb{B}_\epsilon(x) \subset V_i$, since $(V_i)_{i\in I}$ satisfies i) in Definition 1.4.2, and consequently

$$\mathbb{B}_\epsilon(x) \in V_i \cap B_{i_1,\ldots,i_d} = U_{i,i_1,\ldots,i_d}, \tag{1.51}$$

so we see that also $(U_j)_{j\in I\times\mathbb{Z}^d}$ satisfies i) in Definition 1.4.2.

Now we have seen that $(U_j)_{j\in I\times\mathbb{Z}^d}$ satisfies all the assumptions in Definition 1.4.2, and what remains to show is that (1.43) is satisfied.

It is clear by construction that all the sets $(U_j)_{j\in I\times\mathbb{Z}^d}$ are contained in some ball with radius 2ϵ. In order to ensure that each set also contains some ϵ-ball, we can simply consider only the subset $(U_j)_{j\in J}$ of $(U_j)_{j\in I\times\mathbb{Z}^d}$ of sets who do contain an ϵ-ball and leave all the others out.

Note that $(U_j)_{j\in J}$ does still satisfy all the properties from Definition 1.4.2, since $(U_j)_{j\in I\times\mathbb{Z}^d}$ does: sets which to not contain any ϵ-ball cannot help to satisfy condition i), so $(U_j)_{j\in J}$ must also satisfy i), the number of maximum intersection in a single point can only decrease if we leave out sets, so $(U_j)_{j\in J}$ satisfies ii), and finally iii) is a condition on the individual sets, and $(U_j)_{j\in J}$ is a subset of $(U_j)_{j\in I\times\mathbb{Z}^d}$, it also satisfies iii). □

Definition 1.4.4. *We call Lipschitz atlases $((U_j, \xi_j, \kappa_j))_{j\in J}$ for a minimally smooth domain Ω, where the $(U_j)_{j\in J}$ are as in Lemma 1.43, regular.*

In the following, we introduce fractional Sobolev spaces with exponent $p = 2$ on the boundary of minimally smooth domains. First let us consider the simpler case of a special Lipschitz domain $\Omega \subset \mathbb{R}^{d-1}$ with Lipschitz function ξ and rigid body motion $\kappa : \mathbb{R}^d \to \mathbb{R}^d$. We define the corresponding boundary measure and unit outwards vector as

$$\begin{aligned} \mathrm{d}\sigma(x') &= \sqrt{1+|\nabla\xi(x')|^2}\mathrm{d}x', \quad x' \in \mathbb{R}^{d-1}, \\ \nu(x') &= \frac{1}{\sqrt{1+|\nabla\xi(x')|^2}}\begin{pmatrix} -\nabla\xi(x') \\ 1 \end{pmatrix}, \quad x' \in \mathbb{R}^{d-1}, \end{aligned} \tag{1.52}$$

which are well-defined since ξ is almost everywhere differentiable, see Theorem 1.2.11. For $u \in L^2(\partial\Omega, \sigma) = L^2(\partial\Omega)$ (the equality holds since ξ is Lipschitz and thus has bounded gradient) we define

$$u_{\partial\Omega}(x') = u(\kappa^{-1}(x', \xi(x'))), \quad x' \in \mathbb{R}^{d-1}. \tag{1.53}$$

For $s \in [0, 1]$, we define the corresponding Sobolev space as

$$H^s(\partial\Omega) = \{u \in L^2(\partial\Omega) : u_{\partial\Omega} \in H^s(\mathbb{R}^{d-1})\}, \tag{1.54}$$

equipped with the inner product

$$\langle u, v\rangle_{H^s(\partial\Omega)} = \langle u_{\partial\Omega}, v_{\partial\Omega}\rangle_{H^s(\mathbb{R}^{d-1})}, \quad u, v \in H^s(\partial\Omega). \tag{1.55}$$

For $C^{k-1,1}$-domains this definition extends to $s \in [0, k]$, see [25, Chapter 3]. Note that the definition of the inner product depends on the Lipschitz function ξ and the rigid body motion κ used to describe Ω, which is not necessarily unique. Nevertheless, one can show the following result.

Lemma 1.4.5. *Let Ω be a special Lipschitz domain. Then the space $H^s(\partial\Omega)$ as defined in (1.54) is independent of the choice of the defining Lipschitz function and rigid body motion, and different choices lead to equivalent norms. Moreover, the constants between those norms only depend on the Lipschitz constants of the functions and the order $s \in [0, 1]$.*

Proof. [25, Theorem 3.20 and remarks in the section Sobolev spaces on the Boundary in Chapter 3]. □

Now we want to generalise this to minimally smooth domains. To that end, we will first introduce partitions of unity.

Definition 1.4.6. *Let $\Omega \subset \mathbb{R}^d$ be open and non-empty. A sequence $(\phi_i)_{i\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ is called a partition of unity for Ω if the following assertions hold true:*

1. $\phi_i \geq 0, \qquad i \in \mathbb{N},$
2. *for every $x \in \Omega$ there exists an open neighbourhood which intersects with only finitely many* $\operatorname{supp} \phi_i$,
3. $\sum_{i=1}^{\infty} \phi(x) = 1, \qquad x \in \Omega.$

We say that the partition of unity $(\phi_i)_{i\in\mathbb{N}}$ is subordinate to a given open cover $(U_\lambda)_{\lambda\in\Lambda}$ of Ω with index set Λ if for every $i \in \mathbb{N}$ there exists $\lambda \in \Lambda$ such that $\operatorname{supp} \phi_i \subset U_\lambda$.

If $S \subset \mathbb{R}^d$ is not open, we call $(\phi_i)_{i\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ a partition of unity for S, if it is a partition of unity for a open neighbourhood of S.

Lemma 1.4.7. *Let $\Omega \in \mathbb{R}^d$ be open and non-empty. For any open cover $(U_\lambda)_{\lambda\in\Lambda}$ with index set Λ there exists a partition of unity subordinate to it, with the additional property that for every $K \Subset \Omega$ there exists $m_K \in \mathbb{N}$ such that*

$$\sum_{i=1}^{m_K} \phi_i(x) = 1, \quad x \in K. \tag{1.56}$$

If the open cover is countable, w.l.o.g. $\Lambda = \mathbb{N}$, the partition of unity can be chosen such that

$$\operatorname{supp} \phi_i \subset \overline{U_i}, \quad i \in \mathbb{N}. \tag{1.57}$$

Proof. See [25, Theorem 3.21] for the existence of a partition of unity subordinate to $(U_\lambda)_{\lambda\in\Lambda}$.

Now let $K \Subset \Omega$. By definition of a partition of unity, for every $x \in K$ there exists a open neighbourhood V_x of x such that only finitely many functions of the partition of unity are supported in it. Clearly

$$\overline{K} \subset \bigcup_{x\in\overline{K}} V_x, \tag{1.58}$$

and since this is an open cover of a compact set, there exists a finite subcover

$$\overline{K} \subset \bigcup_{j=1}^{n_K} V_{x_j}. \tag{1.59}$$

Since on every V_{x_j} only finitely many ϕ_i are supported and the union is over a finite index set, we can conclude that only finitely many ϕ_i are supported on K. But since

$$\sum_{i=1}^{\infty} \phi_i(x) = 1, \quad x \in \Omega, \tag{1.60}$$

this finally shows that there exists some $m_K \in \mathbb{N}$ such that

$$\sum_{i=1}^{m_K} \phi_i(x) = 1, \quad x \in K. \tag{1.61}$$

For the statement about countable covers see [25, Corollary 3.22]. □

In order to give a good definition of our Sobolev space on the boundary for minimally smooth domains we will need a partition of unity which also has a uniform bound on the gradients. To this end we introduce uniformly subordinate partitions of unity.

Definition 1.4.8. *Let $\Omega \subset \mathbb{R}^d$ be minimally smooth with a regular Lipschitz atlas $((U_j, \xi_j, \kappa_j))_{j\in J}$. We say that a partition of unity $(\phi_j)_{j\in J}$ for $\partial\Omega$ is C^1-uniformly subordinate to the regular Lipschitz atlas if it is subordinate to $(U_j)_{j\in J}$ and there exists $C > 0$ such that*

$$\|\nabla\phi_j\|_{L^\infty} < C, \quad j \in J. \tag{1.62}$$

Lemma 1.4.9. *Let Ω be minimally smooth. Then for any regular Lipschitz atlas of Ω there exists a partition of unity which is C^1-uniformly subordinate to it.*

Proof. Let $((U_j, \xi_j, \kappa_j))_{j\in J}$ be a regular Lipschitz atlas describing Ω. We will show this result by explicitly constructing the partition of unity. Let $j \in J$. First define the function $\delta : \mathbb{R}^d \to \mathbb{R}$ via

$$\delta(x) := \operatorname{dist}(x, \mathbb{R}^d \setminus U_j), \quad x \in \mathbb{R}^d, \tag{1.63}$$

and observe that it is 1-Lipschitz, since

$$|\delta(x) - \delta(y)| = |\operatorname{dist}(x, \mathbb{R}^d \setminus U_j) - \operatorname{dist}(y, \mathbb{R}^d \setminus U_j)| \leq |x - y|. \tag{1.64}$$

Theorem 1.2.11 therefore shows that the function is first-order weakly differentiable and

$$\|\nabla\delta\|_{L^\infty} \leq 1. \tag{1.65}$$

In order to obtain a smooth approximation, consider the standard mollifiers $(\omega_\rho)_{\rho>0}$ as defined in (1.36), and define

$$\delta^{(\rho)} = \delta * \omega_\rho. \tag{1.66}$$

Lemma 1.3.2 shows that $\delta^{(\rho)} \in C^\infty(\mathbb{R}^d)$. From Theorem 1.3.5 we see that

$$|\nabla\delta^{(\rho)}(x)| \leq \int_{\mathbb{R}^d} |\nabla\delta(x-y)|\omega_\rho(y)\mathrm{d}y = 1, \quad x \in \mathbb{R}^d,\ \rho > 0, \tag{1.67}$$

and thus

$$\|\nabla\delta^{(\rho)}\|_{L^\infty} \leq 1, \quad \rho > 0. \tag{1.68}$$

Furthermore, using that δ is 1-Lipschitz, we see that for $\rho > 0$

$$\left|\delta^{(\rho)}(x) - \delta(x)\right| \leq \int_{\mathbb{R}^d} |\delta(x-y) - \delta(x)|\,\omega_\rho(y)\mathrm{d}y \leq \int_{\mathbb{R}^d} |y|\omega_\rho(y)\mathrm{d}y \leq \rho. \tag{1.69}$$

Now we pick a function $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for } x \geq \frac{\epsilon}{2}, \\ 0 & \text{for } x \leq \frac{\epsilon}{4}, \end{cases} \tag{1.70}$$

where $\epsilon > 0$ is as in Definition 1.4.4, and define a function $\hat{\phi}_j : \mathbb{R}^d \to \mathbb{R}$ via

$$\hat{\phi}_j(x) = \eta\left(\delta^{(\frac{\epsilon}{4})}(x)\right), \quad x \in \mathbb{R}^d. \tag{1.71}$$

It is clear that $\hat{\phi}_j \in C^\infty(\mathbb{R}^d)$ since it is a composition of smooth functions and by definition $0 \leq \hat{\phi}_j \leq 1$.

Moreover, if $x \notin U_j$, then we have from (1.69) that

$$\delta^{(\frac{\epsilon}{4})}(x) \leq \frac{\epsilon}{4} + |\delta(x)| = \frac{\epsilon}{4}, \tag{1.72}$$

and thus $\hat{\phi}_j(x) = 0$, which shows that $\operatorname{supp}\hat{\phi}_j \subset \overline{U_j}$ and $\hat{\phi}_j \in C_0^\infty(\mathbb{R}^d)$.

By the same argument we obtain that for $x \in U_j$ with $\operatorname{dist}(x, \mathbb{R}^d \setminus U_j) \geq \frac{3\epsilon}{4}$ we have

$$\delta^{(\frac{\epsilon}{4})}(x) \geq \delta(x) - \frac{\epsilon}{4} \geq \frac{\epsilon}{2}, \tag{1.73}$$

so we see that

$$\hat{\phi}_j(x) = \begin{cases} 1 & \text{for } \operatorname{dist}(x, \mathbb{R}^d \setminus U_j) \geq \frac{3\epsilon}{4}, \\ 0 & \text{for } x \notin U_j. \end{cases} \tag{1.74}$$

Finally we see from the chain rule and (1.68) that

$$\|\nabla \hat{\phi}_j\|_{L^\infty} \leq \|\eta'\|_{L^\infty} \|\nabla \delta^{(\frac{\epsilon}{4})}\|_{L^\infty} \leq \|\eta'\|_{L^\infty}. \tag{1.75}$$

After we did this for every $j \in J$ with the same function η we now want to define our partition of unity. W.l.o.g. assume that $J = \mathbb{N}$ (it is countable by definition of minimally smooth domains), and define

$$\phi_j = \hat{\phi}_j \prod_{i=1}^{j-1} (1 - \hat{\phi}_i), \quad j \in \mathbb{N}. \tag{1.76}$$

Clearly we have that $\operatorname{supp} \phi_j \subset \operatorname{supp} \hat{\phi}_j \subset \overline{U_j}$, and $\phi_j \in C_0^\infty(\mathbb{R}^d)$ for every $j \in J$.

Since Ω is minimally smooth, there is a maximum number N of sets in $(U_j)_{j \in J}$ which may have empty intersection, so at most N functions in $(\hat{\phi}_j)_{j \in J}$ can be supported in a single point, and thus by the product rule we see that

$$\|\nabla \phi_j\|_{L^\infty} \leq N \|\eta'\|_{L^\infty}, \tag{1.77}$$

which shows that the gradients of these functions have a uniform bound.

What remains to show is that these functions sum up to one in an open neighbourhood of $\partial\Omega$.

To prove this we first need to show that

$$\sum_{j=1}^{k} \phi_j(x) = 1 - \prod_{j=1}^{k} (1 - \hat{\phi}_j)(x), \quad x \in \mathbb{R}^d,\, k \in \mathbb{N}. \tag{1.78}$$

This is clear for $k = 1$, since

$$\phi_1(x) = \hat{\phi}_1(x) = 1 - (1 - \hat{\phi}_1)(x), \quad x \in \mathbb{R}^d. \tag{1.79}$$

Moreover, if (1.78) holds true for $k \in \mathbb{N}$, then

$$\begin{aligned} \sum_{j=1}^{k+1} \phi_j(x) = \phi_{k+1}(x) + \sum_{j=1}^{k} \phi_j(x) &= \phi_{k+1}(x) + 1 - \prod_{j=1}^{k} (1 - \hat{\phi}_j)(x) \\ &= \hat{\phi}_{k+1}(x) \prod_{j=1}^{k} (1 - \hat{\phi}_j)(x) + 1 - \prod_{j=1}^{k} (1 - \hat{\phi}_j)(x) \\ &= 1 - \prod_{j=1}^{k+1} (1 - \hat{\phi}_j)(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.80}$$

so (1.78) holds true by induction. Consequently, if there exists $j \in \mathbb{N}$ such that $\hat{\phi}_j(x) = 1$ for some $x \in \mathbb{R}^d$, then

$$\sum_{j=1}^{\infty} \phi(x) = 1 - \prod_{j=1}^{\infty}(1 - \hat{\phi}_j)(x) = 1. \tag{1.81}$$

Now let $x \in \mathbb{R}^d$ such that $\operatorname{dist}(x, \partial\Omega) \leq \frac{\epsilon}{4}$, so since $\partial\Omega$ is closed there exists $y \in \partial\Omega$ such that

$$|y - x| \leq \frac{\epsilon}{4}. \tag{1.82}$$

From the definition of minimally smooth domains we know that there exists $j \in J$ such that $\mathbb{B}_\epsilon(y) \subset U_j$, and consequently

$$\operatorname{dist}(y, \mathbb{R}^d \setminus U_j) \geq \operatorname{dist}(x, \mathbb{R}^d \setminus U_j) - |x - y| \geq \frac{3}{4}, \tag{1.83}$$

hence $\hat{\phi}_j(x) = 1$ and

$$\sum_{j=1}^{\infty} \phi(x) = 1. \tag{1.84}$$

Since this holds for every $\operatorname{dist}(x, \partial\Omega) \leq \frac{\epsilon}{4}$, we see that $(\phi_j)_{j \in J}$ is a partition of unity for $\partial\Omega$. □

Let $\Omega \subset \mathbb{R}^{d-1}$ be minimally smooth, choose a regular Lipschitz atlas $((U_j, \xi_j, \kappa_j))_{j \in J}$, which is possible due to Lemma 1.4.3, and a partition of unity for $\partial\Omega$ which is C^1-uniformly subordinate to the atlas.

We denote the corresponding special Lipschitz domains as in Definition 1.4.2 as $(\Omega_j)_{j \in J}$ and define the space $H^s(\partial\Omega)$ for $s \in [0, 1]$ as

$$\begin{aligned} H^s(\partial\Omega) = \Big\{ u \in L^2(\partial\Omega) : (u\phi_j)_{\partial\Omega_j} \in H^s(\mathbb{R}^{d-1})\, \forall j \in J, \\ \sum_{j \in J} \|(u\phi_j)_{\partial\Omega_j}\|_{H^s}^2 < \infty \Big\}, \end{aligned} \tag{1.85}$$

where $(u\phi_j)_{\partial\Omega_j}$ is defined as in (1.53), and we define the inner product via

$$\langle u, v \rangle_{H^s(\partial\Omega)} = \sum_{j \in J} \langle u\phi_j, v\phi_j \rangle_{H^s(\partial\Omega_j)}, \quad u, v \in H^s(\partial\Omega). \tag{1.86}$$

The norm of the space $H^s(\partial\Omega)$ is the induced norm defined by (1.86). The following Lemma shows that this definition is independent of the chosen regular Lipschitz atlas and partition of unity.

Lemma 1.4.10. *Let $\Omega \subset \mathbb{R}^d$ be a minimally smooth domain. The space $H^s(\partial\Omega)$ as defined in (1.85) is independent of the choice of the regular Lipschitz atlas and the C^1-uniformly subordinate partition of unity.*

In particular, if we have two different choices, then the corresponding inner products as defined in (1.86) yield equivalent norms.

Proof. Let $((U_j, \xi_j, \kappa_j))_{j\in J}$ and $((V_i, \hat{\xi}_i, \hat{\kappa}_i))_{i\in I}$ be regular Lipschitz atlases describing Ω, with C^1-uniformly subordinate partitions of unity $(\phi)_{j\in J}$ and $(\psi)_{i\in I}$, respectively, so we have that

$$\begin{aligned} &\operatorname{supp}\phi_j \subset \overline{U_j}, \quad j \in J,\\ &\operatorname{supp}\psi_i \subset \overline{V_i}, \quad i \in I. \end{aligned} \tag{1.87}$$

We define $H^s_J(\partial\Omega)$ and $H^s_I(\partial\Omega)$ as the sets of functions defined by (1.85) if we use the partitions of unity $((U_j, \xi_j, \kappa_j))_{j\in J}$ and $((V_i, \hat{\xi}_i, \hat{\kappa}_i))_{i\in I}$, respectively.

We need to show that there exists $C > 0$ such that for any $u \in H^s_J(\partial\Omega)$ it holds that

$$\sum_{i\in I} \|(u\psi_i)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^{d-1})} \leq C \sum_{j\in J} \|(u\phi_j)_{\partial\Omega_j}\|^2_{H^s(\mathbb{R}^{d-1})}. \tag{1.88}$$

In the following, we will use the notations

$$u_i = u\psi_i, \quad u^j = u\phi_j, \quad u_i^j = u\psi_i\phi_i, \qquad i \in I, j \in J. \tag{1.89}$$

Note that by definition of the partition of unity $\operatorname{supp}(u_i^j) \subset \overline{V_i} \cap \overline{U_i}$. Lemma 1.4.5 states that different choices of Lipschitz function and rigid body motion yield equivalent H^s-norms on the boundary, and the constant between them only depends on the Lipschitz constants. Since the functions $(\xi_j)_{j\in J}$ and $(\hat{\xi}_{i\in I})$ have a uniform Lipschitz constant $L > 0$, we see that there exists $C_L > 0$ such that

$$\|(u_i^j)_{\partial\Omega_i}\|_{H^s(\mathbb{R}^d)} \leq C_L \|(u_i^j)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)}, \qquad i \in I, j \in J. \tag{1.90}$$

Using that the partitions of unity have a uniform C^1 bound, we call it $C_P > 0$, and using Lemma 1.2.10 we obtain

$$\|(u_i^j)_{\partial\Omega_i}\|_{H^s(\mathbb{R}^d)} \leq C_L \|(u^j\psi_i)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)} \leq C_P C_L \|(u^j)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)}, \qquad i \in I, j \in J. \tag{1.91}$$

Now consider the sets

$$J_i = \{j \in J : \overline{V_i} \cap \overline{U_j} \neq \emptyset\}, \quad i \in I. \tag{1.92}$$

Since we talk about regular Lipschitz atlases, there is a constant $\epsilon_2 > 0$ such that every V_i is contained in a ball of size ϵ_2, and there exists a constant $\epsilon_1 > 0$ such that every set U_j contains a ball of that size. Furthermore, only N of the sets U_j can intersect at a single point, so it is clear that there exists an upper bound M for how many elements any J_i can contain.

Now from $\operatorname{supp}(u_i^j) \subset \overline{V_i} \cap \overline{U_i}$ we see that $u_i^j = 0$ for every $j \notin J_i$, and consequently

$$\|(u_i)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} = \|\sum_{j\in J}(u_i^j)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} = \|\sum_{j\in J_i}(u_i^j)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)}, \quad i \in I, \tag{1.93}$$

and consequently

$$\|(u_i)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} = \|\sum_{j\in J_i}(u_i^j)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} \le M\sum_{j\in J_i}\|(u_i^j)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)}, \quad i \in I, \tag{1.94}$$

where we used that for $a_1, ..., a_M \in \mathbb{R}$

$$\left(\sum_{k=1}^M a_k\right)^2 = \sum_{k,l=1}^M a_k a_l \le \sum_{k,l=1}^M \frac{1}{2}(a_k^2 + a_l^2) = M\sum_{k=1}^M a_k^2. \tag{1.95}$$

Equations (1.91) and (1.94) together yield

$$\|(u_i)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} \le MC_PC_L\sum_{j\in J_i}\|(u^j)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)}, \quad i \in I, \tag{1.96}$$

and summation over all $i \in I$ then yields that

$$\sum_{i\in I}\|(u_i)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} \le MC_PC_L\sum_{i\in I}\sum_{j\in J_i}\|(u^j)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)}. \tag{1.97}$$

We can now define the set

$$I_j = \{i \in I : \overline{V_i} \cap \overline{U}_j \neq \emptyset\} \tag{1.98}$$

and see that by a simple reordering of indices

$$\sum_{i\in I}\sum_{j\in J_i}\|(u^j)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)} = \sum_{j\in J}\sum_{i\in I_j}\|(u^j)_{\partial\Omega_j}\|_{H^s(\mathbb{R}^d)}. \tag{1.99}$$

With the same reasoning as before, there exists a constant $K \in \mathbb{N}$ such that each of the sets I_j contains at most K elements, and consequently we obtain

$$\sum_{i\in I}\|(u_i)_{\partial\Omega_i}\|^2_{H^s(\mathbb{R}^d)} \le MKC_PC_L\sum_{j\in J}\|(u^j)_{\partial\Omega_j}\|^2_{H^s(\mathbb{R}^d)}. \tag{1.100}$$

In particular we now see that also $u \in H_I^s(\partial\Omega)$. Since both atlases were arbitrary, the other direction follows analogously, and the space $H^s(\partial\Omega)$ is therefore well-defined. □

1.5 Traces and Extension Operators

Let $\Omega \in \mathbb{R}^d$ be minimally smooth. For $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$, the operator

$$\gamma_D^0 : C(\overline{\Omega}) \cap H^1(\Omega) \to L^2(\partial\Omega), \quad \phi \mapsto \phi\restriction_{\partial\Omega}, \tag{1.101}$$

is well-defined and bounded if its domain is equipped with the H^1-norm, see [24, Chapter 1]. Since $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $H^1(\Omega)$, see [24, Chapter 1], γ_D^0 can be uniquely extended to a bounded operator

$$\tilde{\gamma}_D : H^1(\Omega) \to L^2(\partial\Omega), \quad \tilde{\gamma}_D\restriction_{C(\overline{\Omega})\cap H^1(\Omega)} = \gamma_D^0. \tag{1.102}$$

We call $\tilde{\gamma}_D$ the Dirichlet trace. As the following theorem states we have that $\mathrm{Ran}\tilde{\gamma}_D \subset H^{\frac{1}{2}}(\partial\Omega)$, therefore we will denote the Dirichlet trace by γ_D if we consider it as an operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$.

Theorem 1.5.1. *Let Ω be minimally smooth. Then the Dirichlet trace*

$$\gamma_D : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \tag{1.103}$$

is bounded, surjective and has a bounded right inverse

$$\mathcal{E} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega). \tag{1.104}$$

Proof. See [24, Theorem 2] for surjectivity and right inverse. For the boundedness, recall that we already know that

$$\tilde{\gamma}_D : H^1(\Omega) \to L^2(\partial\Omega) \tag{1.105}$$

is bounded. As γ_D is well-defined on a closed domain it is sufficient to show that γ_D is closed and conclude boundedness from the Closed Graph Theorem. Consider $(u_n)_n \subset H^1(\Omega)$ such that

$$u_n \to u \text{ in } H^1(\Omega), \quad \gamma_D u_n \to g \text{ in } H^{\frac{1}{2}}(\partial\Omega). \tag{1.106}$$

It is clear that $u \in \mathrm{Dom}\gamma_D = H^1(\Omega)$ as $H^1(\Omega)$ is a Hilbert space. Since $\gamma_D u_n$ converges to g in the $H^{\frac{1}{2}}(\partial\Omega)$-norm it also converges in the weaker $L^2(\partial\Omega)$-norm, and consequently

$$\gamma_D u = \tilde{\gamma}_D u = \lim_{n\to\infty} \tilde{\gamma}_D u_n = g \quad \text{in } L^2(\partial\Omega), \tag{1.107}$$

which proves that γ_D is closed and thus bounded. □

Lemma 1.5.2. *Let Ω be minimally smooth. It holds true that $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $H^1(\Omega)$ and $\mathrm{Ran}\gamma_D\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$.*

Proof. For the first assertion see [24, Chapter 1]. Now let $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $\epsilon > 0$.
Theorem 1.5.1 states that γ_D is surjective, so there exists $u \in H^1(\Omega)$ such that

$$\gamma_D u = g. \tag{1.108}$$

Since $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $H^1(\Omega)$, for any $\epsilon > 0$ we find $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ such that

$$\|u - \phi\|_{H^1} < \epsilon, \tag{1.109}$$

and thus by the boundedness of γ_D, see Theorem 1.5.1, we obtain

$$\|g - \gamma_D\phi\|_{H^{\frac{1}{2}}} = \|\gamma_D(u - \phi)\|_{H^{\frac{1}{2}}} < \epsilon\|\gamma_D\|_{H^1 \to H^{\frac{1}{2}}}, \tag{1.110}$$

which shows the density of $\mathrm{Ran}\gamma_D\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ in $H^{\frac{1}{2}}(\partial\Omega)$. □

Lemma 1.5.3. *Let Ω be a minimally smooth domain. Then there exists a linear, bounded extension operator*

$$E_\Omega : H^1(\Omega) \to H^1(\mathbb{R}^d). \tag{1.111}$$

Proof. See [24, Chapter 1]. □

Let us now consider the space

$$H^1_\Delta(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}. \tag{1.112}$$

On this space we can define the Neumann trace

$$\begin{aligned} &\gamma_N^\Delta : H^1_\Delta(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \\ &(\gamma_N^\Delta u, g)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} = \langle \nabla u, \nabla \mathcal{E} g\rangle_{L^2} + \langle \Delta u, \mathcal{E} g\rangle_{L^2}, \quad g \in H^{\frac{1}{2}}(\partial\Omega). \end{aligned} \tag{1.113}$$

We will later encounter different definitions of the Neumann trace. However, we will see that all those definitions are compatible on the right domains.

1.6 Basic Properties and Calculus of Weak Derivatives

Lemma 1.6.1. *Let Ω be a non-empty, open domain and $f, g \in H^1_{\mathrm{loc}}(\Omega)$. Then it holds true that*

$$\nabla(fg) = f\nabla g + g\nabla f. \tag{1.114}$$

Proof. Since both functions are in $L^2_{\text{loc}}(\Omega)$ we see that $fg \in L^1_{\text{loc}}(\Omega)$ and consequently a gradient exists in the distributional sense.

By the same reasoning we see that also $f\partial_j g, g\partial_j f \in L^1_{\text{loc}}(\Omega)$ for $j \in \{1, ..., d\}$, so they can be viewed as distributions. Let now $\phi \in C_0^\infty(\Omega)$. Then there exists an open set $K \Subset \Omega$ with $\operatorname{supp} \phi \subset K$, and thus it holds true that

$$(f\partial_j g, \phi)_{\mathcal{D}'\times\mathcal{D}} = \int_\Omega (\partial_j g) f\overline{\phi} \mathrm{d}x = \int_K (\partial_j g) f\overline{\phi} \mathrm{d}x, \quad j \in \{1, ..., d\}. \tag{1.115}$$

Note that $(\overline{f}\phi) \restriction_K \in H_0^1(K)$, so since $C_0^\infty(K)$ is dense in $H_0^1(K)$ we can find a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(K)$ such that

$$\psi_n \xrightarrow{n\to\infty} (\overline{f}\phi) \restriction_K \qquad \text{in } H^1(K). \tag{1.116}$$

For an easier notation we will denote the zero-extension to Ω of these functions also by $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$, and it is clear that

$$\psi_n \xrightarrow{n\to\infty} \overline{f}\phi \quad \text{ in } H^1(\Omega). \tag{1.117}$$

Now by definition of distributional derivatives we have that

$$\begin{aligned} \int_K (\partial_j g)\overline{\psi_n} \mathrm{d}x &= (\partial_j g, \psi_n)_{\mathcal{D}'\times\mathcal{D}} = -(g, \partial_j \psi_n)_{\mathcal{D}'\times\mathcal{D}} \\ &= -\int_K g\partial_j \overline{\psi_n} \mathrm{d}x, \quad n \in \mathbb{N}, j \in \{1, ..., d\}. \end{aligned} \tag{1.118}$$

Since $g \in H^1(K)$ the convergences in (1.117) now implies that

$$\int_K (\partial_j g) f\overline{\phi} \mathrm{d}x = -\int_K g\partial_j (f\overline{\phi}) \mathrm{d}x, \quad j \in \{1, ..., d\}. \tag{1.119}$$

Using the Leibniz rule for multiplication with test functions, see Lemma 1.2.10, we consequently obtain that

$$\begin{aligned} \int_\Omega (\partial_j g) f\overline{\phi} \mathrm{d}x = \int_K (\partial_j g) f\overline{\phi} \mathrm{d}x &= -\int_K g(\partial_j f)\overline{\phi} \mathrm{d}x - \int_K gf\partial_j \overline{\phi} \mathrm{d}x \\ &= -\int_\Omega g(\partial_j f)\overline{\phi} \mathrm{d}x - \int_\Omega gf\partial_j \overline{\phi} \mathrm{d}x, \quad j \in \{1, ..., d\}. \end{aligned} \tag{1.120}$$

or rewritten in terms of distributions

$$(f\partial_j g + g\partial_j f, \phi)_{\mathcal{D}'\times\mathcal{D}} = -(fg, \partial_j \phi)_{\mathcal{D}'\times\mathcal{D}}, \quad j \in \{1, ..., d\}. \tag{1.121}$$

Since this holds for every $\phi \in C_0^\infty(\Omega)$ we see by the definition of distributional derivatives that

$$\nabla(fg) = f\nabla g + g\nabla f. \tag{1.122}$$

□

Lemma 1.6.2. *Let $D, D' \subset \mathbb{R}^d$ be non-empty, open sets, let $\kappa : D \to D'$ be a C^1-diffeomorphism, $\Omega \Subset D$ and $\Omega' = \mathrm{Ran}\kappa\restriction_\Omega$, and let $u \in W^{1,p}(\Omega')$ for $p \in [1,\infty)$. Then it holds that $u \circ \kappa \in W^{1,p}(\Omega)$ with*

$$\nabla(u \circ \kappa) = (D\kappa)^T \left((\nabla u) \circ \kappa\right), \tag{1.123}$$

where $D\kappa$ denotes the Jacobi matrix of κ.

Furthermore, if $u_n \xrightarrow{n\to\infty} u$ in $W^{1,p}(\Omega')$, then it also holds that $u_n \circ \kappa \xrightarrow{n\to\infty} u \circ \kappa$ in $W^{1,p}(\Omega)$.

Proof. The mapping κ is continuously differentiable and hence Lipschitz continuous on $\Omega \Subset D$. Since continuous functions map compact sets to compact sets it follows that $\overline{\Omega'}$ is compact. Furthermore since κ is a diffeomorphism κ^{-1} is continuous and injective, so Ω' can be viewed as the pre-image of the open set Ω under κ^{-1} and is therefore open. In summary this gives $\Omega' \Subset D'$ and thus κ^{-1} is Lipschitz continuous on Ω', so $u \circ \kappa \in W^{1,p}(\Omega)$ and (1.123) follow from [30, Theorem 2.2.2].

Now let $(u_n)_n \subset W^{1,p}(\Omega')$ with $u_n \xrightarrow{n\to\infty} u$ in $W^{1,p}(\Omega')$. Then it holds true that

$$\begin{aligned}\|u_n \circ \kappa - u \circ \kappa\|_{L^p(\Omega)} &= \int_\Omega |u_n(\kappa(x)) - u(\kappa(x))|^p \,\mathrm{d}x \\ &= \int_{\Omega'} |u_n(x') - u(x')|^p \,|\det D\kappa^{-1}(x')|\mathrm{d}x' \\ &\le \|\det(D\kappa^{-1})\|_{L^\infty(\Omega')}\|u_n - u\|_{L^p(\Omega')} \xrightarrow{n\to\infty} 0,\end{aligned} \tag{1.124}$$

where we used that κ^{-1} is Lipschitz continuous and thus has bounded derivative, see Theorem 1.2.11.

Together with (1.123) we see by the same arguments that

$$\begin{aligned}&\|\nabla(u_n \circ \kappa) - \nabla(u \circ \kappa)\|_{L^p(\Omega)} = \int_\Omega \left|(D\kappa)^T(x)\left(\nabla u_n(\kappa(x)) - \nabla u(\kappa(x))\right)\right|^p \mathrm{d}x \\ &= \int_{\Omega'} \left|(D\kappa)^T(\kappa^{-1}(x'))\left(\nabla u_n(x') - \nabla u(x')\right)\right|^p |\det D\kappa^{-1}|\mathrm{d}x' \\ &\le \|D\kappa^T\|^p_{L^\infty(\Omega')}\|\det D\kappa^{-1}\|_{L^\infty(\Omega')}\|\nabla(u_n - u)\|_{L^p(\Omega')} \xrightarrow{n\to\infty} 0,\end{aligned} \tag{1.125}$$

which proves that $u_n \circ \kappa \xrightarrow{n\to\infty} u \circ \kappa$ in $W^{1,p}(\Omega)$. □

Remark 1.6.3. *The results of Lemma 1.6.2 can be generalised to $W^{s,p}(\Omega)$ with $s \in [0,1]$. The proof is analogous.*

Lemma 1.6.4. *Let $D, D' \subset \mathbb{R}^d$ be non-empty, open sets, let $\kappa : D \to D'$ be a C^2-diffeomorphism, $\Omega \Subset D$ and $\Omega' = \mathrm{Ran}\kappa\restriction_\Omega$, and let $u \in W^{2,p}(\Omega')$ for $p \in [1,\infty)$. Then it holds that $u \circ \kappa \in W^{2,p}(\Omega)$.*

Furthermore, if $u_n \xrightarrow{n\to\infty} u$ in $W^{2,p}(\Omega')$, then it also holds that $u_n \circ \kappa \xrightarrow{n\to\infty} u \circ \kappa$ in $W^{2,p}(\Omega)$.

Proof. First note that we satisfy all the assumptions of Lemma 1.6.2, so we immediately have that $u \circ \kappa \in W^{1,p}(\Omega)$ and from (1.123) we obtain that

$$\begin{aligned}\partial_i(u \circ \kappa) &= \sum_{j=1}^{d} (D\kappa)_{ji} \left((\partial_j u) \circ \kappa\right) \\ &= \sum_{j=1}^{d} \left[\left((\partial_i \kappa_j) \circ \kappa^{-1}\right) (\partial_j u)\right] \circ \kappa, \quad i \in \{1, ..., d\}.\end{aligned} \tag{1.126}$$

Note that $\Omega' \Subset D'$ since continuous mappings map compact sets to compact sets. Consequently, since κ is a C^2-diffeomorphism, we obtain that

$$D\kappa \circ \kappa^{-1} \in W^{1,\infty}(\Omega'). \tag{1.127}$$

Since by assumption $u \in W^{2,p}(\Omega')$ and thus $\partial_j u \in W^{1,p}(\Omega')$ for each $i \in \{1, ..., d\}$, we now see that

$$\sum_{j=1}^{d} \left[\left((\partial_i \kappa_j) \circ \kappa^{-1}\right) (\partial_j u)\right] \in W^{1,p}(\Omega'). \tag{1.128}$$

This function satisfies the assumptions of Lemma 1.6.2, and with (1.126) we therefore see that $\partial_i(u \circ \kappa) \in W^{1,p}(\Omega)$ for every $i \in \{1, ..., d\}$, so consequently $u \circ \kappa \in W^{2,p}(\Omega)$.

If $u_n \xrightarrow{n\to\infty} u$ in $W^{2,p}(\Omega')$, then it also holds that $\partial_j u_n \xrightarrow{n\to\infty} \partial_j u$ in $W^{1,p}(\Omega')$ for every $j \in \{1, ..., d\}$, and since we already saw that we can apply Lemma 1.6.2 on $\partial_j u$, we have $\partial_j(u_n \circ \kappa) \xrightarrow{n\to\infty} \partial_j(u \circ \kappa)$ in $W^{1,p}(\Omega)$ for every $j \in \{1, ..., d\}$, and hence $u_n \circ \kappa \xrightarrow{n\to\infty} u \circ \kappa$ in $W^{2,p}(\Omega)$. □

Lemma 1.6.5. *Let $D, D' \subset \mathbb{R}^d$ be non-empty, open sets, let $\kappa : D \to D'$ be a C^2-diffeomorphism, $\Omega \Subset D$ and $\Omega' = \mathrm{Ran}\kappa\restriction_\Omega$, and let $u \in H^1(\Omega')$ with $\Delta u \in L^p_{\mathrm{loc}}(\Omega')$, $p \in [1, \infty)$. Then it holds true that if*

$$\langle \nabla u, \nabla \phi \rangle_{L^2(\Omega')} + \int_{\Omega'} \Delta u \overline{\phi} \mathrm{d}x = 0, \tag{1.129}$$

for some $\phi \in H^1(\Omega')$, then

$$\begin{aligned}&\int_\Omega \nabla(u \circ \kappa)(y) \cdot \Big(|\det(D\kappa(y))| \left[(D\kappa(y))^T D\kappa(y)\right]^{-1} \nabla(\overline{\phi} \circ \kappa)(y)\Big) \mathrm{d}y \\ &\quad = -\int_\Omega (\Delta u)(\kappa(y))(\overline{\phi} \circ \kappa)(y) |\det(D\kappa(y))| \mathrm{d}y,\end{aligned} \tag{1.130}$$

where $D\kappa$ denotes the Jacobi matrix of κ. Moreover, we also obtain

$$\mathrm{div}\left[|\det(D\kappa)| \left((D\kappa)^T (D\kappa)\right)^{-1} \nabla(u \circ \kappa)\right] = |\det(D\kappa)|(\Delta u) \circ \kappa \in L^p_{\mathrm{loc}}(\Omega). \tag{1.131}$$

Proof. Let $\phi \in H^1(\Omega')$ such that

$$\langle \nabla u, \nabla \phi \rangle_{L^2(\Omega')} + \int_{\Omega'} \Delta u \overline{\phi} \mathrm{d}x = 0. \tag{1.132}$$

A simple change of coordinates together with Lemma 1.6.2 gives

$$\begin{aligned} 0 &= \int_\Omega (\Delta u)(\kappa(y)) \overline{\phi}(\kappa(y)) |\det(D\kappa(y))| \mathrm{d}y \\ &+ \int_\Omega (\nabla u)(\kappa(y)) \cdot (\nabla \overline{\phi})(\kappa(y)) |\det(D\kappa(y))| \mathrm{d}y \\ &= \int_\Omega (\Delta u)(\kappa(y)) \overline{\phi} \circ \kappa(y) |\det(D\kappa(y))| \mathrm{d}y \\ &+ \int_\Omega \big((D\kappa(y))^{-T} \nabla (u \circ \kappa)(y))\big) \cdot \big((D\kappa(y))^{-T} \nabla (\overline{\phi} \circ \kappa)(y)\big) |\det(D\kappa(y))| \mathrm{d}y, \end{aligned} \tag{1.133}$$

where we used that $D\kappa(y)$ is everywhere invertible since κ is a diffeomorphism. If we rewrite it slightly we obtain (1.130). This shows the first assertion.

For the second one let $\psi \in C_0^\infty(\Omega)$ and define

$$\phi = \psi \circ \kappa^{-1}. \tag{1.134}$$

Then it holds true that $\phi \in C^1(\Omega)$ and, since κ maps compact sets to compact sets, ϕ has compact support, which yields $\phi \in H_0^1(\Omega)$.

By the definition of the Laplacian and the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ it is clear that

$$\langle \nabla u, \nabla \phi \rangle_{L^2(\Omega')} + \int_{\Omega'} \Delta u \overline{\phi} \mathrm{d}x = 0, \tag{1.135}$$

and so by the first part of this Lemma and since $\psi \in \mathcal{D}(\Omega)$ was arbitrary we obtain that

$$\begin{aligned} 0 = &\int_\Omega \nabla (u \circ \kappa)(y)) \cdot \Big(|\det(D\kappa(y))| \left[(D\kappa(y))^T (D\kappa(y)) \right]^{-1} \nabla \overline{\psi}(y) \Big) \mathrm{d}y \\ &+ \int_\Omega |\det(D\kappa(y))| (\Delta u)(\kappa(y)) \overline{\psi}(y) \mathrm{d}y, \qquad \psi \in \mathcal{D}(\Omega). \end{aligned} \tag{1.136}$$

This shows that the second assertion holds in the distributional sense.

Finally note that $D\kappa \in L^\infty(\Omega)$, since it is a continuous mapping on a compact set, and thus

$$|\det(D\kappa)| (\Delta u) \circ \kappa \in L^p_{\mathrm{loc}}(\Omega). \tag{1.137}$$

□

Corollary 1.6.6. *Let $\Omega, \Omega' \subset \mathbb{R}^d$ be non-empty, open, bounded sets and let $\kappa : \mathbb{R}^d \to \mathbb{R}^d$ be rigid body motion as in Definition 1.4.1. Then it holds that for $u, v \in H^1(\Omega')$*

$$\langle u \circ \kappa, v \circ \kappa \rangle_{L^2(\Omega)} = \langle u, v \rangle_{L^2(\Omega')}, \tag{1.138}$$

and if $\Delta u \in L^1_{\text{loc}}(\Omega')$, then

$$\Delta(u \circ \kappa) = (\Delta u) \circ \kappa. \tag{1.139}$$

Proof. Rigid body motions are mappings of the form

$$\kappa(x) = Rx + t, \quad x \in \mathbb{R}^d, \tag{1.140}$$

where $R \in \mathbb{R}^{d\times d}$ is a matrix with $R^T R = RR^T = I$ and $t \in \mathbb{R}^d$. Consequently the Jacobi matrix is given by $D\kappa = R$, and thus we see that

$$(D\kappa)^T (D\kappa) = I, \quad |\det(D\kappa)| = 1. \tag{1.141}$$

The assertion now follows as a consequence of Lemma 1.6.2 and (1.131) in Lemma 1.6.5. □

The following Lemma shows that also the Dirichlet trace has a quite natural transformation under change of variables.

Lemma 1.6.7. *Let $D, D' \subset \mathbb{R}^d$ be non-empty, open sets, let $\kappa : D \to D'$ be a C^1-diffeomorphism, let $\Omega \Subset D$ be a Lipschitz domain, $\Omega' = \text{Ran}\kappa\restriction_\Omega$, and let $u \in H^1(\Omega')$. Then Ω' is also a Lipschitz domain, $u \circ \kappa \in H^1(\Omega)$, and it holds true that*

$$\gamma_D(u \circ \kappa) = (\gamma_D u) \circ \kappa \quad \text{in } H^{\frac{1}{2}}(\partial\Omega). \tag{1.142}$$

Proof. From Lemma 1.6.2 we see that $u \circ \kappa \in H^1(\Omega)$. Additionally we can use [20, Chapter 4.1] to see that Ω' is a Lipschitz domain, so the Dirichlet trace on $\partial\Omega'$ is well defined.

For $u \in C(\overline{\Omega'}) \cap H^1(\Omega')$ it thus holds that $u \circ \kappa \in C(\overline{\Omega}) \cap H^1(\Omega)$ and therefore

$$\gamma_D(u \circ \kappa)(x) = u \circ \kappa(x) = (\gamma_D u) \circ \kappa(x), \quad x \in \partial\Omega, \tag{1.143}$$

holds true pointwise, and so we see that in this case

$$\gamma_D(u \circ \kappa) = (\gamma_D u) \circ \kappa \quad \text{in } H^{\frac{1}{2}}(\partial\Omega). \tag{1.144}$$

Next let $u \in H^1(\Omega')$. Since $C_0^\infty(\mathbb{R}^d)\restriction_{\Omega'}$ is dense in $H^1(\Omega')$, see Lemma 1.5.2, we can find a sequence $(u_n)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_{\Omega'}$ such that

$$u_n \xrightarrow{n\to\infty} u \quad \text{in } H^1(\Omega'). \tag{1.145}$$

Lemma 1.6.2 shows that in this case also

$$u_n \circ \kappa \xrightarrow{n\to\infty} u \circ \kappa \text{ in } H^1(\Omega). \tag{1.146}$$

Since $(u_n)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_{\Omega'} \subset C(\overline{\Omega'}) \cap H^1(\Omega')$ and since the Dirichlet trace on both Ω and Ω' is continuous, see Theorem 1.5.1, we finally arrive at

$$\gamma_D(u \circ \kappa) = \lim_{n\to\infty} \gamma_D(u_n \circ \kappa) = \lim_{n\to\infty} (\gamma_D u_n) \circ \kappa = (\gamma_D u) \circ \kappa \quad \text{ in } H^{\frac{1}{2}}(\partial\Omega), \tag{1.147}$$

where we used Remark 1.6.3 to show that the limits coincide. □

Next we want to generalise the chain rule to weakly differentiable functions. We will just consider the case of real-valued functions for now.

Lemma 1.6.8. *Let $\Omega \subset \mathbb{R}^d$ open, non-empty and bounded, let $f : \mathbb{R} \to \mathbb{R}$ uniformly Lipschitz continuous, $p \in (1, \infty)$ and $u \in W^{1,p}(\Omega)$ real-valued. Then it holds that $f \circ u \in W^{1,p}(\Omega)$ and*

$$\partial_j(f \circ u) = f'(u)\partial_j u, \qquad j \in \{1, ..., d\}. \tag{1.148}$$

If $f(0) = 0$, then the assertion also holds true for unbounded domains Ω.

Proof. See [12, Theorem VI.2.1]. □

Lemma 1.6.9. *Let $\Omega \subset \mathbb{R}^d$ open and non-empty, let $f : \mathbb{R} \to \mathbb{R}$ uniformly Lipschitz continuous, $p \in (1, \infty)$ and $u \in W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$ real-valued. Then it holds that $f \circ u \in W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$ and*

$$\partial_j(f \circ u) = f'(u)\partial_j u, \qquad j \in \{1, ..., d\}. \tag{1.149}$$

Proof. Follows from applying Lemma 1.6.8 on bounded subsets of Ω. □

Above we only considered the chain rule for real-valued functions, since this is enough for what we require in this thesis. However, we will need the derivative of the magnitude of complex-valued functions, so we state it explicitly in the following lemma.

Lemma 1.6.10. *Let $\Omega \subset \mathbb{R}^d$ be open, non-empty, and let $u \in H^1(\Omega)$. Then $|u| \in H^1(\Omega)$ and*

$$\nabla|u| = \mathrm{Re}\left(\mathrm{sgn}(\overline{u})\nabla u\right), \tag{1.150}$$

where

$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases} \tag{1.151}$$

is the complex signum function.

Proof. See [3, Example 4.5.(a)]. □

Since it holds true for every $u \in H^1(\Omega)$ that also $|u| \in H^1(\Omega)$ we can use the density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in $H^1(\Omega)$, see Lemma 1.5.2, to approximate $|u|$ with $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$-functions. However, since $|u|$ is non-negative, the question arises whether we can approximate it even with non-negative $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$-functions. The answer is given by the following lemma.

Lemma 1.6.11. *Let $\Omega \subset \mathbb{R}^d$ be minimally smooth and $u \in H^1(\Omega)$. Then there exists a sequence $(\phi_n)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ such that*

$$\phi_n \geq 0,\ n \in \mathbb{N}, \qquad \phi_n \xrightarrow{n\to\infty} |u| \text{ in } H^1(\Omega). \tag{1.152}$$

Proof. First by Lemma 1.5.3 we see that there exists a $\tilde{u} \in H^1(\mathbb{R}^d)$ such that $\tilde{u} = u$ holds true on Ω. Now Lemma 1.6.10 shows that $|\tilde{u}| \in H^1(\mathbb{R}^d)$.

We choose a function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq \eta \leq 1$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 2, \end{cases} \tag{1.153}$$

and define

$$\varphi_n = \eta(\tfrac{\cdot}{n})|\tilde{u}|. \tag{1.154}$$

It is clear that this sequence converges pointwise to $|\tilde{u}|$ and $0 \leq \varphi_n \leq |\tilde{u}|$ for all $n \in \mathbb{N}$, so by dominated convergence we see that $(\varphi_n)_n$ converges to $|\tilde{u}|$ in $L^2(\mathbb{R}^d)$.

Furthermore using the product rule we obtain

$$\nabla\varphi_n = \eta(\tfrac{\cdot}{n})\nabla|\tilde{u}| + |\tilde{u}|\tfrac{1}{n}(\nabla\eta)(\tfrac{\cdot}{n}). \tag{1.155}$$

The first summand on the right-hand side converges in $L^2(\mathbb{R}^d;\mathbb{R}^d)$ to $\nabla|\tilde{u}|$ by the same dominated convergence argument as above. For the second summand we have that

$$\left\||\tilde{u}|\tfrac{1}{n}(\nabla\eta)(\tfrac{\cdot}{n})\right\|_{L^2} \leq \tfrac{1}{n}\|\nabla\eta\|_{L^\infty}\,\||\tilde{u}|\|_{L^2} \xrightarrow{n\to\infty} 0, \tag{1.156}$$

and thus we see that $(\varphi_n)_n$ converges to $|\tilde{u}|$ in $H^1(\mathbb{R}^d)$.

Now define the functions

$$\varphi_n^{(\epsilon)} = \varphi_n * \omega_\epsilon, \quad n \in \mathbb{N},\ \epsilon > 0, \tag{1.157}$$

where $(\omega_\epsilon)_{\epsilon>0}$ denotes the family of standard mollifiers as defined in (1.36). We constructed $(\varphi_n)_n$ such that $\operatorname{supp}(\varphi_n) \subset \overline{\mathbb{B}(2n)}$ for all $n \in \mathbb{N}$, so Lemma 1.3.2 shows that $\varphi_n^{(\epsilon)} \in C_0^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$ and $\epsilon > 0$.

From Theorem 1.3.3 and Theorem 1.3.5 we see that

$$\varphi_n^{(\epsilon)} \xrightarrow{\epsilon\to 0^+} \varphi_n \text{ in } H^1(\Omega), \quad n \in \mathbb{N}. \tag{1.158}$$

Therefore we can pick a sequence $(\epsilon_n)_n \subset (0,\infty)$ such that

$$\|\varphi_n^{(\epsilon_n)} - \varphi_n\|_{H^1} \leq \frac{1}{n}, \quad n \in \mathbb{N}, \tag{1.159}$$

and define

$$\phi_n = \varphi_n^{(\epsilon_n)}, \quad n \in \mathbb{N}. \tag{1.160}$$

We now have that $(\phi_n \restriction_\Omega)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ and

$$\begin{aligned} \|\phi_n \restriction_\Omega - |u|\|_{H^1(\Omega)} &\leq \|\phi_n - |\tilde{u}|\|_{H^1(\mathbb{R}^d)} = \|\varphi_n^{(\epsilon_n)} - |\tilde{u}|\|_{H^1(\mathbb{R}^d)} \\ &\leq \|\varphi_n^{(\epsilon_n)} - \varphi_n\|_{H^1(\mathbb{R}^d)} + \|\varphi_n - |\tilde{u}|\|_{H^1(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0. \end{aligned} \tag{1.161}$$

So it remains to show that $\phi_n \restriction_\Omega \geq 0$ for every $n \in \mathbb{N}$. For all $x \in \mathbb{R}^d$ it holds true that

$$\phi_n(x) = \int_{\mathbb{B}_{\epsilon_n}(0)} \varphi_n(x-y)\omega_{\epsilon_n}(y)\mathrm{d}y = \int_{\mathbb{B}_{\epsilon_n}(0)} \eta\left(\frac{x-y}{n}\right)|\tilde{u}|(x-y)\omega_{\epsilon_n}(y)\mathrm{d}y \geq 0, \tag{1.162}$$

since all the functions in the integral are non-negative by definition, which concludes the proof. □

1.7 Domains Divided by a Lipschitz Function

Next we will consider some results for domains which are split into different parts by the graph of a Lipschitz function. This situation natural occurs when working with minimally smooth domains, where the boundary can be locally described on domains $(U_j)_{j\in J}$ as the graph of a Lipschitz function, see Definition 1.4.2. Throughout this section we will therefore consider the following situation.

Definition 1.7.1. *Let $U \subset \mathbb{R}^d$ be a non-empty, open set. We say that U is divided into three parts U', U'', Γ by a Lipschitz-continuous function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ if*

$$\begin{aligned} U' &= U \cap \{(x', x_d) \in \mathbb{R}^d : x_d < \xi(x')\}, \\ U'' &= U \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \xi(x')\}, \\ \Gamma &= U \cap \{(x', x_d) \in \mathbb{R}^d : x_d = \xi(x')\}. \end{aligned} \tag{1.163}$$

An illustration of a domain divided into three parts as in Definition 1.7.1 is given in Figure 1.1.

The problem which arises for domains U' and U'' as in Definition 1.7.1 is that they might not be Lipschitz domains, even if U is. Therefore we cannot guarantee the existence of a Dirichlet trace on Γ. However, as we will see in Lemma 1.7.4, we can ensure that existence of a Dirichlet trace on compact subsets of Γ. First we need to show the following result.

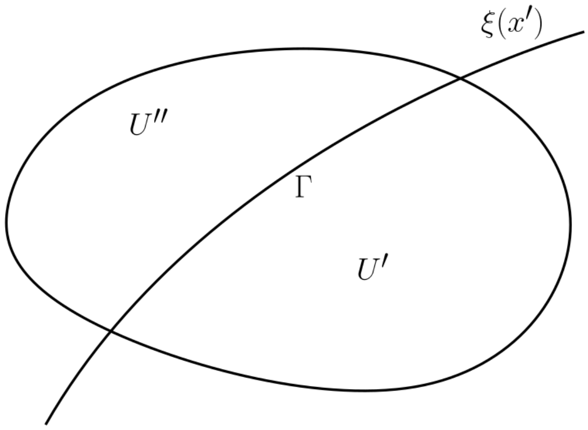

Figure 1.1: A domain divided into three parts as in Definition 1.7.1

Lemma 1.7.2. *Let $U \subset \mathbb{R}^d$ be a non-empty, open set divided into three parts U', U'', Γ by a Lipschitz-continuous function as in Definition 1.7.1, and let $\Gamma_C \subset \Gamma$ be compact. Then for every open neighbourhood $V \Subset U$ of Γ_C there exists a bounded linear operator*

$$\Lambda : H^1(U') \to H^1(\Omega), \tag{1.164}$$

where

$$\Omega = \{x \in \mathbb{R}^d : x_d < \xi(x_1, ..., x_{d-1})\}, \tag{1.165}$$

such that Λ maps $C(\overline{U'})$-functions to $C(\overline{\Omega})$-functions and

$$(\Lambda u) \restriction_{V \cap U'} = u \restriction_{V \cap U'}, \quad u \in H^1(U'), \tag{1.166}$$

holds true.

Proof. Since $V \Subset U$ we can find an open neighbourhood $W \Subset U$ of $\overline{V}$ and a function $\eta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in V \cap U', \\ 0 & \text{for } x \notin W. \end{cases} \tag{1.167}$$

The whole setting is illustrated in Figure 1.2. Lemma 1.2.10 shows that $\eta u \in H^1(U')$ for every $u \in H^1(U')$.

Let us now define the extension $\Lambda : H^1(U') \to H^1(\Omega)$ via

$$\Lambda u = \begin{cases} \eta u & \text{on } U', \\ 0 & \text{else,} \end{cases} \qquad u \in H^1(U'). \tag{1.168}$$

This is clearly linear and maps $C(\overline{U'})$-functions to $C(\overline{\Omega})$ functions. Next we need to show that it is well-defined. To this end let $\phi \in C_0^\infty(\Omega)$. Then it is clear that $\phi\eta$ is a smooth function, and furthermore

$$\operatorname{supp}(\eta\phi) \subset \operatorname{supp} \eta \cap \operatorname{supp} \phi \subset U \cap \Omega = U', \tag{1.169}$$

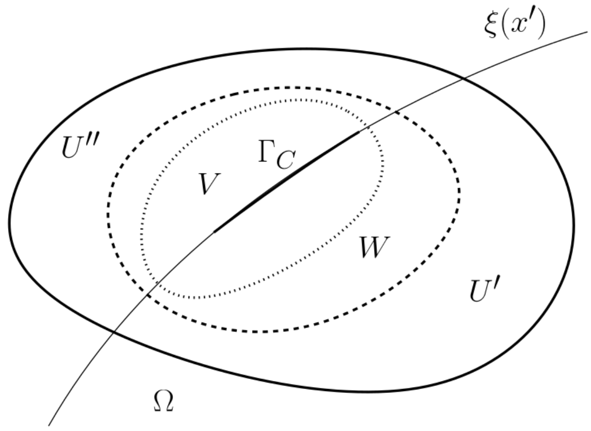

Figure 1.2: Setting of Lemma 1.7.2

which shows that $(\phi\eta)\restriction_{U'} \in C_0^\infty(U')$. Using this and the definition of weak derivatives we obtain

$$\begin{aligned}\int_\Omega \Lambda u \nabla\phi \mathrm{d}x &= \int_{U'} u\eta\nabla\phi \mathrm{d}x = \int_{U'} u\nabla(\eta\phi)\mathrm{d}x - \int_{U'} u(\nabla\eta)\phi \mathrm{d}x \\ &= -\int_{U'} (\nabla u)\eta\phi \mathrm{d}x - \int_{U'} u(\nabla\eta)\phi\mathrm{d}x \\ &= -\int_{U'} (\eta\nabla u + u\nabla\eta)\,\phi\mathrm{d}x, \qquad\qquad u \in H^1(U').\end{aligned} \tag{1.170}$$

This shows that

$$\nabla(\Lambda u) = \begin{cases} \eta\nabla u + u\nabla\eta & \text{on } U', \\ 0 & \text{else}, \qquad u \in H^1(U'), \end{cases} \tag{1.171}$$

hence we have $\Lambda u \in H^1(\Omega)$, so the operator is well-defined.

Lemma 1.2.10 shows the existence of a $C > 0$ such that

$$\|\Lambda u\|_{H^1(\Omega)} = \|\eta u\|_{H^1(U')} \leq C\|\eta\|_{W^{1,\infty}(\mathbb{R}^d)}\|u\|_{H^1(U')}, \quad u \in H^1(U'). \tag{1.172}$$

Therefore Λ is bounded, which concludes the proof. □

Corollary 1.7.3. *In the setting of Lemma 1.7.2 let $u \in H^1(U')$. Then for every open neighbourhood $V \Subset U$ of Γ_C we can find a sequence Cauchy sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\restriction_{U'}$ in $H^1(U')$ such that*

$$\phi_n\restriction_{U'\cap V} \xrightarrow{n\to\infty} u\restriction_{U'\cap V} \quad \text{in } H^1(U'\cap V). \tag{1.173}$$

Proof. From Lemma 1.7.2 we obtain $\Lambda u \in H^1(\Omega)$ with

$$\Lambda u = u \quad \text{on } V, \tag{1.174}$$

where Ω is a Lipschitz hypograph. Therefore by Lemma 1.5.2 we obtain a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ such that

$$\phi_n \xrightarrow{n\to\infty} \Lambda u \quad \text{in } H^1(\Omega). \tag{1.175}$$

This yields

$$\phi_n \restriction_{U'\cap V} \xrightarrow{n\to\infty} \Lambda u \restriction_{U'\cap V} = u \restriction_{U'\cap V} \qquad \text{in } H^1(U'\cap V), \tag{1.176}$$

which shows that $(\phi_n \restriction_{U'})_{n\to\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\restriction_{U'}$ satisfied all the required properties. □

Since Ω as in (1.165) is a Lipschitz hypograph and thus minimally smooth, the Dirichlet trace $\gamma_D : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is well-defined, see Theorem 1.5.1.

Therefore, in the setting of Lemma 1.7.2, we can define a Dirichlet trace on Γ_C as

$$\gamma_{D,\Gamma_C} : H^1(U') \to H^{\frac{1}{2}}(\Gamma_C), \quad \gamma_{D,\Gamma_C} u = (\gamma_D \Lambda u)\restriction_{\Gamma_C}, \quad u \in H^1(U'). \tag{1.177}$$

Lemma 1.7.4. *In the setting of Lemma 1.7.2 the Dirichlet trace γ_{D,Γ_C} from (1.177) is well-defined, linear, bounded, independent of the specific choice of Λ, and satisfies*

$$\gamma_{D,\Gamma_C}\phi = \phi\restriction_{\Gamma_C}, \quad \phi \in C(\overline{U'}) \cap H^1(U'). \tag{1.178}$$

Proof. With Theorem 1.5.1 and Lemma 1.7.2 we immediately see that this is a well-defined, bounded and linear operator.

Let now $V \Subset \Omega$ be an open neighbourhood of Γ_C and let Λ be as in Lemma 1.7.2. For $\phi \in C(\overline{U'})$ we have that $\Lambda\phi \in C(\overline{\Omega})$, see Lemma 1.7.2. It holds true that

$$\Lambda u = u \quad \text{on } V \cap U', \qquad u \in H^1(U'), \tag{1.179}$$

and consequently since ϕ and $\Lambda\phi$ are continuous and $\Gamma_C \subset V$ also

$$\Lambda\phi = \phi \quad \text{on } \Gamma_C. \tag{1.180}$$

The Dirichlet trace γ_D of continuous function up to the boundary is just the restriction to the boundary, see (1.102), so we see that

$$\begin{aligned} \gamma_{D,\Gamma_C}\phi &= (\gamma_D \Lambda\phi)\restriction_{\Gamma_C} \\ &= ((\Lambda\phi)\restriction_{\partial\Omega})\restriction_{\Gamma_C} = (\Lambda\phi)\restriction_{\Gamma_C} = \phi\restriction_{\Gamma_C} . \end{aligned} \tag{1.181}$$

It remains to show that γ_{D,Γ_C} is independent of the concrete choice of Λ. To this end, assume that for two open neighbourhoods $V_1, V_2 \Subset U$ we have two extension operators Λ_1, Λ_2 as in Lemma 1.7.2, and let $u \in H^1(U')$.

It holds true that $\Lambda_1 u, \Lambda_2 u \in H^1(\Omega)$, and since Ω is minimally smooth there exists sequences $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ and $(\psi_n \in \mathbb{N}) \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ such that

$$\phi_n \xrightarrow{n\to\infty} \Lambda_1 u \text{ in } H^1(\Omega), \quad \psi_n \xrightarrow{n\to\infty} \Lambda_2 u \text{ in } H^1(\Omega), \tag{1.182}$$

see Lemma 1.5.2.

Also by assumption on Λ_1, Λ_2 we have that

$$(\Lambda_1 u)\restriction_{V_1} = u\restriction_{V_1}, \quad (\Lambda_2 u)\restriction_{V_2} = u\restriction_{V_2}. \tag{1.183}$$

Consequently $V = V_1 \cap V_2$ of Γ_C is also a open neighbourhood of Γ_C and it holds true that

$$(\Lambda_1 u)\restriction_V = (\Lambda_2 u)\restriction_V. \tag{1.184}$$

We now choose an open neighbourhood $\tilde{V} \Subset V$ of Γ_C and a function $\eta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in \tilde{V}, \\ 0 & \text{for } x \notin V. \end{cases} \tag{1.185}$$

From (1.184) we see that

$$\eta\Lambda_1 u = \eta\Lambda_2 u, \tag{1.186}$$

and Lemma 1.2.10 shows that

$$\begin{aligned} \|\eta(\phi_n - \psi_n)\|_{H^1} &\leq \|\eta\phi_n - \eta\Lambda_1 u\|_{H^1} + \|\eta\Lambda_1 u - \eta\Lambda_2 u\|_{H^1} + \|\eta\Lambda_2 u - \eta\psi_n\|_{H^1} \\ &\leq C\|\phi_n - \Lambda_1 u\|_{H^1} + C\|\psi_n - \Lambda_2 u\|_{H^1} \xrightarrow{n\to\infty} 0, \end{aligned} \tag{1.187}$$

with some $C > 0$.

Note that $\eta = 1$ in an neighbourhood of Γ_C, so since η, $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ are smooth functions it holds true that

$$(\gamma_D \eta\phi_n)\restriction_{\Gamma_C} = (\eta\phi_n)\restriction_{\Gamma_C} = \phi_n\restriction_{\Gamma_C} = (\gamma_D\phi_n)\restriction_{\Gamma_C}, \quad n \in \mathbb{N}, \tag{1.188}$$

and

$$(\gamma_D \eta\psi_n)\restriction_{\Gamma_C} = (\eta\psi_n)\restriction_{\Gamma_C} = \psi_n\restriction_{\Gamma_C} = (\gamma_D\psi_n)\restriction_{\Gamma_C}, \quad n \in \mathbb{N}. \tag{1.189}$$

With the continuity of the Dirichlet trace, see Theorem 1.5.1, we now conclude that

$$\begin{aligned} (\gamma_D\Lambda_1 u)\restriction_{\Gamma_C} &= \left(\lim_{n\to\infty}\gamma_D\phi_n\right)\restriction_{\Gamma_C} = \lim_{n\to\infty}(\gamma_D\phi_n\restriction_{\Gamma_C}) = \lim_{n\to\infty}(\gamma_D\eta\phi_n\restriction_{\Gamma_C}) \\ &= \lim_{n\to\infty}(\gamma_D\eta\psi_n\restriction_{\Gamma_C}) = \lim_{n\to\infty}(\gamma_D\psi_n\restriction_{\Gamma_C}) = \left(\lim_{n\to\infty}\gamma_D\psi_n\right)\restriction_{\Gamma_C} \\ &= (\gamma_D\Lambda_2 u)\restriction_{\Gamma_C}, \end{aligned} \tag{1.190}$$

where the second and the next-to-last equality holds due to Remark 1.6.3. This proves the uniqueness. □

Corollary 1.7.5. *In the setting of Lemma 1.7.4, if we have $u, v \in H^1(U')$ such that there exists an open neighbourhood N of Γ_C with $u = v$ on $U' \cap N$, then it holds true that*

$$\gamma_{D,\Gamma_C} u = \gamma_{D,\Gamma_C} v. \tag{1.191}$$

Proof. Lemma 1.7.4 states that γ_{D,Γ_C} is independent of the choice of the extension operator Λ in (1.177). The idea of the proof is to construct an extension operator for which the assertion follows naturally.

Since N is an open neighbourhood of the compact set Γ_C we can find an open neighbourhood $\tilde{N}$ of Γ_C with $\tilde{N} \Subset N$. Now we choose $\eta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in \tilde{N}, \\ 0 & \text{for } x \notin N, \end{cases} \tag{1.192}$$

and define the extension operator

$$\Lambda : H^1(U') \to H^1(\Omega) \tag{1.193}$$

with

$$\Lambda f = \begin{cases} \eta f & \text{on } \tilde{U}', \\ 0 & \text{else}, \end{cases} \qquad f \in H^1(U'). \tag{1.194}$$

Notice that this is exactly the type of construction for Λ which we also used in the proof of Lemma 1.7.2, so by the analogous arguments we see that Λ is an extension which does have all the properties claimed in Lemma 1.7.2, and therefore by Lemma 1.7.4 we see that

$$\gamma_{D,\Gamma_C} = \gamma_D \Lambda. \tag{1.195}$$

We have $u = v$ on N and η is zero outside of N, thus it follows that $\Lambda u = \Lambda v$, so consequently

$$\gamma_{D,\Gamma_C} u = \gamma_D \Lambda u = \gamma_D \Lambda v = \gamma_{D,\Gamma_C} v, \tag{1.196}$$

which concludes the proof. □

Next we want to show how this Dirichlet traces transform under change of variables. For this to make sense we will be interested in diffeomorphisms which preserve the structure of the domain, i.e. we want the image to be again divided into three parts by a Lipschitz function as in Definition 1.7.1.

Definition 1.7.6. *Let $D, D' \subset \mathbb{R}^d$ be non-empty, open sets, let $U \Subset D$ be a domain divided into three parts by a Lipschitz function as in Definition 1.7.1. If $\kappa : D \to D'$ is a C^1-diffeomorphism such that $V = \mathrm{Ran}\kappa \restriction_U$ is also divided into three parts by a Lipschitz function with $V' = \mathrm{Ran}\kappa \restriction_{U'}$, $\Gamma^{(V)} = \mathrm{Ran}\kappa \restriction_\Gamma$ and $V'' = \mathrm{Ran}\kappa \restriction_{U''}$, or alternatively $V' = \mathrm{Ran}\kappa \restriction_{U''}$, $\Gamma^{(V)} = \mathrm{Ran}\kappa \restriction_\Gamma$ and $V'' = \mathrm{Ran}\kappa \restriction_{U'}$, we say that κ is structure preserving for U.*

A visualisation of a structure preserving diffeomorphism as in Definition 1.7.6 is given in Figure 1.3.

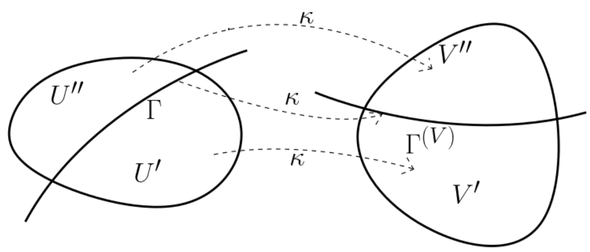

Figure 1.3: Structure preserving diffeomorphism as in Definition 1.7.6

Lemma 1.7.7. *Let $D, D' \subset \mathbb{R}^d$ be non-empty, open sets, let $U \Subset D$ be a domain divided into three parts by a Lipschitz function as in Definition 1.7.1 and let $\kappa : D \to D'$ be a structure preserving C^1-diffeomorphism for U.*

Then it holds true that for every compact set $\Gamma_C \subset \Gamma$ and $u \in H^1(\mathrm{Ran}\kappa\restriction_{U'})$ that

$$\gamma_{D,\Gamma_C}(u \circ \kappa) = (\gamma_{D,\mathrm{Ran}\kappa\restriction_{\Gamma_C}} u) \circ \kappa. \tag{1.197}$$

Proof. Since continuous functions map compact sets to compact sets it holds true that $\mathrm{Ran}\kappa\restriction_{\Gamma_C}$ is compact and it is a subset of $\Gamma^V := \mathrm{Ran}\kappa\restriction_{\Gamma}$.

By assumption Γ^V is part of the graph of a Lipschitz function and it is the boundary between the sets

$$V' = \mathrm{Ran}\kappa\restriction_{U'}, \quad V'' = \mathrm{Ran}\kappa\restriction_{U''}, \tag{1.198}$$

so the notion of Dirichlet trace on the left side of (1.197) makes sense.

In this notation let $u \in H^1(V')$. Due to Corollary 1.7.3 we can find a Cauchy sequence $(\phi_n)_{n\to\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\restriction_{V'}$ in $H^1(V')$ such that in some open neighbourhood N of Γ_C it holds true that

$$\phi_n\restriction_{N\cap U'} \xrightarrow{n\to\infty} u\restriction_{N\cap U'} \qquad \text{on } H^1(N\cap V'). \tag{1.199}$$

Since it is a Cauchy sequence in $H^1(V')$ is has a limit, and (1.199) shows that this limit has to coincide on $N \cap U'$ with u, therefore we can use Corollary 1.7.5 and the boundedness of $\gamma_{D,\mathrm{Ran}\kappa\restriction_{\Gamma_C}}$ from Lemma 1.7.4 to obtain

$$\gamma_{D,\mathrm{Ran}\kappa\restriction_{\Gamma_C}} u = \lim_{n\to\infty} \gamma_{D,\mathrm{Ran}\kappa\restriction_{\Gamma_C}} \phi_n = \lim_{n\to\infty} \phi_n\restriction_{\mathrm{Ran}\kappa\restriction_{\Gamma_C}}, \tag{1.200}$$

where we used in the last step that the Dirichlet trace $\gamma_{D,\mathrm{Ran}\kappa\restriction_{\Gamma_C}}$ of continuous functions is just their restriction to the boundary, see Lemma 1.7.4.

Now since κ is a C^1-diffeomorphism we see that $(\phi_n \circ \kappa)_{n\in\mathbb{N}} \subset C(\overline{U'})$. Furthermore Lemma 1.6.2 states that $(\phi_n \circ \kappa)_{n\in\mathbb{N}} \subset H^1(U')$ and

$$\phi_n \circ \kappa \xrightarrow{n\to\infty} u \circ \kappa \quad \text{on } H^1(\mathrm{Ran}\kappa^{-1} \restriction_{N\cap V'}). \tag{1.201}$$

Since N is an open neighbourhood of $\mathrm{Ran}\kappa \restriction_{\Gamma_C}$ and κ is a C^1-diffeomorphism, we see that $\mathrm{Ran}\kappa^{-1} \restriction_N$ is an open neighbourhood of Γ_C, and so by the same arguments as above we obtain

$$\gamma_{D,\Gamma_C}(u\circ\kappa) = \lim_{n\to\infty} \gamma_{D,\Gamma_C} u(\phi_n \circ \kappa) = \lim_{n\to\infty} \phi_n \circ \kappa \restriction_{\Gamma_C}. \tag{1.202}$$

In summary, (1.200) and (1.202) together show that

$$\gamma_{D,\Gamma_C}(u\circ\kappa) = \lim_{n\to\infty} \phi_n \circ \kappa \restriction_{\Gamma_C} = \lim_{n\to\infty} \phi_n \restriction_{\mathrm{Ran}\kappa\restriction_{\Gamma_C}} \circ\kappa = (\gamma_{D,\mathrm{Ran}\kappa\restriction_{\Gamma_C}} u)\circ\kappa, \tag{1.203}$$

where the limits coincide due to Remark 1.6.3. □

Next we want to show a special case of the divergence theorem in the weak sense.

Lemma 1.7.8. *Let $U \subset \mathbb{R}^d$ be a bounded domain divided into three parts as in Definition 1.7.1, and let $u \in H^1(U')$. Then for every $\phi \in C_0^\infty(U)$ it holds true that* $\mathrm{supp}\,\phi \cap \Gamma$ *is compact and*

$$\int_{U'} \partial_j(u\phi)\mathrm{d}x = \int_{\mathrm{supp}\,\phi\cap\Gamma} (\gamma_{D,\mathrm{supp}\,\phi\cap\Gamma} u)\phi\nu_j \mathrm{d}\sigma(x'), \quad j \in \{1, ..., d\}, \tag{1.204}$$

where σ and ν denote the boundary measure and the unit outwards vector from (1.52), and $\gamma_{D,\mathrm{supp}\,\phi\cap\Gamma}$ is as defined in (1.177).

Proof. First we need to show that $\mathrm{supp}\,\phi \cap \Gamma$ is compact. Let

$$\partial\Omega = \{x \in \mathbb{R}^d : \, x_d = \xi(x_1, ..., x_d)\}, \tag{1.205}$$

where $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ is as in Definition 1.7.1. Γ is defined as

$$\Gamma = \{x \in U : \, x_d = \xi(x_1, ..., x_d)\} = U \cap \partial\Omega, \tag{1.206}$$

and since $\mathrm{supp}\,\phi \subset U$ we see that

$$\mathrm{supp}\,\phi \cap \Gamma = \mathrm{supp}\,\phi \cap \partial\Omega, \tag{1.207}$$

which is certainly closed as an intersection of closed sets, and bounded since $\mathrm{supp}\,\phi$ is, so in conclusion we see that $\mathrm{supp}\,\phi \cap \Gamma$ is compact. Therefore the Dirichlet trace $\gamma_{D,\mathrm{supp}\,\phi\cap\Gamma}$ is well-defined.

For now assume that $u \in C_0^\infty(\mathbb{R}^d)\restriction_{U'}$ and $\phi \in C_0^\infty(U)$. For $\Gamma_C = \Gamma \cap \operatorname{supp}\phi$ and Ω from (1.165) we pick an extension operator $\Lambda : H^1(U') \to H^1(\Omega)$ as in Lemma 1.7.2 which satisfies

$$\Lambda v = v \quad \text{on } \operatorname{supp}\phi \cap U', \quad v \in H^1(U'). \tag{1.208}$$

Notice that by construction in the proof of Lemma 1.7.2, Λ maps smooth functions to smooth function. This yields that $\Lambda u \in C^\infty(\overline{\Omega})$.

Now we define

$$F^{(j)} : \Omega \to \mathbb{R}^d, \quad F_i^{(j)} = \begin{cases} (\Lambda u)\phi & \text{for } i = j, \\ 0 & \text{else,} \end{cases} \qquad i, j \in \{1, ..., d\}, \tag{1.209}$$

where we implicitly assumed a zero-extension of ϕ. Since both Λu and ϕ are smooth and ϕ has compact support it holds true that the $(F^{(j)})_{j\in\{1,...,d\}}$ are C^∞ vector fields with compact support and thus we obtain from [25, Theorem 3.34, the case of Lipschitz hypographs is explicitly treated in the proof] that

$$\int_\Omega \operatorname{div} F^{(j)} \mathrm{d}x = \int_{\partial\Omega} F^{(j)} \cdot \nu \mathrm{d}\sigma(x'), \quad j \in \{1, ..., d\}. \tag{1.210}$$

Plugging in the definition of the vector fields $(F^{(j)})_{j\in\{1,...,d\}}$ gives

$$\int_\Omega \partial_j((\Lambda u)\phi)\mathrm{d}x = \int_{\partial\Omega} (\Lambda u)\phi\nu_j \mathrm{d}\sigma(x'), \quad j \in \{1, ..., d\}. \tag{1.211}$$

Using (1.208) we see that

$$\begin{aligned} \int_{U'} \partial_j(u\phi)\mathrm{d}x &= \int_{\Gamma\cap\operatorname{supp}\phi} u\phi\nu_j \mathrm{d}\sigma(x') \\ &= \int_{\Gamma\cap\operatorname{supp}\phi} (\gamma_{D,\Gamma\cap\operatorname{supp}\phi} u)\phi\nu_j \mathrm{d}\sigma(x'), \quad j \in \{1, ..., d\}, \end{aligned} \tag{1.212}$$

where the last equation follows from Lemma 1.7.4 since u is smooth.

Now let $u \in H^1(U')$. Due to Corollary 1.7.3 we can find a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\restriction_{U'}$ such that

$$\psi_n \xrightarrow{n\to\infty} u \quad \text{in } H^1(U' \cap \operatorname{supp}\phi). \tag{1.213}$$

From this we see that

$$\partial_j(\psi_n\phi) = \psi_n\partial_j\phi + \phi\partial_j\psi_n \xrightarrow{n\to\infty} u\partial_j\phi + \phi\partial_j u = \partial_j(u\phi) \quad \text{in } L^2(U' \cap \operatorname{supp}\phi), \tag{1.214}$$

and therefore also in $L^1(U' \cap \operatorname{supp}\phi)$, since $\operatorname{supp}\phi$ is bounded. Furthermore Corollary 1.7.5 shows that

$$\gamma_{D,\Gamma\cap\operatorname{supp}\phi}\psi_n \xrightarrow{n\to\infty} \gamma_{D,\Gamma\cap\operatorname{supp}\phi} u \quad \text{in } H^{\frac{1}{2}}(\operatorname{supp}\phi \cap \Gamma). \tag{1.215}$$

Since (1.212) holds true for the functions $(\psi_n)_{n\in\mathbb{N}}$ we finally arrive at

$$\begin{aligned}\int_{U'} \partial_j(u\phi)\mathrm{d}x &= \lim_{n\to\infty}\int_{U'} \partial_j(\psi_n\phi)\mathrm{d}x = \lim_{n\to\infty}\int_{\operatorname{supp}\phi\cap\Gamma} (\gamma_{D,\operatorname{supp}\phi\cap\Gamma}\psi_n)\phi\nu_j\mathrm{d}\sigma(x')\\ &= \int_{\operatorname{supp}\phi\cap\Gamma} (\gamma_{D,\operatorname{supp}\phi\cap\Gamma}u)\phi\nu_j\mathrm{d}\sigma(x'),\end{aligned} \tag{1.216}$$

which concludes the proof. □

The next Lemma shows that functions which are piecewise in H^1 are H^1-functions if the Dirichlet trace between the pieces agree.

Lemma 1.7.9. *Let $U \in \mathbb{R}^d$ be a non-empty, open and bounded set, divided into three parts as in Definition 1.7.1, let $p \in [2,\infty]$ and $u \in L^p(U)$ with*

$$\partial_j(u\restriction_{U'}) \in L^p(U'), \quad \partial_j(u\restriction_{U''}) \in L^p(U''), \quad j \in \{1,...,d\}, \tag{1.217}$$

and assume that for every compact subset Γ_C of Γ we have that

$$\gamma_{D,\Gamma_C}(u\restriction_{U'}) = \gamma_{D,\Gamma_C}(u\restriction_{U''}) \quad \text{on } \Gamma_C, \tag{1.218}$$

with γ_{D,Γ_C} from (1.177). Then it holds true that $u \in W^{1,p}(U)$ and

$$\nabla u = \begin{cases}\nabla(u\restriction_{U'}) & \text{on } U',\\ \nabla(u\restriction_{U''}) & \text{on } U''.\end{cases} \tag{1.219}$$

Proof. We will show this directly with the definition of a weak derivative. To this end, let $\phi \in C_0^\infty(U)$. Lemma 1.7.8 shows that $\Gamma\cap\operatorname{supp}\phi$ is compact, hence by (1.218) we have that

$$\gamma_{D,\Gamma\cap\operatorname{supp}\phi}(u\restriction_{U'}) = \gamma_{D,\Gamma\cap\operatorname{supp}\phi}(u\restriction_{U''}) \quad \text{on } \Gamma_C, \tag{1.220}$$

Since $p \in [2,\infty]$ and U is bounded we have that $u\restriction_{U'}\in H^1(U')$, so we can apply Lemma 1.7.8 to obtain that

$$\int_{U'} \nabla(u\restriction_{U'}\phi)\mathrm{d}x = \int_{\operatorname{supp}\phi\cap\Gamma} (\gamma_{D,\Gamma\cap\operatorname{supp}\phi}(u\restriction_{U'}))\phi\nu\mathrm{d}\sigma(x'). \tag{1.221}$$

Analogously one obtains

$$\int_{U''} \nabla(u\restriction_{U''}\phi)\mathrm{d}x = \int_{\operatorname{supp}\phi\cap\Gamma} (\gamma_{D,\Gamma\cap\operatorname{supp}\phi}(u\restriction_{U''}))\phi\nu\mathrm{d}\sigma(x'), \tag{1.222}$$

so with (1.218) we arrive at

$$\int_{U'} \nabla(u\restriction_{U'}\phi)\mathrm{d}x - \int_{U''} \nabla(u\restriction_{U''}\phi)\mathrm{d}x = 0. \tag{1.223}$$

Using the product rule and rearranging everything finally gives

$$\begin{aligned}\int_U u\nabla\phi \mathrm{d}x &= \int_{U'} u\restriction_{U'} \nabla\phi \mathrm{d}x + \int_{U''} u\restriction_{U''} \nabla\phi \mathrm{d}x \\ &= -\int_{U'} (\nabla u\restriction_{U'})\phi \mathrm{d}x - \int_{U''} (\nabla u\restriction_{U''})\phi \mathrm{d}x.\end{aligned} \tag{1.224}$$

Since this holds true for every $\phi \in C_0^\infty(U)$ this gives

$$\nabla u = \begin{cases} \nabla(u\restriction_{U'}) & \text{on } U', \\ \nabla(u\restriction_{U''}) & \text{on } U'', \end{cases} \tag{1.225}$$

and thus $u \in W^{1,p}(U)$ holds true. □

1.8 Differential Geometry, Tubular Mapping and Reflections along the Normal Vector

In order to state the necessary results to proceed, we need to introduce some basic notations from differential geometry. To this end, we will mostly follow [23].

Let $\Omega \subset \mathbb{R}^d$ be a C^∞ hypograph defined by a function ξ. Clearly the boundary $\partial\Omega$ is an embedded submanifold of $\mathbb{R}^d$ since it is globally the graph of a smooth function (although we would only need this property locally). Our global chart from $\mathbb{R}^{d-1}$ to the embedding into $\mathbb{R}^d$ will be

$$F : \mathbb{R}^{d-1} \to \partial\Omega, \quad F(x') = (x', \xi(x')). \tag{1.226}$$

Since $\mathbb{R}^{d-1}$ is a $(d-1)$-dimensional manifold, its tangent space $T_{x'}\mathbb{R}^{d-1}$ at any point $x' \in \mathbb{R}^{d-1}$ is also $(d-1)$-dimensional and in $\mathbb{R}^{d-1}$ a basis is given by

$$\left(\frac{\partial}{\partial x_i}|_{x'}\right)_{i\in\{1,\dots,d-1\}}, \tag{1.227}$$

see [23, Proposition 3.10 and Proposition 3.15]. Using the transformation rule from [23, pages 63 and 64], we see that the vectors

$$\left(\partial F_{x'}\left(\frac{\partial}{\partial x_i}|_{x'}\right)\right)_{i\in\{1,\dots,d-1\}}, \tag{1.228}$$

given by

$$\begin{aligned}\partial F_{x'}\left(\frac{\partial}{\partial x_i}|_{x'}\right) &= \left(\sum_{j=1}^d \frac{\partial F^j}{\partial x_i}\frac{\partial}{\partial y_j}|_{F(x')}\right) \\ &= \frac{\partial}{\partial y_i}|_{F(x')} + \frac{\partial \xi}{\partial x_i}|_{x'}\frac{\partial}{\partial y_d}|_{F(x')}, \qquad i \in \{1,\dots,d-1\},\end{aligned} \tag{1.229}$$

form a basis of the tangent space $T_{F(x')}\partial\Omega$. In the canonical embedding into $\mathbb{R}^d$ we can write them as vectors $(v^i)_{i\in\{1,...,d-1\}}$ with

$$v^i_j = \begin{cases} 1 & \text{for } j = i, \\ \frac{\partial \xi}{\partial x_i}(x') & \text{for } j = d, \\ 0 & \text{else,} \end{cases} \qquad i \in \{1, ..., d-1\}. \tag{1.230}$$

The space normal space $N_x\partial\Omega$ at a point $x \in \partial\Omega$ is the space of vectors orthogonal to $(v^i)_{i\in\{1,...,d-1\}}$, which is only 1-dimensional and spanned by

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla\xi(x_1, ..., x_{d-1})|^2}}(-\nabla\xi(x_1, ..., x_{d-1}), 1), \tag{1.231}$$

which is the normal vector at $x \in \partial\Omega$, see (1.52).

A normal bundle to a submanifold M of $\mathbb{R}^d$ is given by

$$NM = \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d : x \in M, v \in N_xM\}. \tag{1.232}$$

The normal bundle of $\partial\Omega$ is therefore given by

$$\begin{aligned} N\partial\Omega &= \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d : x \in \partial\Omega, v \in N_x\partial\Omega\} \\ &= \{(x, c\nu(x)) \in \mathbb{R}^d \times \mathbb{R}^d : x \in \partial\Omega, c \in \mathbb{R}\}. \end{aligned} \tag{1.233}$$

Furthermore, we consider mappings

$$E : NM \to \mathbb{R}^d, \quad E(x, v) = x + v, \tag{1.234}$$

which in our case is given by

$$E : N\partial\Omega \to \mathbb{R}^d, \quad E(x, c\nu(x)) = x + c\nu(x), \quad c \in \mathbb{R}. \tag{1.235}$$

With all of that, we can now define tubular neighbourhoods.

Definition 1.8.1. *Let M be a smooth submanifold of $\mathbb{R}^d$ and assume that there is a smooth diffeomorphism between the set*

$$V = \{(x, v) \in NM : |v| < \delta(x)\}, \tag{1.236}$$

where $\delta : M \to \mathbb{R}$ is a positive, continuous function, and its image under E. Then we call $E(V)$ a tubular neighbourhood of M.

Theorem 1.8.2 (Tubular Neighbourhood Theorem)**.** *Every embedded submanifold of $\mathbb{R}^d$ has a tubular neighbourhood.*

Proof. [23, Theorem 6.24]. □

Given our particular setting, we can rewrite this Theorem to state the following result.

Corollary 1.8.3. *Let $\Omega \subset \mathbb{R}^d$ be a C^∞-hypograph defined by a function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$. Then there exists a continuous, positive function $\delta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that the tubular mapping*

$$\begin{aligned} &\mathcal{T} : \mathrm{Dom}\mathcal{T} = \{(x', t) \in \mathbb{R}^{d-1} \times \mathbb{R} : \, |t| < \delta(x')\} \to \mathrm{Ran}\mathcal{T}, \\ &\mathcal{T}(x', t) = (x', \xi(x')) + t\nu((x', \xi(x'))), \end{aligned} \tag{1.237}$$

is a smooth diffeomorphism.

In other words, for a C^∞-hypograph with boundary Γ, there exists a neighbourhood in which every point can be uniquely written in the from $x + t\nu(x)$, where $x \in \Gamma, t \in \mathbb{R}$ and $\nu(x)$ is the unit normal vector outwards of the hypograph. We can use this in order to define the reflection along the normal vector in the tubular neighbourhood.

Definition 1.8.4. *In the setting of Corollary 1.8.3 the reflection along the normal vector $\mathcal{R}$ is given by*

$$\begin{aligned} &\mathcal{R} : \mathrm{Dom}\mathcal{R} = \mathrm{Ran}\mathcal{T} \to \mathrm{Ran}\mathcal{T}, \\ &\mathcal{R}(x) = \tilde{\mathcal{T}} \circ \mathrm{diag}(1, ..., 1, -1) \circ \tilde{\mathcal{T}}^{-1}(x). \end{aligned} \tag{1.238}$$

The interpretation is that we take a point in the form $x + t\nu(x)$, where $x \in \Gamma, t \in \mathbb{R}$ and $\nu(x)$ is the unit normal vector outwards of the hypograph, and map it to $x - t\nu(x)$. So if a point $y \in \mathrm{Dom}\mathcal{R}$ was inside the hypograph, then $\mathcal{R}(y)$ is outside and vice versa. The reflection $\mathcal{R}$ is illustrated in Figure 1.4.

Some of the basic properties of this reflection will be summarised in the following Lemma.

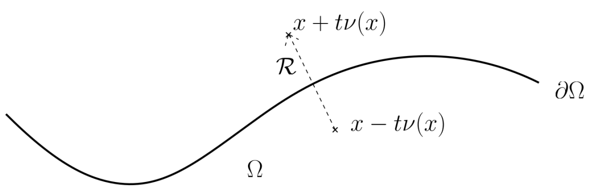

Figure 1.4: Reflection across the normal vector $\mathcal{R}$ as in Definition 1.8.4

Lemma 1.8.5. *Let $\mathcal{R}$ be as in Definition 1.8.4. Then the following assertions hold true.*

1. $\mathcal{R}^{-1} = \mathcal{R}$.
2. $\mathrm{Dom}\mathcal{R}$ *is open and* $\mathcal{R}$ *is a smooth diffeomorphism,*
3. *For all $x \in \Gamma$ we have $\mathcal{R}(x) = x, \det(D\mathcal{R}(x)) = -1$ and $D\mathcal{R}(x)$ is orthogonal.*

Proof. 1. Let $x \in \mathrm{Dom}\mathcal{R}$. Then we have that

$$\begin{aligned}\mathcal{R} \circ \mathcal{R}(x) &= \tilde{\mathcal{T}} \circ \mathrm{diag}(1, ..., 1, -1) \circ \tilde{\mathcal{T}}^{-1} \circ \tilde{\mathcal{T}} \circ \mathrm{diag}(1, ..., 1, -1) \circ \tilde{\mathcal{T}}^{-1}(x) \\ &= \tilde{\mathcal{T}} \circ \mathrm{diag}(1, ..., 1, -1) \circ \mathrm{diag}(1, ..., 1, -1) \circ \tilde{\mathcal{T}}^{-1}(x) \\ &= \tilde{\mathcal{T}} \circ \tilde{\mathcal{T}}^{-1}(x) = x.\end{aligned} \tag{1.239}$$

2. By Definition we have that $\mathrm{Dom}\mathcal{R} = \mathrm{Ran}\mathcal{T}$ and since $\mathcal{T}$ is a diffeomorphism, its range is also open. Since $\mathcal{R}$ is defined as a composition of smooth functions, it is itself smooth and as the first item shows that $\mathcal{R}^{-1} = \mathcal{R}$ it is a smooth diffeomorphism.
3. Let $x \in \Gamma$. Then we can write $x = (x', \xi(x'))$ for some $x' \in \mathbb{R}^{d-1}$, and thus we have that $x = \mathcal{T}(x', 0)$, especially $x \in \mathrm{Ran}\mathcal{T}$. We calculate

$$\begin{aligned}\mathcal{R}(x) &= \tilde{\mathcal{T}} \circ \mathrm{diag}(1, ..., 1, -1) \circ \tilde{\mathcal{T}}^{-1}(x) \\ &= \tilde{\mathcal{T}} \circ \mathrm{diag}(1, ..., 1, -1)(x', 0) = \tilde{\mathcal{T}}(x', 0) = (x', \xi(x')) = x,\end{aligned} \tag{1.240}$$

so $\mathcal{R}(x) = x$.

Let $x \in \Gamma$, so we can write it as $x = (x', \xi(x'))$. By definition it holds true that for $t \in \mathbb{R}$ with $|t| < \delta(x')$ that

$$\mathcal{R}(x + t\nu(x)) = x - t\nu(x), \tag{1.241}$$

so we see that

$$D\mathcal{R}(x)\nu(x) = \lim_{t \to 0} \frac{\mathcal{R}(x + t\nu(x)) - \mathcal{R}(x)}{t} = -\nu(x), \tag{1.242}$$

so $\nu(x)$ is an eigenvector of $D\mathcal{R}(x)$ with eigenvalue -1.

Now for all $i \in \{1, ..., d-1\}$ we define curves and vectors

$$\gamma_i : \mathbb{R} \to \mathbb{R}^d, \quad \gamma_i(s) = (x' + se_i, \xi(x' + se_i)), \quad v_i = \gamma_i'(0), \tag{1.243}$$

where $(e_i)_{i \in \{1,...,d-1\}}$ denotes the canonical basis in $\mathbb{R}^{d-1}$. By definition we have that $\mathrm{Ran}\gamma_i \subset \Gamma$ for all $i \in \{1, ..., d-1\}$. Now we see that, since $\mathcal{R}$ is just the identity on Γ,

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[\mathcal{R}(\gamma_i(s))\big]|_{s=0} = \gamma_i'(0) = v_i, \quad i \in \{1, ..., d-1\}, \tag{1.244}$$

and using the chain rule it also holds true that

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[\mathcal{R}(\gamma_i(s))\big]|_{s=0} = D\mathcal{R}(\gamma_i(0))\gamma_i'(0) = D\mathcal{R}(x)v_i, \quad i \in \{1, ..., d-1\}. \tag{1.245}$$

Thus

$$D\mathcal{R}(x)v_i = v_i, \quad i \in \{1, ..., d-1\}, \tag{1.246}$$

and therefore all the v_i are eigenvectors of $D\mathcal{R}$ with eigenvalue 1.
Finally, we calculate that

$$v_i^j = \begin{cases} 1 & \text{for } j = i, \\ \partial_i \xi(x') & \text{for } j = d, \\ 0 & \text{else}, \end{cases} \qquad i \in \{1, ..., d-1\}, \tag{1.247}$$

so the $(v_i)_{i\in\{1,\dots,d-1\}}$ are clearly linearly independent and also orthogonal to

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla \xi(x')|^2}}(-\nabla \xi(x'), 1), \tag{1.248}$$

so $D\mathcal{R}$ has only the eigenvalues ± 1 and we found orthogonal eigenspaces which together form a basis of $\mathbb{R}^d$. Therefore $D\mathcal{R}$ is orthogonal.
Finally, since the determinant is the product of the eigenvalues, we have

$$\det(D\mathcal{R}(x)) = -1. \tag{1.249}$$

□

The following results about $\mathcal{R}$ will be very helpful when we construct extensions of functions via reflection along the normal vector.

Lemma 1.8.6. *Let $U \subset \mathbb{R}^d$ be an open, non-empty and connected set, divided into three parts U', U'', Γ by a C^∞-function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ as in Definition 1.7.1, and let $\Gamma_C \subset \Gamma$ be compact. Furthermore let $\mathcal{R}$ be the reflection along the normal vector from Definition 1.8.4. Then there exists an open neighbourhood V around Γ_C such that*

1. *$V \Subset \mathrm{Dom}\mathcal{R} \cap U$,*
2. *the Jacobi matrix $D\mathcal{R}$ satisfies*

$$|\det(D\mathcal{R})|\big((D\mathcal{R})^T D\mathcal{R}\big)^{-1} \restriction_V \in W^{1,\infty}(V), \tag{1.250}$$

3. *$v \cdot \Big(|\det(D\mathcal{R}(x))|\big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1} v\Big) \geq \frac{1}{2}|v|^2 \quad v \in \mathbb{R}^d, x \in V.$*

Proof. We can first find open neighbourhoods of Γ_C which satisfy the all items individually, and their intersection will than satisfy all three.

1. It is clear that $\Gamma \subset \mathrm{Dom}\mathcal{R} \cap U$ holds true by the definition of Γ and the tubular mapping. Since it is a intersection of two open domains, $\mathrm{Dom}\mathcal{R} \cap U$ is open and by the compactness of Γ_C there exists a open neighbourhood V_1 of Γ_C with

$$V_1 \Subset \mathrm{Dom}\mathcal{R} \cap U. \tag{1.251}$$

2. We know from Lemma 1.8.5 that on $\Gamma_C \subset \Gamma$ it holds true that $D\mathcal{R}$ is orthogonal with $\det(D\mathcal{R}(x)) = -1$.

   Since $D\mathcal{R}$ is smooth and well-defined in an open neighbourhood around Γ_C we see that there exists an open neighbourhood $\tilde{V}_2$ of Γ_C such that $D\mathcal{R}$ is invertible in $\tilde{V}_2$, and in this neighbourhood we consequently have that

$$|\det(D\mathcal{R}(x))|\big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1} \restriction_{\tilde{V}_2} \in C^\infty(\tilde{V}_2). \tag{1.252}$$

   Now we can choose any open neighbourhood V_2 of Γ_C with $V_2 \Subset \tilde{V}_2$ and obtain that

$$|\det(D\mathcal{R}(x))|\big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1} \restriction_{V_2} \in W^{1,\infty}(V_2), \tag{1.253}$$

   since continuous functions are bounded on compact sets.

3. Since $D\mathcal{R}$ is orthogonal on $\Gamma_C \subset \Gamma$, see Lemma 1.8.5 it holds that

$$|\det(D\mathcal{R}(x))|\big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1} = I, \quad x \in \Gamma_C, \tag{1.254}$$

   and therefore all eigenvalues of $|\det(D\mathcal{R}(x))|\big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1}$ are 1 on Γ_C.

   In the previous item we concluded that this matrix is continuous in some open neighbourhood V_2 around $\Gamma \cap \overline{N}$, and so there exists a smaller open neighbourhood V_3 around Γ_C on which all eigenvalues are larger than $\frac{1}{2}$, or in other terms

$$v^T |\det(D\mathcal{R}(x))|\big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1} v \geq \frac{1}{2}|v|^2 \quad v \in \mathbb{R}^d, x \in V_3. \tag{1.255}$$

Finally it is clear that the set $V = V_1 \cap V_2 \cap V_3$ satisfies all the properties above. □

Lemma 1.8.7. *Let $U \subset \mathbb{R}^d$ be an open, non-empty and connected set, divided into three parts U', U'', Γ by a C^∞-function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ as in Definition 1.7.1, and let $\Gamma_C \subset \Gamma$ be compact. Furthermore let $\mathcal{R}$ be the reflection along the normal vector from Definition 1.8.4. Then for any open neighbourhood $W \subset U \cap \mathrm{Dom}\mathcal{R}$ of Γ_C we have that the set*

$$V = \{x \in W : \mathcal{R}x \in W\} \tag{1.256}$$

is also an open neighbourhood of Γ_C and $\mathcal{R}\restriction_V : V \to V$ is a bijection and a smooth diffeomorphism.

Proof. Since $W \subset \mathrm{Dom}\mathcal{R}$ the set is well defined. From Lemma 1.8.5 we see that $\mathcal{R}$ is just the identity mapping on $\Gamma_C \subset W$, and hence $\Gamma_C \subset V$.

What remains to show is that V is indeed open. To this end, recall from Lemma 1.8.5 that $\mathcal{R} = \mathcal{R}^{-1}$ and let $x \in V \subset W$. Since W is open, there exists some open neighbourhood U_x of x such that $U_x \subset W$.

By definition of V we know that $\mathcal{R}(x) \in V \subset W$, and consequently by the same argument as above there also exists an open neighbourhood $U_{\mathcal{R}(x)}$ around $\mathcal{R}(x)$ which satisfies $U_{\mathcal{R}(x)} \subset W$.

$\mathcal{R}^{-1} = \mathcal{R}$ is defined on W, and since $\mathcal{R}$ is continuous it holds that the preimage $\mathcal{R}^{-1}(U_{\mathcal{R}(x)})$ is an open neighbourhood around x, thus we have that

$$U'_x = U_x \cap \mathcal{R}^{-1}(U_{\mathcal{R}(x)}) \tag{1.257}$$

is also an open neighbourhood around x.

For $x \in U'_x$ we have $x \in U_x \subset W$ and $\mathcal{R}(x) \in U_{\mathcal{R}(x)} \subset W$, thus $x \in V$, and so $U'_x \subset V$, which proves that V is open.

By the definition of V it is clear that $\mathcal{R}$ maps V into V. Also for $y \in V$ we see that with $x = \mathcal{R}(y) \in V$ we have that

$$\mathcal{R}(x) = \mathcal{R} \circ \mathcal{R}(y) = \mathcal{R} \circ \mathcal{R}^{-1}(y) = y, \tag{1.258}$$

since Lemma 1.8.5 states that $\mathcal{R} = \mathcal{R}^{-1}$. Therefore $\mathcal{R} : V \to V$ is surjective, and since $\mathcal{R}^2$ is just the identity mapping it is also injective.

Now we now that $\mathcal{R}$ and its inverse are smooth and so is every restriction, thus $\mathcal{R} : V \to V$ is a smooth diffeomorphism. □

1.9 Elliptic regularity result

Lemma 1.9.1. *Let $p \in (1, 2]$ and $u \in H^1_0(\mathbb{R}^d)$ with compact essential support and*

$$\operatorname{div}[A\nabla u] \in L^p(\mathbb{R}^d), \tag{1.259}$$

where $A \in W^{1,\infty}(\mathbb{R}^d)$ is a $d \times d$-matrix which satisfies the uniform ellipticity condition

$$\exists \delta > 0 : \quad y \cdot A(x)y > \delta |y|^2, \quad \forall x, y \in \mathbb{R}^d. \tag{1.260}$$

Then it holds that $u \in W^{2,p}(\mathbb{R}^d)$.

Proof. First note that since $u \in H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d)$ and the essential support of u is compact, we have that also $u \in W^{1,p}(\mathbb{R}^d)$. The rest follows from [26, Corollary 2.3], where we use $m = k = 1$ and $\Omega = \mathbb{R}^d$ and rewrite (HD1) as the ellipticity condition (1.260). □

1.10 Dual Spaces, Bidual Spaces and Adjoints

In this section we summarise the definitions and basic properties of dual and bidual spaces. They will play an important role throughout the thesis.

Definition 1.10.1. *Let B be a Banach space. Then its dual space B^* is defined as the space of all anti-linear and bounded functionals on B, i.e.*

$$B^* = \{f : B \to \mathbb{C},\ f \text{ anti-linear and bounded}\}, \tag{1.261}$$

equipped with the dual norm

$$\|f\|_{B^*} = \sup_{0 \neq b \in B} \frac{|f(b)|}{\|b\|_B}, \quad f \in B^*. \tag{1.262}$$

We usually write $(f, b)_{B^* \times B}$ instead of $f(b)$ for $b \in B$ and $f \in B^*$.

Lemma 1.10.2. *Let B be a Banach space. Then B^* is a Banach space.*

Proof. See [27, Theorem 4.3]. □

Lemma 1.10.3. *Let B be a Banach space and $b \in B$. Then there exists $f \in B^*$ such that*

$$(f, b)_{B^* \times B} = \|b\|_B, \quad \|f\|_{B^*} = 1. \tag{1.263}$$

Consequently we have that $b = 0$ holds if and only if $(f, b)_{B^* \times B} = 0$ for all $f \in B^*$.

Proof. See [27, Corollary of Theorem 3.3]. □

Lemma 1.10.4. *Let $\mathcal{H}$ be a Hilbert space. Then the mapping*

$$\iota_R : \mathcal{H} \to \mathcal{H}^*, \quad (\iota_R u, v)_{\mathcal{H}^* \times \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}, \qquad u, v \in \mathcal{H}, \tag{1.264}$$

is an isometric isomorphism, the so called Riesz isomorphism. In particular we see that $\mathcal{H}^*$ can be interpreted as a Hilbert space with the inner product

$$\langle f, h \rangle_{\mathcal{H}^*} = \langle \iota_R^{-1} f, \iota_R^{-1} h \rangle_{\mathcal{H}}, \qquad f, h \in \mathcal{H}^*. \tag{1.265}$$

Proof. See [27, Theorem 12.5]. □

Definition 1.10.5. *The bidual space B^{**} of B is the dual space of B^*. Furthermore we define the bidual mapping $J : B \to B^{**}$ via*

$$(Jb, f)_{B^{**} \times B^*} = \overline{(f, b)_{B^* \times B}}, \quad f \in B^*,\ b \in B. \tag{1.266}$$

Lemma 1.10.6. *The bidual mapping J from Definition 1.10.5 is injective, linear, isometric and has closed range.*

Proof. See [27, Remarks in Chapter 4.5]. □

The question remains whether J from Definition 1.10.5 is also surjective. In general this is not true, as remarked in [27, Chapter 4.5], so the following definition makes sense.

Definition 1.10.7. *We call a Banach space B reflexive if the bidual mapping $J : B \to B^{**}$ from Definition 1.10.5 is an isometric isomorphism.*

Lemma 1.10.8. *Hilbert spaces are reflexive.*

Proof. Let $\mathcal{H}$ be a Hilbert space. Then from Lemma 1.10.4 we obtain that there exists the isometric Riesz isomorphism ι_R which satisfies

$$(f,u)_{\mathcal{H}^*\times\mathcal{H}} = \langle \iota_R^{-1} f, u\rangle_{\mathcal{H}} = \langle f, \iota_R u\rangle_{\mathcal{H}^*} = \overline{\langle \iota_R u, f\rangle_{\mathcal{H}^*}}, \quad u \in \mathcal{H},\ f \in \mathcal{H}^*. \tag{1.267}$$

Lemma 1.10.4 also states that $\mathcal{H}^*$ is itself a Hilbert space and so there is also another Riesz isomorphism $\iota_R' : \mathcal{H}^* \to \mathcal{H}^{**}$, which satisfies

$$(\iota_R' g, f)_{\mathcal{H}^{**}\times\mathcal{H}^*} = \langle g, f\rangle_{\mathcal{H}^*}, \quad g, f \in \mathcal{H}^*, \tag{1.268}$$

and thus with (1.267) we obtain that

$$(\iota_R'\iota_R u, f)_{\mathcal{H}^{**}\times\mathcal{H}^*} = \langle \iota_R u, f\rangle_{\mathcal{H}^*} = \overline{(f,u)_{\mathcal{H}^*\times\mathcal{H}}}, \quad u \in \mathcal{H},\ f \in \mathcal{H}^*, \tag{1.269}$$

which shows that $J = \iota_R'\iota_R$ with the bidual mapping J as in Definition 1.10.5.

Since both Riesz isomorphisms are isomorphic, so is J and thus $\mathcal{H}$ is reflexive. □

Definition 1.10.9. *Let B, Y be Banach spaces and $T : \mathrm{Dom}T \subset B \to Y$ be a densely defined operator. The adjoint $T^* : \mathrm{Dom}T^* \subset Y^* \to B^*$ of T is given by*

$$\begin{aligned} \mathrm{Dom}T^* &= \{f \in Y^* : \exists \eta \in B^* \ s.t.\ (\eta, b)_{B^*\times B} = (f, Tb)_{Y^*\times Y}\ \forall b \in B\}, \\ T^* &= \eta. \end{aligned} \tag{1.270}$$

Lemma 1.10.10. *Let B, Y be Banach spaces and $T : \mathrm{Dom}T \subset B \to Y$ be a densely defined operator. If T^* is surjective, then T is injective.*

Proof. In this given setting consider $b \in \mathrm{Dom}T$ with $Tb = 0$. Then by Lemma 1.10.3 we see that

$$(f, Tb)_{Y^*\times Y} = 0, \quad f \in Y^*. \tag{1.271}$$

Since T^* is surjective, for every $g \in B^*$ there exists $f \in \mathrm{Dom}T^*$ such that $T^* f = g$, so we see that

$$(g, b)_{B^*\times B} = (T^* f, b)_{B^*\times B} = (f, Tb)_{Y^*\times Y} = 0, \tag{1.272}$$

and since this holds for every $g \in B^*$, Lemma 1.10.3 shows that $b = 0$, thus T is injective. □

In the same form as we defined a bidual space as the dual of the dual space, we might also be interested in the adjoint of the adjoint. As the following lemma shows it has a particularly nice form.

Lemma 1.10.11. *Let B, Y be reflexive Banach spaces and $T : \mathrm{Dom}T \subset B \to Y$ be a densely defined operator. Then T^* is densely defined if and only if T is closable, and in this case it holds true that*

$$T^{**} = J_Y \overline{T} J_B^{-1}, \tag{1.273}$$

where $J_B : B \to B^{**}$ and $J_Y : Y \to Y^{**}$ denote the bidual mappings from Definition 1.10.5.

Proof. See [12, Theorem III.1.5]. There a slightly different notation is used, where the bidual mappings are not written explicitly. □

1.11 Rigged Hilbert spaces

Rigged Hilbert spaces will play an important role in constructing boundary triples and showing that they satisfy all the properties we need. Therefore, we will introduce some basic concepts and results, more details can be found in [6].

Definition 1.11.1. *Let $\mathcal{G}$, $\mathcal{H}$ be Hilbert spaces such that $\mathcal{G}$ is continuously and densely embedded in $\mathcal{H}$. Then we call $\{\mathcal{G}, \mathcal{H}, \mathcal{G}^*\}$ a rigged Hilbert space.*

Lemma 1.11.2. *Let $\{\mathcal{G}, \mathcal{H}, \mathcal{G}^*\}$ be a rigged Hilbert space and $\iota : \mathcal{G} \to \mathcal{H}$ is the continuous embedding of $\mathcal{G}$ in $\mathcal{H}$ with dense range. Then the dual $\iota^* : \mathcal{H} \to \mathcal{G}^*$, defined by*

$$(\iota^* h, g)_{\mathcal{G}^* \times \mathcal{G}} = \langle h, \iota g \rangle_{\mathcal{H}}, \quad h \in \mathcal{H},\ g \in \mathcal{G}, \tag{1.274}$$

is also a continuous embedding with dense range. Furthermore $\iota^{**} : \mathcal{G}^{**} \to \mathcal{H}$ is given by

$$\iota^{**} = \iota J^{-1}, \tag{1.275}$$

where $J : \mathcal{G} \to \mathcal{G}^{**}$ is the natural embedding of $\mathcal{G}$ into its bidual space from Definition 1.10.5.

Proof. See [6, Chapter 8.1]. □

Lemma 1.11.3. *Let $\{\mathcal{G}, \mathcal{H}, \mathcal{G}^*\}$ be a rigged Hilbert space. Then there exist isometric isomorphisms $\iota_+ : \mathcal{G} \to \mathcal{H}$, $\iota_- : \mathcal{H} \to \mathcal{G}^*$, such that*

$$(u, v)_{\mathcal{G}^* \times \mathcal{G}} = \langle \iota_- u, \iota_+ v \rangle_{\mathcal{H}}, \quad u \in \mathcal{G}^*,\ v \in \mathcal{G}. \tag{1.276}$$

Proof. See [6, Lemma 8.1.2 (ii)]. □

1.12 Representation Theorem

In the following let us consider Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}$ and $\mathcal{H}_2$ such that there exist continuous embeddings

$$\iota_1 : \mathcal{H}_1 \to \mathcal{H}, \quad \iota_2 : \mathcal{H} \to \mathcal{H}_2, \tag{1.277}$$

with dense range. In this situation, we can formulate the following representation theorem that will be one of our main tools for proving results in later chapters.

Theorem 1.12.1. *Let $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be boundedly invertible. We define an operator in $\mathcal{H}$ by*

$$\begin{aligned} \mathrm{Dom} A &= \{u \in \mathrm{Ran}\iota_1 : \hat{A}\iota_1^{-1}u \in \mathrm{Ran}\iota_2\}, \\ A &= \iota_2^{-1}\hat{A}\iota_1^{-1}. \end{aligned} \tag{1.278}$$

Then the following holds true:

1. $\mathrm{Dom}A$ *is dense in* $\mathcal{H}$ *and* $\iota_1^{-1}\mathrm{Dom}A$ *is dense in* $\mathcal{H}_1$*,*
2. $0 \in \rho(A)$ *and*
$$A^{-1} = \iota_1\hat{A}^{-1}\iota_2, \tag{1.279}$$
3. *if one of the embeddings ι_1, ι_2 is compact, then A has compact resolvent,*
4. *the adjoint of A is given by*

$$\begin{aligned} \mathrm{Dom} A^* &= \{u \in \mathrm{Ran}\iota_2^* : \hat{A}^*(\iota_2^*)^{-1}u \in \mathrm{Ran}\iota_1^*\}, \\ A^* &= (\iota_1^*)^{-1}\hat{A}(\iota_2^*)^{-1}. \end{aligned} \tag{1.280}$$

Lemma 1.12.2. *Let $\mathcal{G}, \mathcal{H}$ be Hilbert spaces and $T : \mathrm{Dom}T \to \mathcal{H}$, $\mathrm{Dom}T \subset G$ be a continuous operator with dense range. If $\mathcal{G}_0 \subset \mathrm{Dom}T$ is a dense subspace of $\mathcal{G}$, then it holds true that $T\restriction_{\mathcal{G}_0}$ has dense range.*

Proof. Let $h \in \mathcal{H}$ and $\epsilon > 0$. Since T has dense range, there exists $g \in \mathrm{Dom}T \subset \mathcal{G}$ with

$$\|h - Tg\|_{\mathcal{H}} < \frac{\epsilon}{2}. \tag{1.281}$$

Furthermore, we have that $\mathcal{G}_0$ is dense in $\mathcal{G}$ and T is bounded, so there exists $g_0 \in \mathcal{G}_0$ such that

$$\|g - g_0\|_{\mathcal{G}} < \frac{\epsilon}{2\|T\|}. \tag{1.282}$$

Putting everything together we obtain that

$$\|h - Tg_0\|_{\mathcal{H}} \le \|h - Tg\|_{\mathcal{H}} + \|Tg_0 - Tg\|_{\mathcal{H}} \le \|h - Tg\|_{\mathcal{H}} + \|T\|\|g_0 - g\|_{\mathcal{G}} < \epsilon, \tag{1.283}$$

so $T\restriction_{\mathcal{G}_0}$ has dense range. □

Proof of Theorem 1.12.1. 1. We can also write the domain of A as defined above as

$$\mathrm{Dom}A = \mathrm{Ran}\iota_1\hat{A}^{-1}\iota_2, \quad \iota_1^{-1}\mathrm{Dom}A = \mathrm{Ran}\hat{A}^{-1}\iota_2. \tag{1.284}$$

By assumption ι_2 has dense range and $\hat{A}^{-1}$ is everywhere defined, continuous and also has dense range, so by Lemma 1.12.2 we see that $\mathrm{Ran}\hat{A}^{-1}\iota_2$ is dense in $\mathcal{H}_1$. Since ι_1 is continuous and has dense range, again using Lemma 1.12.2 yields that $\mathrm{Dom}A$ is dense in $\mathcal{H}$.

2. Since ι_1 and ι_2 are everywhere defined and injective, the inverse mappings ι_1 and ι_2 are well-defined, surjective and injective (although not everywhere defined). Clearly the operator
$$A = \iota_2^{-1}\hat{A}\iota_1^{-1} \tag{1.285}$$
is injective as a composition of injective maps.
If $h \in \mathcal{H}$, then by surjectiviy of ι_2^{-1} there exists $h_2 \in \mathcal{H}_2$ such that $\iota_2^{-1}h_2 = h$. Since also ι_1^{-1} is surjective, there exists $h_1 \in \mathcal{H}$ such that $\iota_1^{-1}h_1 = \hat{A}^{-1}h_2$.
Note that by definition $h_1 \in \mathrm{Ran}\iota_1$ and $\hat{A}\iota_1^{-1}h_1 = h_2 \in \mathcal{H}_2$, so $h_1 \in \mathrm{Dom}A$ with
$$Ah_1 = \iota_2^{-1}\hat{A}\iota_1^{-1}h_1 = \iota_2^{-1}h_2 = h, \tag{1.286}$$
which shows that A is surjective, and the inverse is given by
$$A^{-1} = \iota_1\hat{A}^{-1}\iota_2. \tag{1.287}$$
As a composition of continuous functions the inverse is also continuous, and thus $0 \in \rho(A)$.

3. This is clear since the inverse is a composition of three continuous operators, so if one of them is compact, so is A.

4. A simple calculation gives
$$A^* = \left(\iota_2^{-1}\hat{A}\iota_1^{-1}\right)^* = (\iota_1^{-1})^*\hat{A}^*(\iota_2^{-1})^* \tag{1.288}$$
with the appropriate domain
$$\begin{aligned}\mathrm{Dom}A^* &= \{u \in \mathrm{Ran}\iota_2^* : \hat{A}^*(\iota_2^*)^{-1}u \in \mathrm{Ran}\iota_1^*\},\\ A^* &= (\iota_1^*)^{-1}\hat{A}(\iota_2^*)^{-1}.\end{aligned} \tag{1.289}$$

□

With Theorem 1.12.1 in mind, we are interested in conditions under which operators are boundedly invertible. One of the standard results is the following Lemma.

Lemma 1.12.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. An operator $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is bijective and boundedly invertible if and only if the following conditions holds true:*

(r1)

$$\exists \alpha > 0 \quad \text{s.t.} \quad \|\hat{A}u\|_{\mathcal{H}_2} \geq \alpha \|u\|_{\mathcal{H}_1}, \qquad u \in \mathcal{H}_1, \tag{1.290}$$

(r2) Ran$\hat{A}$ *is dense in* $\mathcal{H}_2$.

Proof. If (r1) holds true, we see that

$$\hat{A}u = 0 \Rightarrow \|u\|_{\mathcal{H}_1} \leq \frac{1}{\alpha}\|\hat{A}u\|_{\mathcal{H}_2} = 0 \Rightarrow u = 0, \tag{1.291}$$

so $\hat{A}$ is injective. Next, we want to show that (r1) also implies a closed range. Let $(v_n)_n \subset \mathrm{Ran}\hat{A}$ be a Cauchy sequence in $\mathcal{H}_2$ with limit $v \in \mathcal{H}_2$. Then there exist $(u_n)_n \subset \mathrm{Dom}\hat{A}$ such that

$$\hat{A}u_n = v_n, \qquad n \in \mathbb{N}. \tag{1.292}$$

(r1) implies that

$$\|u_n - u_m\|_{\mathcal{H}_1} \leq \frac{1}{\alpha}\|\hat{A}(u_n - u_m)\|_{\mathcal{H}_2} = \frac{1}{\alpha}\|v_n - v_m\|_{\mathcal{H}_2}, \tag{1.293}$$

thus $(u_n)_n$ is a Cauchy sequence in $\mathcal{H}_1$, which has a limit $u \in \mathcal{H}_1$, and since $\hat{A}$ is bounded it follows that

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \hat{A}u_n = \hat{A}u, \tag{1.294}$$

thus $v \in \mathrm{Ran}\hat{A}$ and Ran$\hat{A}$ is closed. But as (r2) states that the range is also dense, $\hat{A}$ must also be surjective. The inverse is bounded since $\hat{A}$ is bounded and bijective between Banach spaces, see [10, Lemma 1.2.3]. □

We now define sesquilinear forms since they are a helpful tool in analysing whether certain operators are boundedly invertible.

Definition 1.12.4. *A map* $\mathbf{a} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ *is called a sesquilinear form if for all* $u, u_1, u_2 \in \mathcal{H}_1$, $v, v_1, v_2 \in \mathcal{H}_2$ *and* $\alpha \in \mathbb{C}$

1. $\mathbf{a}(u_1 + u_2, v) = \mathbf{a}(u_1, v) + \mathbf{a}(u_2, v)$,
2. $\mathbf{a}(u, v_1 + v_2) = \mathbf{a}(u, v_1) + \mathbf{a}(u, v_2)$,
3. $\mathbf{a}(\alpha u, v) = \alpha\mathbf{a}(u, v)$,
4. $\mathbf{a}(u, \alpha v) = \overline{\alpha}\mathbf{a}(u, v)$.

If $\mathbf{b} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ *is another sesquilinear form, we an add them via*

$$(\mathbf{a} + \mathbf{b})(u, v) = \mathbf{a}(u, v) + \mathbf{b}(u, v), \qquad u \in \mathcal{H}_1, v \in \mathcal{H}_2, \tag{1.295}$$

and for $\beta \in \mathbb{C}$ *we can define a scalar multiplication*

$$(\beta\mathbf{a})(u, v) = \beta\mathbf{a}(u, v), \qquad u \in \mathcal{H}_1, v \in \mathcal{H}_2. \tag{1.296}$$

Furthermore, we also define the adjoint form $\mathbf{a}^* : \mathcal{H}_2 \times \mathcal{H}_1 \to \mathbb{C}$ *of* $\mathbf{a}$ *as*

$$\mathbf{a}^*(v,u) = \overline{\mathbf{a}(u,v)}, \qquad u \in \mathcal{H}_1, v \in \mathcal{H}_2. \tag{1.297}$$

We call $\mathbf{a}$ *a bounded sesquilinear form, if there exists* $C > 0$ *such that*

$$|\mathbf{a}(u,v)| \le C\|u\|_{\mathcal{H}_1}\|v\|_{\mathcal{H}_2}, \qquad u \in \mathcal{H}_1, v \in \mathcal{H}_2. \tag{1.298}$$

Lemma 1.12.5. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be Hilbert spaces. Every operator* $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *defines a unique bounded sesquilinear form*

$$\mathbf{a} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}, \quad \mathbf{a}(u,v) = \langle \hat{A}u, v\rangle_{\mathcal{H}_2}. \tag{1.299}$$

On the other hand, every bounded sesquilinear form $\mathbf{a} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ *defines a unique bounded linear operator* $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *which satisfies (1.299).*

Proof. [22, Chapter I.6.4]. □

Corollary 1.12.6. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be Hilbert spaces. An operator* $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *is bijective if and only if the following conditions on the corresponding form* $\mathbf{a} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ *holds true:*

(rf1)

$$\exists \alpha > 0 \quad \text{s.t.} \quad \inf_{0 \ne u \in \mathcal{H}_1} \sup_{0 \ne v \in \mathcal{H}_2} \frac{|\boldsymbol{a}(u,v)|}{\|u\|_{\mathcal{H}_1}\|v\|_{\mathcal{H}_2}} \ge \alpha, \tag{1.300}$$

(rf2)

$$(\boldsymbol{a}(u,v) = 0 \quad \forall u \in \mathcal{H}_1) \quad \Rightarrow \quad v = 0. \tag{1.301}$$

In this case, there also exists a bounded inverse for $\hat{A}$.

Proof. We aim to show that (rf1)⇔ (r1) and (rf2)⇔(r2). Using (1.299) we obtain

$$\begin{aligned}
\text{(rf1)} \Leftrightarrow \exists \alpha > 0 \quad \text{s.t.} \quad & \inf_{0 \ne u \in \mathcal{H}_1} \sup_{0 \ne v \in \mathcal{H}_2} \frac{|\mathbf{a}(u,v)|}{\|u\|_{\mathcal{H}_1}\|v\|_{\mathcal{H}_2}} \ge \alpha \\
\Leftrightarrow \exists \alpha > 0 \quad \text{s.t.} \quad & \inf_{0 \ne u \in \mathcal{H}_1} \sup_{0 \ne v \in \mathcal{H}_2} \frac{|\langle \hat{A}u, v\rangle_{\mathcal{H}_2}|}{\|u\|_{\mathcal{H}_1}\|v\|_{\mathcal{H}_2}} \ge \alpha \\
\Leftrightarrow \exists \alpha > 0 \quad \text{s.t.} \quad & \|\hat{A}u\|_{\mathcal{H}_2} \ge \alpha\|u\|_{\mathcal{H}_1}, \qquad u \in \mathcal{H}_1 \Leftrightarrow \text{(r1)},
\end{aligned} \tag{1.302}$$

which proves the first equivalency. For the second we see that

$$\begin{aligned}
\text{(rf1)} &\Leftrightarrow (\mathbf{a}(u,v) = 0 \quad \forall u \in \mathcal{H}_1) \quad \Rightarrow \quad v = 0 \\
&\Leftrightarrow \Big(\langle \hat{A}u, v\rangle_{\mathcal{H}_2} = 0 \quad \forall u \in \mathcal{H}_1\Big) \quad \Rightarrow \quad v = 0 \\
&\Leftrightarrow (\text{Ran}\hat{A})^\perp = \{0\} \Leftrightarrow \text{Ran}\hat{A} \text{ is dense in } \mathcal{H}_2 \Leftrightarrow \text{(r1)},
\end{aligned} \tag{1.303}$$

which concludes the proof. □

Definition 1.12.7. *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{a} : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ be a bounded sesquilinear form. If there exists a $c > 0$ such that*

$$|\mathbf{a}(u, u)| \geq c\|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{H}, \tag{1.304}$$

then we say that $\mathbf{a}$ is coercive.

Lemma 1.12.8. *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{a} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ coercive. Then $\mathbf{a}$ satisfies* (rf1) *and* (rf2)*.*

Proof. Let $\mathbf{a}$ be coercive with coercivity constant $c > 0$. Then we have that

$$\inf_{0\neq u\in\mathcal{H}} \sup_{0\neq v\in\mathcal{H}} \frac{|\mathbf{a}(u, v)|}{\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}} \geq \inf_{0\neq u\in\mathcal{H}} \frac{|\mathbf{a}(u, u)|}{\|u\|_{\mathcal{H}}^2} \geq c > 0, \tag{1.305}$$

so (rf1) is satisfied.

Now let $v \in \mathcal{H}$ be such that

$$\mathbf{a}(u, v) = 0 \quad \forall u \in \mathcal{H}. \tag{1.306}$$

Then we especially have for the choice $u = v$

$$\|v\|_{\mathcal{H}}^2 \leq \frac{1}{c}\mathbf{a}(v, v) = 0, \tag{1.307}$$

so $v = 0$, which proves that also (rf2) holds true. □

Definition 1.12.9. *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{a} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ a bounded sesquilinear form. If there exist operators $\Theta_1, \Theta_2 \in \mathcal{B}(\mathcal{H})$ such that*

(AH1)

$$\exists\alpha_1 > 0 \quad \text{s.t. } |\mathbf{a}(u, u)| + |\mathbf{a}(u, \Theta_1 u)| \geq \alpha_1\|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{H}, \tag{1.308}$$

(AH2)

$$\exists\alpha_2 > 0 \quad \text{s.t. } |\mathbf{a}(v, v)| + |\mathbf{a}(\Theta_2 v, v)| \geq \alpha_2\|v\|_{\mathcal{H}}^2, \quad v \in \mathcal{H}, \tag{1.309}$$

then we call $\mathbf{a}$ Almog-Helffer-coercive.

Lemma 1.12.10. *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{a} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be Almog-Helffer coercive. Then $\mathbf{a}$ satisfies* (rf1) *and* (rf2)*, in particular (AH1) $\Rightarrow$* (rf1) *and (AH2) $\Rightarrow$* (rf2)*.*

Proof. We first show that (AH1) $\Rightarrow$ (rf1). Let $u \in \mathcal{H}$, $u \neq 0$. Define

$$\tilde{u} = \operatorname{sgn}(\mathbf{a}(u, u))u + \operatorname{sgn}(\mathbf{a}(u, \Theta_1 u))\Theta_1 u, \tag{1.310}$$

where sgn is the complex-valued signum function

$$\operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{for } z \neq 0, \\ 0 & \text{else.} \end{cases} \tag{1.311}$$

Then due to (AH1) it holds true that

$$\begin{aligned} \mathbf{a}(u, \tilde{u}) &= \overline{\operatorname{sgn}(\mathbf{a}(u, u))}\mathbf{a}(u, u) + \overline{\operatorname{sgn}(\mathbf{a}(u, \Theta_1 u))}\mathbf{a}(u, \Theta_1 u) \\ &= |\mathbf{a}(u, u)| + |\mathbf{a}(u, \Theta_1 u)| \geq \alpha_1 \|u\|_{\mathcal{H}}^2 > 0, \end{aligned} \tag{1.312}$$

in particular this also shows $\tilde{u} \neq 0$. Furthermore it holds true that

$$\begin{aligned} \|\tilde{u}\|_{\mathcal{H}} &= \| \operatorname{sgn}(\mathbf{a}(u, u))u + \operatorname{sgn}(\mathbf{a}(u, \Theta_1 u))\Theta_1 u\|_{\mathcal{H}} \\ &\leq \|u\|_{\mathcal{H}} + \|\Theta_1 u\|_{\mathcal{H}} \leq (1 + \|\Theta_1\|)\|u\|_{\mathcal{H}}. \end{aligned} \tag{1.313}$$

Together this shows that

$$\sup_{0 \neq v \in \mathcal{H}} \frac{|\mathbf{a}(u, v)|}{\|v\|_{\mathcal{H}}} \geq \frac{|\mathbf{a}(u, \tilde{u})|}{\|\tilde{u}\|_{\mathcal{H}}} \geq \frac{\alpha_1 \|u\|_{\mathcal{H}}}{1 + \|\Theta_1\|}, \tag{1.314}$$

and since this holds true for every $0 \neq u \in \mathcal{H}$ (rf1) follows.

Now assume that (AH2) holds true and let $v \in \mathcal{H}$ be such that

$$\mathbf{a}(u, v) = 0, \quad u \in \mathcal{H}. \tag{1.315}$$

This especially holds true for the choices $u = v$ and $u = \Theta_2 v$, and thus (AH2) gives

$$\alpha_2 \|v\|_{\mathcal{H}}^2 \leq |\mathbf{a}(v, v)| + |\mathbf{a}(\Theta_2 v, v)| = 0, \tag{1.316}$$

so $v = 0$ and we see that (rf2) holds true. □

1.13 Quasi Boundary Triples for Dual Pairs

Definition 1.13.1. *Let $\mathcal{H}$ be a separable Hilbert space and $S, \tilde{S}$ closed and densely defined operators in $\mathcal{H}$ such that*

$$\tilde{S} \subset S^*, \quad S \subset \tilde{S}^*. \tag{1.317}$$

Then we call $(S, \tilde{S})$ a dual pair. We call operators $T \subset S^*$, $\tilde{T} \subset \tilde{S}^*$ cores of S^* and $\tilde{S}^*$ if they satisfy

$$T^* = S, \quad \tilde{T}^* = \tilde{S}. \tag{1.318}$$

Definition 1.13.2. *Let* $(S, \tilde{S})$ *be a dual pair in* $\mathcal{H}$ *with cores* $T, \tilde{T}$ *of* $S^*, \tilde{S}^*$*, respectively. A triple* $(\mathcal{G}, (\Gamma_0, \Gamma_1), (\tilde{\Gamma}_0, \tilde{\Gamma}_1))$*, consisting of a Hilbert space* $\mathcal{G}$ *and linear mappings*

$$\Gamma_0, \Gamma_1 : \mathrm{Dom} T \to \mathcal{G}, \quad \tilde{\Gamma}_0, \tilde{\Gamma}_1 : \mathrm{Dom} \tilde{T} \to \mathcal{G}, \tag{1.319}$$

is called a quasi boundary triple for the dual pair $(S, \tilde{S})$*, if it satisfies the following properties:*

(G) $(Tf, g)_{\mathcal{H}} - (f, \tilde{T} g)_{\mathcal{H}} = (\Gamma_1 f, \tilde{\Gamma}_0 g)_{\mathcal{G}} - (\Gamma_0 f, \tilde{\Gamma}_1 g)_{\mathcal{G}}, \quad f \in \mathrm{Dom} T,\ g \in \mathrm{Dom} \tilde{T},$

(DD) $\mathrm{Ran}(\Gamma_0, \Gamma_1)^T$ *and* $\mathrm{Ran}(\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T$ *are dense in* $\mathcal{G} \oplus \mathcal{G}$*,*

(M) the operators $A_0 = T \restriction_{\ker \Gamma_0}$ *and* $\tilde{A}_0 = \tilde{T} \restriction_{\ker \tilde{\Gamma}_0}$ *satisfy*

$$A_0^* = \tilde{A}_0, \quad \tilde{A}_0^* = A_0.$$

It is often more convenient to begin with the cores T and $\tilde{T}$ and construct the boundary triple without explicitly introducing a dual pair $(S, \tilde{S})$. The following theorem states that, under suitable assumptions, the resulting construction nevertheless produces a dual pair.

Theorem 1.13.3. *Let* $\mathcal{H}$ *and* $\mathcal{G}$ *be Hilbert spaces and* $T, \tilde{T}$ *be operators in* $\mathcal{H}$*. Let the triple* $(\mathcal{G}, (\Gamma_0, \Gamma_1), (\tilde{\Gamma}_0, \tilde{\Gamma}_1))$ *with*

$$\Gamma_0, \Gamma_1 : \mathrm{Dom} T \to \mathcal{G}, \quad \tilde{\Gamma}_0, \tilde{\Gamma}_1 : \mathrm{Dom} \tilde{T} \to \mathcal{G}, \tag{1.320}$$

satisfy (G), (DD) and (M), and assume that $\ker \Gamma_0 \cap \ker \Gamma_1$ *and* $\ker \tilde{\Gamma}_0 \cap \ker \tilde{\Gamma}_1$ *are dense in* $\mathcal{H}$*. Then the pair of operators* $(S, \tilde{S})$*, defined by*

$$\begin{aligned} Sf &= \tilde{T} f, \quad f \in \mathrm{Dom} S = \ker \tilde{\Gamma}_0 \cap \ker \tilde{\Gamma}_1, \\ \tilde{S} f &= T f, \quad f \in \mathrm{Dom} \tilde{S} = \ker \Gamma_0 \cap \ker \Gamma_1, \end{aligned} \tag{1.321}$$

are closed and form an dual pair, such that $(\mathcal{G}, (\Gamma_0, \Gamma_1), (\tilde{\Gamma}_0, \tilde{\Gamma}_1))$ *form a quasi boundary triple for it.*

Proof. See [5, Theorem 2.7]. □

2 Sectorial and Almog-Helffer Coercive Schrödinger Operators

2.1 Assumptions

In this chapter we want to consider sectorial and Almog-Helffer coercive Schrödinger operators, i.e. we want that the following assumptions are satisfied.

Assumption 2.1.1. Let $\Omega \subset \mathbb{R}^d$ be minimally smooth, $q \in L^1_{\text{loc}}(\overline{\Omega})$ and $\operatorname{Re} q \geq 1$ a.e., such that it fulfils one of the following conditions:

(SEC) q is sectorial, i.e. $\operatorname{Re} q \geq 1$ a.e. and there exists $\theta \in [0, \frac{\pi}{2})$ such that

$$|\operatorname{Im} q| \leq \tan(\theta) \operatorname{Re} q \text{ a.e.}. \tag{2.1}$$

(AHc1) There exists a real-valued $\phi \in L^\infty(\Omega)$ such that $\nabla\phi \in L^2_{\text{loc}}(\Omega)$,

$$|\nabla\phi| \leq C(|q|+1)^{\frac{1}{2}}, \tag{2.2}$$

for some $C > 0$ there exists a measurable function $W : \Omega \to [0, \infty)$ and constants $C_W > 0$, $\alpha > 0$, $\epsilon_W \in (0, 1]$ such that for all $f \in H^1(\Omega) \cap \operatorname{Dom}|q|^{\frac{1}{2}}$ it holds true that

$$\int_\Omega \operatorname{Im} q\phi|f|^2 \mathrm{d}x \geq \alpha \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x - \int_\Omega W|f|^2 \mathrm{d}x, \tag{2.3}$$

and

$$\begin{aligned} \||W|^{\frac{1}{2}} f\|^2_{L^2} \leq & C_W \left(\|f\|^2_{L^2} + \|\nabla f\|^2_{L^2} + \||\operatorname{Re} q|^{\frac{1}{2}} f\|^2_{L^2} \right) \\ & + (1-\epsilon_W)\alpha \||\operatorname{Im} q|^{\frac{1}{2}} f\|^2_{L^2}. \end{aligned} \tag{2.4}$$

(AHc2) $q \in W^{1,\infty}_{\text{loc}}(\Omega)$ and there exists $C > 0$ such that

$$|\nabla \operatorname{Im} q| \leq C(1 + |\operatorname{Im} q|^2)^{\frac{3}{2}}(1 + |q|)^{\frac{1}{2}}. \tag{2.5}$$

Lemma 2.1.2. *The assumption (AHc2) implies (AHc1).*

Proof. First define the real-valued function

$$\phi = \frac{\operatorname{Im} q}{(1 + (\operatorname{Im} q)^2)^{\frac{1}{2}}}. \tag{2.6}$$

Clearly it holds true that $|\phi| \leq 1$, thus $\phi \in L^\infty(\Omega)$. In order to calculate the gradient of this function, note that $\operatorname{Im} q \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \subset H^1_{\mathrm{loc}}(\Omega)$, and therefore we can use Lemma 1.6.9 to calculate that

$$\nabla\phi = \frac{\nabla \operatorname{Im} q}{(1 + (\operatorname{Im} q)^2)^{\frac{1}{2}}} - \frac{(\operatorname{Im} q)^2 \nabla \operatorname{Im} q}{(1 + (\operatorname{Im} q)^2)^{\frac{3}{2}}} = \frac{\nabla \operatorname{Im} q}{(1 + (\operatorname{Im} q)^2)^{\frac{3}{2}}} \leq C(1 + |q|)^{\frac{1}{2}}. \tag{2.7}$$

so $\nabla\phi \in L^2_{\mathrm{loc}}(\Omega)$ and consequently $\phi : \Omega \to \mathbb{R}$ satisfies (2.2). Furthermore we have that

$$\phi \operatorname{Im} q = \frac{(\operatorname{Im} q)^2}{(1 + (\operatorname{Im} q)^2)^{\frac{1}{2}}} = (1 + (\operatorname{Im} q)^2)^{\frac{1}{2}} - \frac{1}{(1 + (\operatorname{Im} q)^2)^{\frac{1}{2}}} \geq |\operatorname{Im} q| - 1, \tag{2.8}$$

so we see that with $W = 1$ all the assumptions in (AHc1) are fulfilled, which concludes the proof. □

2.2 Form Domain $\mathcal{V}$

Under Assumptions 2.1.1 let us consider the space

$$\mathcal{V} := \left(H^1(\Omega) \cap \operatorname{Dom}|q|^{\frac{1}{2}}, \|\cdot\|^2_{\mathcal{V}} = \|\cdot\|^2_{H^1} + \||q|^{\frac{1}{2}} \cdot\|^2_{L^2} \right), \tag{2.9}$$

where

$$\operatorname{Dom}|q|^{\frac{1}{2}} = \{f \in L^2(\Omega) : \, |q|^{\frac{1}{2}} f \in L^2(\Omega)\}. \tag{2.10}$$

Lemma 2.2.1. *The space $\mathcal{V}$ together with the inner product*

$$\langle u, v\rangle_{\mathcal{V}} = \langle u, v\rangle_{H^1} + \int_\Omega |q| u\overline{v} \mathrm{d}x, \quad u, v \in \mathcal{V}, \tag{2.11}$$

is a Hilbert space, the inner product induces the norm in (2.9), $C_0^\infty(\mathbb{R}^d) \restriction_\Omega \subset \mathcal{V}$ and $\mathcal{V}$ is dense in $L^2(\Omega)$.

Proof. First note that

$$\langle u, u\rangle_{\mathcal{V}} = \|u\|^2_{H^1} + \int_\Omega |q||u|^2 \mathrm{d}x = \|u\|^2_{H^1} + \||q|^{\frac{1}{2}} u\|^2_{L^2}, \qquad u \in \mathcal{V}, \tag{2.12}$$

so this inner product indeed induces the norm in (2.9).

We need to show that the space is complete. To this end, let $(f_n)_n \subset \mathcal{V}$ be a Cauchy sequence. Then we see by definition of the norms that

$$\|f_n - f_m\|_{H^1} \leq \|f_n - f_m\|_{\mathcal{V}} \xrightarrow{n,m\to\infty} 0, \tag{2.13}$$

and since $H^1(\Omega)$ is a Hilbert spaces, there exists a limit $f = \lim_{n\to\infty} f_n$ in $H^1(\Omega)$.

From [7, Example 4.3.3] we see that the multiplication operator

$$|q|^{\frac{1}{2}} : \mathrm{Dom}|q|^{\frac{1}{2}} \subset L^2(\Omega) \to L^2(\Omega) \tag{2.14}$$

is self-adjoint in $L^2(\Omega)$ and thus closed, so $\mathrm{Dom}|q|^{\frac{1}{2}}$ together with the graph-norm is complete. Consequently, using

$$\frac{1}{2}\left(\|f_n - f_m\|_{L^2} + \||q|^{\frac{1}{2}}(f_n - f_m)\|_{L^2}\right) \leq \|f_n - f_m\|_{\mathcal{V}} \xrightarrow{n,m\to\infty} 0, \tag{2.15}$$

we conclude that there also exists a limit $f' = \lim_{n\to\infty} f_n$ in the graph norm induced by the multiplication operator $|q|^{\frac{1}{2}}$. Finally, as

$$\begin{aligned} &\|f_n - f\|_{L^2} \leq \|f_n - f\|_{H^1}, \\ &\|f_n - f'\|_{L^2} \leq \|f_n - f'\|_{L^2} + \||q|^{\frac{1}{2}}(f_n - f')\|_{L^2}, \qquad n \in \mathbb{N}, \end{aligned} \tag{2.16}$$

we see that $f = \lim_{n\to\infty} f_n = f'$ in $L^2(\Omega)$, so we obtain $f = f'$ and

$$\|f_n - f\|_{\mathcal{V}}^2 = \|f_n - f\|_{H^1}^2 + \||q|^{\frac{1}{2}}(f_n - f)\|_{L^2}^2 \xrightarrow{n\to\infty} 0, \tag{2.17}$$

which proves the completeness.

Next we want to show that $\mathcal{V}$ is dense in $L^2(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it suffices if we can show that $C_0^\infty(\Omega) \subset \mathcal{V}$.

To this end, let $\phi \in C_0^\infty(\Omega)$, so there exists $K \Subset \Omega$ with $\operatorname{supp}\phi \subset K$. Then it holds true that

$$\int_\Omega \left||q|^{\frac{1}{2}}\phi\right|^2 \mathrm{d}x = \int_K |q||\phi|^2 \mathrm{d}x \leq \|q\|_{L^1(K)}\|\phi\|_{L^\infty(\Omega)}^2, \tag{2.18}$$

so $\phi \in \mathrm{Dom}|q|^{\frac{1}{2}}$ and thus since $C_0^\infty(\Omega) \subset H^1(\Omega)$ we conclude that $\phi \in \mathcal{V}$. □

Lemma 2.2.2. *The space $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $\mathcal{V}$.*

Proof. Let $f \in \mathcal{V}$. We want to show that there exists a sequence $(\phi_n)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ with $\phi_n \xrightarrow{n\to\infty} f$ in $\mathcal{V}$.

Without loss of generality, we can assume that f is real-valued, since by definition of $\mathcal{V}$ we have that also $\overline{f} \in \mathcal{V}$, and thus

$$\operatorname{Re} f = \frac{1}{2}(f + \overline{f}) \in \mathcal{V}, \quad \operatorname{Im} f = \frac{1}{2i}(f - \overline{f}) \in \mathcal{V}, \tag{2.19}$$

so it suffices to approximate real and imaginary part individually.

Lemma 1.5.3 gives us an extension $\tilde{f} \in H^1(\mathbb{R}^d)$ with $\tilde{f}\restriction_\Omega = f$. Without loss of generality we can also assume that it is real-valued (just ignore any possible imaginary part from the extension operator).

Next we try to find approximations of $\tilde{f}$ which have compact support and are in $L^\infty(\mathbb{R}^d)$. To this end consider functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ with

$$\varphi_n(x) = \begin{cases} -n & \text{for } x \in (-\infty, -n), \\ x & \text{for } x \in [-n, n], \\ n & \text{for } x \in (n, \infty), \end{cases} \tag{2.20}$$

and a cut-off function $\psi \in C_0^\infty(\mathbb{R}^d)$, $0 \le \psi \le 1$, with

$$\psi(x) = \begin{cases} 1 & \text{for } x \in \mathbb{B}_1(0), \\ 0 & \text{for } x \notin \mathbb{B}_2(0), \end{cases} \tag{2.21}$$

We define our approximations as

$$\tilde{f}_n = \varphi_n(\tilde{f})\psi(\tfrac{\cdot}{n}), \quad n \in \mathbb{N}. \tag{2.22}$$

It is easy to see that $|\tilde{f}_n| \le n$ and $\operatorname{supp} \tilde{f}_n \subset \mathbb{B}_{2n}(0)$.

By definition of these functions it is also clear that $\tilde{f}_n$ converges pointwise to $\tilde{f}$ and that $|\tilde{f}_n| \le |\tilde{f}|$, so dominated convergence shows that

$$\tilde{f}_n \xrightarrow{n\to\infty} \tilde{f} \text{ in } L^2(\mathbb{R}^d). \tag{2.23}$$

Note that

$$\varphi_n'(x) = \begin{cases} 1 & \text{for } x \in [-n, n], \\ 0 & \text{else}, \end{cases} \qquad n \in \mathbb{N}. \tag{2.24}$$

Therefore with Lemma 1.6.9 and with the product rule we see that

$$\begin{aligned} \nabla \tilde{f}_n &= \varphi_n'(\tilde{f})(\nabla \tilde{f})\psi(\tfrac{\cdot}{n}) + \frac{1}{n}\varphi_n(\tilde{f})(\nabla\psi)(\tfrac{\cdot}{n}) \\ &= \chi_{\tilde{f}\in[-n,n]}(\nabla \tilde{f})\psi(\tfrac{\cdot}{n}) + \frac{1}{n}\varphi_n(\tilde{f})(\nabla\psi)(\tfrac{\cdot}{n}), \end{aligned} \tag{2.25}$$

where

$$\chi_{\tilde{f}\in[-n,n]}(x) = \begin{cases} 1 & \text{for } f(x) \in [-n, n], \\ 0 & \text{else}, \end{cases} \qquad n \in \mathbb{N}. \tag{2.26}$$

The first summand converges pointwise to $\nabla \tilde{f}$, and the second summand is zero inside $\mathbb{B}_n(0)$, since ψ is constant on $\mathbb{B}_1(0)$, and consequently converges pointwise to 0.

Furthermore, we find that

$$|\nabla \tilde{f}_n| = |\chi_{\tilde{f}\in[-n,n]}(\nabla \tilde{f})\psi(\tfrac{\cdot}{n}) + \frac{1}{n}\varphi_n(\tilde{f})(\nabla\psi)(\tfrac{\cdot}{n})| \leq |\nabla f| + |f|\|\nabla\psi\|_{L^\infty}, \tag{2.27}$$

which is an L^2-integrable upper bound, so $\nabla \tilde{f}_n \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ and by dominated convergence

$$\nabla \tilde{f}_n \xrightarrow{n\to\infty} \nabla \tilde{f} \text{ in } L^2(\mathbb{R}^d, \mathbb{R}^d). \tag{2.28}$$

Similarly, on Ω we have that $|q|^{\frac{1}{2}}|\tilde{f}_n| \leq |q|^{\frac{1}{2}}|\tilde{f}| = |q|^{\frac{1}{2}}|f|$, and $|q|^{\frac{1}{2}}\tilde{f}_n$ converges pointwise to $|q|^{\frac{1}{2}}f$, so dominated convergence gives us

$$|q|^{\frac{1}{2}}\tilde{f}_n\restriction_\Omega \xrightarrow{n\to\infty} |q|^{\frac{1}{2}}f \text{ in } L^2(\Omega). \tag{2.29}$$

Putting everything together, we obtain that

$$\tilde{f}_n\restriction_\Omega \xrightarrow{n\to\infty} f \text{ in } \mathcal{V}. \tag{2.30}$$

The next step is to mollify the functions $\tilde{f}_n$ in order to have smooth approximations. To this end, let $n \in N$ be fixed and define

$$\phi_n^{(\rho)} = \tilde{f}_n * \omega_\rho, \quad \rho > 0, \tag{2.31}$$

where $(\omega_\rho)_{\rho>0}$ is the family of standard mollifiers as defined in (1.36).

Theorem 1.3.3 and Theorem 1.3.5 show that

$$\phi_n^{(\rho)} \xrightarrow{\rho\to 0^+} \tilde{f}_n \quad \text{in } H^1(\mathbb{R}^d). \tag{2.32}$$

In particular we can choose a subsequence which converges pointwise to $\tilde{f}_n$, w.l.o.g. assume that the whole sequence does.

It holds true that

$$\begin{aligned} |\phi_n^{(\rho)}(x)| &= \left|\int_{\mathbb{R}^d} \tilde{f}_n(x-y)\omega_\rho(y)\mathrm{d}y\right| \\ &\leq \|\tilde{f}_n\|_{L^\infty}\|\omega_\rho\|_{L^1} \leq \|\tilde{f}_n\|_{L^\infty}, \quad x \in \mathbb{R}^d,\, \rho \in (0,1). \end{aligned} \tag{2.33}$$

Since $\operatorname{supp}\tilde{f}_n \subset \mathbb{B}_{2n}(0)$, it follows for all $\rho \in (0,1)$ that $\operatorname{supp}\phi_n^{(\rho)} \subset \mathbb{B}_{2n+1}(0)$, and since convolutions with mollifiers are smooth, see Theorem 1.3.3, we have that $(\phi_n^{(\rho)})_{\rho\in(0,1),\, n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$, and

$$\int_\Omega |q||\tilde{f}_n - \phi_n^{(\rho)}|^2\mathrm{d}x = \int_{\Omega\cap\mathbb{B}_{2n+1}(0)} |q||\tilde{f}_n - \phi_n^{(\rho)}|^2\mathrm{d}x, \quad \rho \in (0,1). \tag{2.34}$$

Since $\Omega \cap \mathbb{B}_{2n+1}(0)$ is bounded and $q \in L^1_{\mathrm{loc}}(\overline{\Omega})$, we see that

$$|q||\tilde{f}_n - \phi_n^{(\rho)}|^2 \leq 4\|\tilde{f}_n\|^2_{L^\infty}|q| \in L^1(\Omega\cap\mathbb{B}_{2n+1}(0)), \tag{2.35}$$

so we have an integrable upper bound, thus the pointwise convergence for $\rho \to 0^+$ and dominated convergence gives

$$|q|^{\frac{1}{2}}\phi_n^{(\rho)} \xrightarrow{\rho\to 0^+} |q|^{\frac{1}{2}}\tilde{f}_n \quad \text{in } L^2(\Omega). \tag{2.36}$$

Putting everything together, we finally see that

$$\phi_n^{(\rho)}\restriction_\Omega \xrightarrow{\rho\to 0^+} \tilde{f}_n\restriction_\Omega \quad \text{in } \mathcal{V}, \tag{2.37}$$

which concludes the proof. □

2.3 Dirichlet Trace $\gamma_D^{\mathcal{V}}$

Next we define suitable spaces on the boundary. As seen in Theorem 1.5.1 the Dirichlet trace γ_D from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ is bounded, surjective and has a bounded right-inverse. Our goal is to define a space $\mathcal{W}$ on the boundary such that the Dirichlet trace from $\mathcal{V} \subset H^1(\Omega)$ to $\mathcal{W}$ also satisfies these properties.

In order to guarantee surjectivity we will focus on the set

$$\mathcal{W} = \mathrm{Ran}\gamma_D\restriction_{\mathcal{V}}, \tag{2.38}$$

and we need to find a proper norm such that both the Dirichlet trace and a bounded right-inverse are bounded.

First note that the following holds true.

Lemma 2.3.1. *The space $\mathcal{W}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$.*

Proof. As $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $H^1(\Omega)$, see [24, Chapter 1], and γ_D is bounded and surjective, see Theorem 1.5.1, it follows that $\mathrm{Ran}\gamma_D\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$.

Lemma 2.2.1 shows that $C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathcal{V}$ and therefore by Lemma 1.12.2 $\mathcal{W} = \mathrm{Ran}\gamma_D\restriction_{\mathcal{V}}$ is also dense in $H^{\frac{1}{2}}(\partial\Omega)$. □

We will from now on work with the surjective Dirichlet trace

$$\gamma_D^{\mathcal{V}} : \mathcal{V} \to \mathcal{W}, \quad \gamma_D^{\mathcal{V}} f = \gamma_D f, \quad f \in \mathcal{V}. \tag{2.39}$$

Lemma 2.3.2. *It holds true that* $\ker \gamma_D^{\mathcal{V}}$ *is closed, and therefore we can write the form domain $\mathcal{V}$ as*

$$\mathcal{V} = \ker \gamma_D^{\mathcal{V}} \oplus (\ker \gamma_D^{\mathcal{V}})^\perp. \tag{2.40}$$

Proof. We want to show that the kernel is closed, the decomposition then follows from [27, Theorem 12.4].

To this end, let $(f_n)_n \subset \ker \gamma_D^{\mathcal{V}}$ be a Cauchy sequence in $\mathcal{V}$. Since $\mathcal{V}$ is a Hilbert space, see Lemma 2.2.1, the existence of a limit $f \in \mathcal{V}$ is clear.

Now note that

$$\|f_n - f\|_{H^1} \leq \|f_n - f\|_{\mathcal{V}}, \quad n \in \mathbb{N}, \tag{2.41}$$

by definition of the norm, so $(f_n)_n$ converges also in H^1 to f. Since γ_D is continuous in H^1, see Theorem 1.5.1, we obtain that

$$\|\gamma_D f\|_{H^{\frac{1}{2}}(\partial\Omega)} = \lim_{n\to\infty} \|\gamma_D f - \gamma_D f_n\|_{H^{\frac{1}{2}}(\partial\Omega)} = 0, \tag{2.42}$$

thus $\gamma_D^{\mathcal{V}} f = \gamma_D f = 0$, which proves $f \in \ker \gamma_D^{\mathcal{V}}$ and it follows that the kernel is closed. □

Lemma 2.3.3. *The mapping*

$$\gamma_D^{\mathcal{V}} : (\ker \gamma_D^{\mathcal{V}})^{\perp} \to \mathcal{W} \tag{2.43}$$

is a bijection.

Proof. Let $\psi \in \mathcal{W} = \mathrm{Ran}(\gamma_D^{\mathcal{V}})$. Then there exists $f \in \mathcal{V}$ with $\gamma_D^{\mathcal{V}} f = \psi$. Due to Lemma 2.3.2 there exist f_1, f_2 such that

$$f = f_1 + f_2, \quad f_1 \in \ker \gamma_D^{\mathcal{V}},\ f_2 \in (\ker \gamma_D^{\mathcal{V}})^{\perp}, \tag{2.44}$$

and therefore we see that

$$\gamma_D^{\mathcal{V}} f_2 = \gamma_D^{\mathcal{V}} f - \gamma_D^{\mathcal{V}} f_1 = \gamma_D^{\mathcal{V}} f = \psi, \tag{2.45}$$

so we found $f_2 \in (\ker \gamma_D^{\mathcal{V}})^{\perp}$ with $\gamma_D^{\mathcal{V}} f_2 = \psi$, which shows the surjectivity.

Now assume that $\gamma_D^{\mathcal{V}} f = 0$ for $f \in (\ker \gamma_D^{\mathcal{V}})^{\perp}$. This immediately shows

$$f \in \ker \gamma_D^{\mathcal{V}} \cap (\ker \gamma_D^{\mathcal{V}})^{\perp} = \{0\}, \tag{2.46}$$

which proves the injectivity. □

Due to Lemma 2.3.3 we see that the mapping

$$\mathcal{E}^{\mathcal{V}} : \mathcal{W} \to \mathcal{V}, \quad \mathcal{E}^{\mathcal{V}} \psi = \left((\gamma_D^{\mathcal{V}}) \restriction_{(\ker \gamma_D^{\mathcal{V}})^{\perp}} \right)^{-1} \psi, \quad \psi \in W, \tag{2.47}$$

is a well-defined and linear bijection. Consequently, we can define the inner product

$$\langle u, v \rangle_{\mathcal{W}} = \langle \mathcal{E}^{\mathcal{V}} u, \mathcal{E}^{\mathcal{V}} v \rangle_{\mathcal{V}}, \quad u, v \in \mathcal{W}, \tag{2.48}$$

on $\mathcal{W}$.

With these spaces we can finally show our desired result.

Theorem 2.3.4. *The space $\mathcal{W}$ is a Hilbert space, the Dirichlet trace*

$$\gamma_D^{\mathcal{V}} : \mathcal{V} \to \mathcal{W} \tag{2.49}$$

is surjective, bounded and $\mathcal{E}^{\mathcal{V}}$ as in (2.47) is a bounded right-inverse for $\gamma_D^{\mathcal{V}}$.

Proof. The inner product from (2.48) induces the norm

$$\|u\|_{\mathcal{W}} = \|\mathcal{E}^{\mathcal{V}} u\|_{\mathcal{V}}, \tag{2.50}$$

so we see that $\mathcal{E}^{\mathcal{V}}$ is by definition a isometric isomorphism between $\mathcal{W}$ and $(\ker \gamma_D^{\mathcal{V}})^{\perp}$, and since the later is a closed subspace of a Hilbert space, we see that $\mathcal{W}$ is closed and hence a Hilbert space.

By construction $\gamma_D^{\mathcal{V}}$ is surjective and $\mathcal{E}^{\mathcal{V}}$ a right-inverse with

$$\|\mathcal{E}^{V} \psi\|_{\mathcal{V}} = \|\psi\|_{\mathcal{W}}, \quad \psi \in \mathcal{W}. \tag{2.51}$$

What remains to show is that $\gamma_D^{\mathcal{V}}$ is bounded. Let $f \in \mathcal{V}$. Due to Lemma 2.3.2 there exist f_1, f_2 such that

$$f = f_1 + f_2, \quad f_1 \in \ker \gamma_D^{\mathcal{V}},\ f_2 \in (\ker \gamma_D^{\mathcal{V}})^{\perp}. \tag{2.52}$$

Then we have that

$$\gamma_D^{\mathcal{V}} f_2 = \gamma_D^{\mathcal{V}} f - \gamma_D^{\mathcal{V}} f_1 = \gamma_D^{\mathcal{V}} f, \tag{2.53}$$

with $f_2 \in (\ker \gamma_D^{\mathcal{V}})^{\perp}$. Clearly this yields

$$\mathcal{E}^{\mathcal{V}}(\gamma_D^{\mathcal{V}} f) = f_2 \tag{2.54}$$

and so

$$\|\gamma_D^{\mathcal{V}} f\|_{\mathcal{W}} = \|\mathcal{E}^{\mathcal{V}}(\gamma_D^{\mathcal{V}} f)\|_{\mathcal{V}} = \|f_2\|_{\mathcal{V}} \leq \|f_1\|_{\mathcal{V}} + \|f_2\|_{\mathcal{V}} = \|f\|_{\mathcal{V}}, \tag{2.55}$$

which concludes the proof. □

Corollary 2.3.5. *It holds true that* $\mathrm{Ran}\gamma_D^{\mathcal{V}} \restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ *is dense in $\mathcal{W}$.*

Proof. Let $\psi \in \mathcal{W}$ and $\epsilon > 0$. Since $C_0^\infty(\mathbb{R}^d) \restriction_\Omega$ is dense in $\mathcal{V}$, see Lemma 2.2.2, there exists $\phi \in C_0^\infty(\mathbb{R}^d) \restriction_\Omega$ such that

$$\|\mathcal{E}^{\mathcal{V}} \psi - \phi\|_{\mathcal{V}} < \epsilon. \tag{2.56}$$

Theorem 2.3.4 shows that the Dirichlet trace is bounded, and thus

$$\|\psi - \gamma_D^{\mathcal{V}} \phi\|_{\mathcal{W}} = \|\gamma_D^{\mathcal{V}} \mathcal{E}^{\mathcal{V}} \psi - \gamma_D^{\mathcal{V}} \phi\|_{\mathcal{W}} < \epsilon \|\gamma_D^{\mathcal{V}}\|. \tag{2.57}$$

Since $\gamma_D^{\mathcal{V}} \phi \in \mathrm{Ran}\gamma_D^{\mathcal{V}} \restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ this concludes the proof. □

Lemma 2.3.6. *The natural embeddings of $\mathcal{W}$ in $H^{\frac{1}{2}}(\partial\Omega)$ and $L^2(\partial\Omega)$ are continuous and have dense range.*

Proof. The space $\mathcal{W}$ is by definition contained in $H^{\frac{1}{2}}(\partial\Omega)$ and thus $L^2(\partial\Omega)$, so the embeddings exist.

Lemma 2.3.1 already shows that $\mathcal{W}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$. We can view the natural embedding as

$$\iota : \mathcal{W} \to H^{\frac{1}{2}}(\partial\Omega), \quad \iota f = \gamma_D \iota_0 \mathcal{E}^{\mathcal{V}} f, \tag{2.58}$$

with the natural embedding $\iota_0 : \mathcal{V} \to H^1(\Omega)$. Clearly it holds that

$$\|\iota_0 f\|_{H^1} = \|f\|_{H^1} \leq \|f\|_{\mathcal{V}}, \tag{2.59}$$

hence ι_0 is continuous, and so is ι as composition of continuous mappings. Consequently the embedding of $\mathcal{W}$ into $H^{\frac{1}{2}}(\partial\Omega)$ is continuous with dense range.

Since the same holds true for the embedding of $H^{\frac{1}{2}}(\partial\Omega)$ into $L^2(\partial\Omega)$, we see that $\mathcal{W}$ is also continuously and densely embedded into $L^2(\partial\Omega)$. □

2.4 Neumann Trace γ_N^q

We define a sesquilinear form

$$\mathbf{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}, \quad \mathbf{a}(u,v) = \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega q u \overline{v} \mathrm{d}x, \quad u,v \in \mathcal{V}, \tag{2.60}$$

and the corresponding operator

$$\hat{A} : \mathcal{V} \to \mathcal{V}^*, \quad (\hat{A}u, v)_{\mathcal{V}^* \times \mathcal{V}} = \mathbf{a}(u,v), \quad u,v \in \mathcal{V}. \tag{2.61}$$

Additionally, let

$$\hat{A}^C : \mathcal{V} \to \mathcal{V}^*, \quad (\hat{A}^C u, v)_{\mathcal{V}^* \times \mathcal{V}} = \overline{\mathbf{a}(v,u)} = \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega \overline{q} u \overline{v} \mathrm{d}x, \quad u,v \in \mathcal{V}. \tag{2.62}$$

These forms and operators are of course motivated by the definition of the $(-\Delta+q)u$ in the distributional sense for $u \in \mathcal{V}$. However, in order to define a Neumann trace we need also a L^2-realisation of $(-\Delta+q)u$, similar to the definition of the usual Neumann trace γ_N^Δ in (1.113), where we needed L^2 realisations of the Laplacian. To this end we define the operators

$$\begin{aligned} &\mathrm{Dom}T = \{u \in \mathcal{V} :\, (-\Delta+q)u \in L^2(\Omega)\}, \quad Tu = (-\Delta+q)u, \quad u \in \mathrm{Dom}T, \\ &\mathrm{Dom}\tilde{T} = \{u \in \mathcal{V} :\, (-\Delta+\overline{q})u \in L^2(\Omega)\}, \quad \tilde{T}u = (-\Delta+\overline{q})u, \quad u \in \mathrm{Dom}\tilde{T}. \end{aligned} \tag{2.63}$$

We want that the Neumann trace γ_N^q maps $\mathrm{Dom}T$ into $\mathcal{W}^*$. In order to ensure that we will need the following Lemma.

Lemma 2.4.1. *Let $u \in \mathrm{Dom}T$. Then it holds true that*

$$\left|(\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| \leq (2\|u\|_{\mathcal{V}} + \|Tu\|_{L^2})\|\psi\|_{\mathcal{W}}, \quad \psi \in \mathcal{W}. \tag{2.64}$$

The analogous statement holds true for $u \in \mathrm{Dom}\tilde{T}$, $\tilde{T}$ and $\hat{A}^C$.

Proof. Let $u \in \mathrm{Dom}T$ and $\psi \in \mathcal{W}$. Straightforward estimates show that

$$\begin{aligned}\left|(\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| &\leq \left|(\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}}\right| + \left|\langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| \\ &\leq \left|\langle \nabla u, \nabla\mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| + \left|\int_\Omega qu\overline{\mathcal{E}^{\mathcal{V}}\psi}\mathrm{d}x\right| + \left|\langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| \\ &\leq \|u\|_{H^1}\|\mathcal{E}^{\mathcal{V}}\psi\|_{H^1} + \||q|^{\frac{1}{2}}u\|_{L^2}\||q|^{\frac{1}{2}}\mathcal{E}^{\mathcal{V}}\psi\|_{L^2} + \|Tu\|_{L^2}\|\mathcal{E}^{\mathcal{V}}\psi\|_{L^2}.\end{aligned} \tag{2.65}$$

Now note that by definition of the norm on $\mathcal{V}$ we have that

$$\|v\|_{L^2} \leq \|v\|_{H^1} \leq \|v\|_{\mathcal{V}}, \quad \||q|^{\frac{1}{2}}v\|_{L^2} \leq \|v\|_{\mathcal{V}}, \quad v \in \mathcal{V}, \tag{2.66}$$

and consequently

$$\left|(\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| \leq (2\|u\|_{\mathcal{V}} + \|Tu\|_{L^2})\|\mathcal{E}^{\mathcal{V}}\psi\|_{\mathcal{V}}. \tag{2.67}$$

Finally we know from (2.51) that

$$\|\mathcal{E}^V\psi\|_{\mathcal{V}} = \|\psi\|_{\mathcal{W}}, \tag{2.68}$$

and thus

$$\left|(\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}\right| \leq (2\|u\|_{\mathcal{V}} + \|Tu\|_{L^2})\|\psi\|_{\mathcal{W}}. \tag{2.69}$$

The statement for $u \in \mathrm{Dom}\tilde{T}$ follows analogously. □

With these preparations we can finally define a Neumann trace.

Definition 2.4.2. *The Neumann traces $\gamma_N^q : \mathrm{Dom}T \to \mathcal{W}^*$ and $\gamma_N^{\overline{q}} : \mathrm{Dom}\tilde{T} \to \mathcal{W}^*$ are defined via*

$$\begin{aligned}(\gamma_N^q u,\psi)_{\mathcal{W}^*\times\mathcal{W}} &= \langle \nabla u, \nabla\mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} + \int_\Omega qu\overline{\mathcal{E}^{\mathcal{V}}\psi}\mathrm{d}x + \langle(-\Delta + q)u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} \\ &= (\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle Tu, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}, \qquad u \in \mathrm{Dom}T,\ \psi \in \mathcal{W},\end{aligned} \tag{2.70}$$

and

$$\begin{aligned}(\gamma_N^{\overline{q}} u,\psi)_{\mathcal{W}^*\times\mathcal{W}} &= \langle \nabla u, \nabla\mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} + \int_\Omega \overline{q}u\overline{\mathcal{E}^{\mathcal{V}}\psi}\mathrm{d}x + \langle(-\Delta + \overline{q})u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} \\ &= (\hat{A}^C u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle \tilde{T}u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}, \qquad u \in \mathrm{Dom}\tilde{T},\ \psi \in \mathcal{W},\end{aligned} \tag{2.71}$$

respectively.

The following Lemma will show that these definitions of Neumann traces coincide with the standard definition whenever both definitions make sense.

Lemma 2.4.3. *Assume that $u \in \mathrm{Dom}T$ and $\Delta u \in L^2(\Omega)$. Then it holds true that*

$$(\gamma_N^{\Delta} u, \psi)_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} = (\gamma_N^{q} u, \psi)_{\mathcal{W}^*\times\mathcal{W}}, \quad \psi \in \mathcal{W}, \tag{2.72}$$

where γ_N^{Δ} denotes the standard definition of the Neumann trace given by as defined in (1.113) and $\mathcal{E}$ a the bounded right-inverse of the usual Dirichlet trace γ_D as in Theorem 1.5.1.

The analogous statement holds for $u \in \mathrm{Dom}\tilde{T}$.

Proof. First note that by definition we have that

$$\gamma_D \mathcal{E}\psi = \psi = \gamma_D^{\mathcal{V}} \mathcal{E}^{\mathcal{V}}\psi = \gamma_D \mathcal{E}^{\mathcal{V}}\psi, \quad \psi \in \mathcal{W}. \tag{2.73}$$

Since $u \in \mathcal{V} \subset H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, the Neumann trace $\gamma_N^{\Delta} u$ is well-defined, and therefore we obtain

$$\begin{aligned}\langle \nabla u, \nabla \mathcal{E}\psi\rangle_{L^2} + \langle \Delta u, \mathcal{E}\psi\rangle_{L^2} &= (\gamma_N^{\Delta} u, \psi)_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} \\ &= (\gamma_N^{\Delta} u, \gamma_D \mathcal{E}^{\mathcal{V}}\psi)_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} \\ &= \langle \nabla u, \nabla \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} + \langle \Delta u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2}, \quad \psi \in \mathcal{W}.\end{aligned} \tag{2.74}$$

Using that we finally see that

$$\begin{aligned}&(\gamma_N^{\Delta} u, \psi)_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} = \langle \nabla u, \nabla \mathcal{E}\psi\rangle_{L^2} + \langle \Delta u, \mathcal{E}\psi\rangle_{L^2} \\ &\quad= \langle \nabla u, \nabla \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} + \langle \Delta u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} + \int_\Omega q u \overline{\mathcal{E}^{\mathcal{V}}\psi}\mathrm{d}x - \int_\Omega q u \overline{\mathcal{E}^{\mathcal{V}}\psi}\mathrm{d}x \\ &\quad= \langle \nabla u, \nabla \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} + \int_\Omega q u \overline{\mathcal{E}^{\mathcal{V}}\psi}\mathrm{d}x - \langle (-\Delta + q)u, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} \\ &\quad= (\gamma_N^{q} u, \psi)_{\mathcal{W}^*\times\mathcal{W}}, \quad \psi \in \mathcal{W}.\end{aligned} \tag{2.75}$$

The statement for $u \in \mathrm{Dom}\tilde{T}$ follows analogously. □

Remark 2.4.4. *Note that the Dirichlet trace $\gamma_D^{\mathcal{V}}$ only depends on the space $\mathcal{V}$ and thus the modulus $|q|$, whereas the Neumann trace γ_N^{q} can also depend on the argument of q.*

2.5 Invertibility of $\hat{A}$ in the Sectorial Case

Our goal is now to use the Theorem 1.12.1 in order to find Neumann realisations with non-empty resolvent sets for $(-\Delta+q)$. To this end, we need to show that $\hat{A}$ is bounded and bijective with a bounded inverse.

We will start with the sectorial case, which is rather straightforward.

Lemma 2.5.1. *If q satisfies (SEC), then the operator $\hat{A}$ from (2.61) is bounded and boundedly invertible.*

Proof. Using Cauchy Schwartz and Hölder inequalities it is easy to see that the sesquilinear form $\mathbf{a}$ satisfies

$$\begin{aligned} |\mathbf{a}(u,v)| = \left| \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega q u \overline{v} \mathrm{d}x \right| &\leq \|u\|_{H^1} \|v\|_{H^1} + \||q|^{\frac{1}{2}} u\|_{L^2} \||q|^{\frac{1}{2}} v\|_{L^2} \\ &\leq 2\|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \quad u,v \in \mathcal{V}. \end{aligned} \tag{2.76}$$

So be Lemma 1.12.5 we see that the associated operator $\hat{A}$ is bounded.

With $\operatorname{Re} q \geq 1$ we obtain that

$$\begin{aligned} |\mathbf{a}(u,u)| &= \left| \langle \nabla u, \nabla u \rangle_{L^2} + \int_\Omega q |u|^2 \mathrm{d}x \right| \geq \left| \operatorname{Re} \left(\|\nabla u\|_{L^2}^2 + \int_\Omega q |u|^2 \mathrm{d}x \right) \right| \\ &= \|\nabla u\|_{L^2}^2 + \int_\Omega \operatorname{Re} q |u|^2 \mathrm{d}x \geq \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2 \mathrm{d}x + \frac{1}{2} \int_\Omega \operatorname{Re} q |u|^2 \mathrm{d}x \\ &\geq \frac{1}{2} \|u\|_{H^1}^2 + \frac{1}{2} \int_\Omega \operatorname{Re} q |u|^2 \mathrm{d}x, \qquad u \in \mathcal{V}. \end{aligned} \tag{2.77}$$

Now using the triangle inequality and the assumption (SEC) shows that

$$|q| \leq |\operatorname{Re} q| + |\operatorname{Im} q| \leq (1 + \tan(\theta)) \operatorname{Re} q, \tag{2.78}$$

so consequently we obtain

$$\begin{aligned} |\mathbf{a}(u,u)| &\geq \frac{1}{2} \left(\|u\|_{H^1}^2 + \frac{1}{1+\tan(\theta)} \int_\Omega |q||u|^2 \mathrm{d}x \right) \\ &\geq \frac{1}{2(1+\tan(\theta))} \|u\|_{\mathcal{V}}^2, \qquad u \in \mathcal{V}. \end{aligned} \tag{2.79}$$

So we see that $\mathbf{a}$ is coercive and thus Lemma 1.12.8 and Corollary 1.12.6 show that $\hat{A}$ is boundedly invertible. □

2.6 Invertibility of $\hat{A}$ in the Almog-Helffer Coercive Case

In order to show the same result also for potentials satisfying (AHc1) we will need the following Lemma.

Lemma 2.6.1. *Let $q \in L^1_{\mathrm{loc}}(\overline{\Omega})$, $\phi \in L^\infty(\Omega)$ with $\nabla \phi \in L^2_{\mathrm{loc}}(\Omega)$, and $C > 0$ such that*

$$|\nabla \phi| \leq C(|q|+1)^{\frac{1}{2}}. \tag{2.80}$$

Then the corresponding multiplication operator $\Phi : \mathcal{V} \to \mathcal{V}$ is well-defined and bounded.

Proof. Let $f \in \mathcal{V}$. First note that

$$\|f\nabla\phi\|_{L^2}^2 = \int_\Omega |f|^2|\nabla\phi|^2\mathrm{d}x \leq \int_\Omega C^2(|q|+1)|f|^2\mathrm{d}x \leq C^2\|f\|_{\mathcal{V}}^2, \tag{2.81}$$

and

$$\|\phi\nabla f\|_{L^2}^2 = \int_\Omega |\phi|^2|\nabla f|^2\mathrm{d}x \leq \|\phi\|_{L^\infty}\|f\|_{L^2}^2. \tag{2.82}$$

Note that $L^\infty(\Omega) \subset L^2_{\mathrm{loc}}(\Omega)$, hence we can use the product rule form Lemma 1.6.1 to see that

$$\nabla(\phi f) = f\nabla\phi + \phi\nabla f. \tag{2.83}$$

By definition of the norm on $\mathcal{V}$ this then yields

$$\begin{aligned}\|\Phi f\|_{\mathcal{V}}^2 &= \|\nabla(\phi f)\|_{L^2}^2 + \|\phi f\|_{L^2}^2 + \int_\Omega |q||\phi f|^2\mathrm{d}x \\ &\leq 2\left(\|f\nabla\phi\|_{L^2}^2 + \|\phi\nabla f\|_{L^2}^2\right) + \|\phi f\|_{L^2}^2 + \int_\Omega |q||\phi f|^2\mathrm{d}x \\ &\leq 2C^2\|f\|_{\mathcal{V}}^2 + \|\phi\|_{L^\infty}^2\left(2\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega |q||f|^2\mathrm{d}x\right) \\ &\leq 2(C^2 + \|\phi\|_{L^\infty}^2)\|f\|_{\mathcal{V}}^2,\end{aligned} \tag{2.84}$$

so $\Phi : \mathcal{V} \to \mathcal{V}$ is well-defined and bounded. □

Lemma 2.6.2. *If q satisfies (AHc1), then the operator $\hat{A}$ is bounded and boundedly invertible.*

Proof. That the form $\mathbf{a}$ is bounded was already shown in the proof of Lemma 2.5.1. We will now show that the form $\mathbf{a}$ in satisfies (AH1), and analogously it then follows that it also satisfies (AH2). Together with Lemma 1.12.10 and Corollary 1.12.6 this proves that the associated operator $\hat{A}$ is bounded and boundedly invertible.

Let $f \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$. Since $\operatorname{Re} q \geq 1$ it holds true that

$$\begin{aligned}|\mathbf{a}(f,f)| \geq |\operatorname{Re}\mathbf{a}(f,f)| &= \|\nabla f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2\mathrm{d}x \\ &\geq \frac{1}{2}\left(\|\nabla f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2\mathrm{d}x + \|f\|_{L^2}^2\right).\end{aligned} \tag{2.85}$$

Lemma 2.6.1 shows that $\phi f \in \mathcal{V}$ and that the corresponding multiplication operator $\Phi : \mathcal{V} \to \mathcal{V}$ is bounded. Thus we obtain that for every $\beta > 0$

$$\begin{aligned}|\mathbf{a}(f,\beta\phi f)| \geq \operatorname{Im}\mathbf{a}(f,\beta\phi f) &= \operatorname{Im}\left(\beta\langle\nabla f, \nabla(\phi f)\rangle_{L^2} + \beta\int_\Omega q\phi|f|^2\mathrm{d}x\right) \\ &= \operatorname{Im}\left(\beta\langle\nabla f, f\nabla\phi\rangle_{L^2} + \beta\langle\nabla f, \phi\nabla f\rangle_{L^2} + \beta\int_\Omega q\phi|f|^2\mathrm{d}x\right).\end{aligned} \tag{2.86}$$

Note that ϕ is by assumption real-valued, thus the second summand in the last line is real, and so we obtain

$$|\mathbf{a}(f, \beta\phi f)| \geq \operatorname{Im} \beta \langle \nabla f, f \nabla \phi \rangle_{L^2} + \beta \int_\Omega \operatorname{Im} q \phi |f|^2 \mathrm{d}x. \tag{2.87}$$

In order to proceed we need estimates on both terms on the right-hand side. First from

$$2ab \leq a^2 + b^2, \qquad a, b \in \mathbb{R}, \tag{2.88}$$

we see that for every $\delta > 0$

$$\begin{aligned} |\operatorname{Im} \beta \langle \nabla f, f \nabla \phi \rangle_{L^2}| &\leq \beta \|\nabla f\|_{L^2} \|f \nabla \phi\|_{L^2} = \beta \left(\delta^{\frac{1}{2}} \|f \nabla \phi\|_{L^2} \right) \left(\frac{1}{\delta^{\frac{1}{2}}} \|\nabla f\|_{L^2} \right) \\ &\leq \beta \frac{\delta}{2} \|f \nabla \phi\|_{L^2}^2 + \beta \frac{1}{2\delta} \|\nabla f\|_{L^2}^2. \end{aligned} \tag{2.89}$$

From (2.81) we see that there exists $M > 0$ such that

$$\|f \nabla \phi\|_{L^2} < M \|f\|_{\mathcal{V}}, \tag{2.90}$$

and consequently we obtain

$$|\mathbf{a}(f, \beta\phi f)| \geq -\beta \frac{\delta}{2} M \|f\|_{\mathcal{V}}^2 - \beta \frac{1}{2\delta} \|\nabla f\|_{L^2}^2 + \beta \int_\Omega \operatorname{Im} q \phi |f|^2 \mathrm{d}x. \tag{2.91}$$

Note that $|q| \leq |\operatorname{Im} q| + \operatorname{Re} q$, so we see that

$$\begin{aligned} \|f\|_{\mathcal{V}}^2 &= \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega |q||f|^2 \mathrm{d}x \\ &\leq \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q |f|^2 \mathrm{d}x + \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x, \end{aligned} \tag{2.92}$$

and thus

$$\begin{aligned} |\mathbf{a}(f, \beta\phi f)| \geq &- \beta \frac{\delta}{2} M \left(\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q |f|^2 \mathrm{d}x + \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x \right) \\ &- \beta \frac{1}{2\delta} \|\nabla f\|_{L^2}^2 + \beta \int_\Omega \operatorname{Im} q \phi |f|^2 \mathrm{d}x. \end{aligned} \tag{2.93}$$

For the last term (2.3) gives that

$$\int_\Omega \operatorname{Im} q \phi |f|^2 \mathrm{d}x \geq \alpha \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x - \int_\Omega W |f|^2 \mathrm{d}x, \tag{2.94}$$

and combined with (2.4) this shows that

$$\begin{aligned}\int_\Omega \operatorname{Im} q\phi|f|^2 \mathrm{d}x \geq& \alpha\epsilon_W \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x \\ &- C_W \left(\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2 \mathrm{d}x \right).\end{aligned} \tag{2.95}$$

Plugging (2.95) into (2.93) and rearranging everything then gives

$$\begin{aligned}|\mathbf{a}(f, \beta\phi f)| \geq& \beta \left(\alpha\epsilon_W - M\frac{\delta}{2} \right) \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x \\ &- \beta \left(\frac{1}{2\delta} + M\frac{\delta}{2} + C_W \right) \|\nabla f\|_{L^2}^2 \\ &- \beta \left(M\frac{\delta}{2} + C_W \right) \left(\|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2 \mathrm{d}x \right).\end{aligned} \tag{2.96}$$

This holds for all $\delta > 0$, so we can choose

$$\delta = \frac{\alpha\epsilon_W}{M} > 0 \tag{2.97}$$

in order to obtain

$$\begin{aligned}|\mathbf{a}(f, \beta\phi f)| \geq& \beta\frac{\alpha\epsilon_W}{2} \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x - \beta \left(\frac{M}{2\alpha\epsilon_W} + \frac{\alpha\epsilon_W}{2} + C_W \right) \|\nabla f\|_{L^2}^2 \\ &- \beta \left(\frac{\alpha\epsilon_W}{2} + C_W \right) \left(\|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2 \mathrm{d}x \right).\end{aligned} \tag{2.98}$$

Now we choose the parameter β as

$$\beta = \frac{1}{3} \left(\frac{M}{2\alpha\epsilon_W} + \frac{\alpha\epsilon_W}{2} + C_W \right)^{-1} > 0 \tag{2.99}$$

and so see that

$$|\mathbf{a}(f, \beta\phi f)| \geq \beta\frac{\alpha\epsilon_W}{2} \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x - \frac{1}{3} \left(\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2 \mathrm{d}x \right). \tag{2.100}$$

Together with (2.85) this yields

$$\begin{aligned}|\mathbf{a}(f, f)| + |\mathbf{a}(f, \beta\phi f)| \geq& \beta\frac{\alpha\epsilon_W}{2} \int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x \\ &+ \frac{1}{6} \left(\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega \operatorname{Re} q|f|^2 \mathrm{d}x \right),\end{aligned} \tag{2.101}$$

so with

$$m = \min\left(\tfrac{1}{6}, \beta\frac{\alpha\epsilon_W}{2}\right) > 0 \tag{2.102}$$

we see that

$$\begin{aligned}|\mathbf{a}(f,f)| + |\mathbf{a}(f,\beta\phi f)| \geq & m\|\nabla f\|_{L^2}^2 + m\|f\|_{L^2}^2 \\ & + m\int_\Omega \operatorname{Re} q|f|^2 \mathrm{d}x + m\int_\Omega |\operatorname{Im} q||f|^2 \mathrm{d}x,\end{aligned} \tag{2.103}$$

and now $|q| \leq |\operatorname{Im} q| + |\operatorname{Re} q|$ yields that

$$|\mathbf{a}(f,f)| + |\mathbf{a}(f,\beta\phi f)| \geq m\left(\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 + \int_\Omega |q||f|^2\mathrm{d}x\right) = m\|f\|_{\mathcal{V}}^2. \tag{2.104}$$

Since $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $\mathcal{V}$ and both $\mathbf{a}$ and the multiplication operator Φ are bounded the result holds true for every $f \in \mathcal{V}$, which shows that $\mathbf{a}$ satisfies (AH1) and thus concludes the proof. □

2.7 Neumann Realisation A_N of $(-\Delta + q)$

All together, Lemma 2.5.1, Lemma 2.6.2 and Lemma 2.1.2 show the following result.

Corollary 2.7.1. *If q satisfies Assumptions 2.1.1, then the operator $\hat{A}$ is bounded and boundedly invertible.*

Now we define the operator

$$\begin{aligned}\mathrm{Dom} A_N &= \{f \in \mathcal{V} :\ (-\Delta + q)f \in L^2(\Omega),\ \gamma_N^q f = 0\}, \\ A_N f &= (-\Delta + q)f.\end{aligned} \tag{2.105}$$

As we will see in the following Theorem, A_N is the L^2-realisation of the operator $\hat{A}$ from (2.61) which we obtain from Theorem 1.12.1. Furthermore it is clear by definition that

$$\mathrm{Dom} A_N = \ker \gamma_N^q, \tag{2.106}$$

so it is what we call the Neumann realisation of the Schrödinger operator.

Theorem 2.7.2. *Let A_N be as in (2.105). Then $\mathrm{Dom} A_N$ is dense in $\mathcal{V}$ and $L^2(\Omega)$ and A_N is a closed operator with $0 \in \rho(A_N)$. Furthermore, the adjoint is given by*

$$\begin{aligned}\mathrm{Dom} A_N^* &= \{f \in \mathcal{V} :\ (-\Delta + \overline{q})f \in L^2(\Omega),\ \gamma_N^{\overline{q}} f = 0\}, \\ A_N^* f &= (-\Delta + \overline{q})f.\end{aligned} \tag{2.107}$$

Proof. From Lemma 2.2.1 we know that $\mathcal{V}$ is a dense subset of $L^2(\Omega)$, and by the definition of the norm on $\mathcal{V}$ it is clear that the natural embedding embedding of $\mathcal{V}$ into $L^2(\Omega)$ is continuous.

Lemma 1.11.2 then shows that $L^2(\Omega)$ is consequently a dense subset of the dual space $\mathcal{V}^*$, and the embedding is also continuous.

Since Corollary 2.7.1 states that $\hat{A} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*)$, we can use Theorem 1.12.1 to obtain an operator

$$\begin{aligned} \mathrm{Dom}A &= \{f \in \mathcal{V} : \exists \eta \in L^2(\Omega) : \langle \eta, v\rangle_{L^2} = (\hat{A}f, v)_{\mathcal{V}^* \times \mathcal{V}} \ \forall v \in \mathcal{V}\}, \\ Af &= \eta, \end{aligned} \tag{2.108}$$

which is closed, satisfies $0 \in \rho(A)$ and $\mathrm{Dom}A$ is dense in both $\mathcal{V}$ and $L^2(\Omega)$.

The goal is now to show that $A_N = A$. To this end, let $f \in \mathrm{Dom}A$. Since $\mathcal{D}(\Omega) \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathcal{V}$, see Lemma 2.2.1, we have

$$\langle Af, \phi\rangle_{L^2} = (\hat{A}f, \phi)_{\mathcal{V}^* \times \mathcal{V}} = \mathbf{a}(f, \phi) = \langle \nabla f, \nabla \phi\rangle_{L^2} + \int_\Omega q f \overline{\phi} \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega). \tag{2.109}$$

Therefore

$$Af = (-\Delta + q)f \quad \text{ in } \mathcal{D}'(\Omega), \tag{2.110}$$

and thus $(-\Delta + q)f \in L^2(\Omega)$, so $f \in \mathrm{Dom}T$. Furthermore, since $\mathcal{E}^{\mathcal{V}}\psi \in \mathcal{V}$ for every $\psi \in \mathcal{W}$, we see from (2.108) that

$$\begin{aligned} (\gamma_N^q f, \psi)_{\mathcal{W}^* \times \mathcal{W}} &= (\hat{A}f, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^* \times \mathcal{V}} - \langle (-\Delta + q)f, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} \\ &= (\hat{A}f, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^* \times \mathcal{V}} - \langle Af, \mathcal{E}^{\mathcal{V}}\psi\rangle_{L^2} = 0, \quad \psi \in \mathcal{W}, \end{aligned} \tag{2.111}$$

and therefore $\gamma_N^q f = 0$. This shows that $f \in \mathrm{Dom}A_N$ and $A_N f = Af$.

If on the other hand $f \in \mathrm{Dom}A_N$, one sets

$$\eta = A_N f = (-\Delta + q)f \in L^2(\Omega) \tag{2.112}$$

and obtains from (2.70) with $\gamma_N^q f = 0$ that

$$\begin{aligned} \langle \eta, v\rangle_{L^2} = \langle (-\Delta + q)f, v\rangle_{L^2} &= (\hat{A}f, v)_{\mathcal{V}^* \times \mathcal{V}} - (\gamma_N^q f, \gamma_D^{\mathcal{V}} v)_{\mathcal{W}^* \times \mathcal{W}} \\ &= (\hat{A}f, v)_{\mathcal{V}^* \times \mathcal{V}}, \qquad v \in \mathcal{V}, \end{aligned} \tag{2.113}$$

so consequently $f \in \mathrm{Dom}A$ and $Af = A_N f$, which proves $A_N = A$.

Now consider the operator

$$\begin{aligned} \mathrm{Dom}B &= \{f \in \mathcal{V} : (-\Delta + \overline{q})f \in L^2(\Omega), \gamma_N^{\overline{q}} f = 0\}, \\ Bf &= (-\Delta + \overline{q})f, \quad f \in \mathrm{Dom}B. \end{aligned} \tag{2.114}$$

We want to show that $B = A_N^*$.

Let $f \in \mathrm{Dom} A_N^*$. By definition of the adjoint this means that there exists $\eta \in L^2(\Omega)$ such that

$$\langle \eta, v\rangle_{L^2} = \overline{\langle A_N v, f\rangle_{L^2}} = \overline{(\hat{A}v, f)_{\mathcal{V}^*\times\mathcal{V}}}, \quad v \in \mathrm{Dom} A_N, \tag{2.115}$$

and since we already saw that $\mathrm{Dom} A_N$ is dense in both $L^2(\Omega)$ and $\mathcal{V}$, this also yields that

$$\langle \eta, v\rangle_{L^2} = \overline{(\hat{A}v, f)_{\mathcal{V}^*\times\mathcal{V}}} = \overline{\mathbf{a}(v, f)}, \quad v \in \mathcal{V}. \tag{2.116}$$

Now we can use again that $\mathcal{D}(\Omega) \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathcal{V}$, see Lemma 2.2.1, to see that

$$\langle \eta, \phi\rangle_{L^2} = \overline{\mathbf{a}(\phi, f)} = \langle \nabla f, \nabla\phi\rangle_{L^2} + \int_\Omega \overline{q} f \overline{\phi} \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega), \tag{2.117}$$

which gives

$$(-\Delta + \overline{q})f = \eta \in L^2(\Omega). \tag{2.118}$$

Additionally it holds true that

$$\begin{aligned}(\gamma_N^{\overline{q}} f, \psi)_{\mathcal{W}^*\times\mathcal{W}} &= (\hat{A}^C f, \mathcal{E}^{\overline{q}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle(-\Delta + \overline{q})f, \mathcal{E}^{\overline{q}}\psi\rangle_{L^2} \\ &= \overline{\mathbf{a}(\mathcal{E}^{\overline{q}}\psi, f)} - \langle \eta f, \mathcal{E}^{\overline{q}}\psi\rangle_{L^2} = 0, \quad \psi \in \mathcal{W},\end{aligned} \tag{2.119}$$

so $\gamma_N^{\overline{q}} f = 0$ and thus $f \in \mathrm{Dom} B$, $Bf = A_N^* f$.

If on the other hand $f \in \mathrm{Dom} B$, then it holds true that

$$\begin{aligned}\langle Bf, v\rangle_{L^2} &= \langle(-\Delta + \overline{q})f, v\rangle_{L^2} = (\hat{A}^C f, v)_{\mathcal{V}^*\times\mathcal{V}} \\ &= \overline{\mathbf{a}(v, f)} = \overline{(\hat{A}v, f)_{\mathcal{V}^*\times\mathcal{V}}}, \qquad v \in \mathcal{V},\end{aligned} \tag{2.120}$$

and since $\mathrm{Dom} A_N \subset \mathcal{V}$ this yields that also

$$\langle Bf, v\rangle_{L^2} = \overline{(\hat{A}v, f)_{\mathcal{V}^*\times\mathcal{V}}} = \overline{\langle A_N v, f\rangle_{L^2}}, \quad v \in \mathrm{Dom} A_N, \tag{2.121}$$

so in this case we see that $f \in \mathrm{Dom} A_N^*$ with $A_N^* f = Bf$, so in conclusion $A_N^* = B$. □

Corollary 2.7.3. *Let $\gamma_D^{\mathcal{V}}$ be as in (2.39). Then $\mathrm{Ran}\gamma_D^{\mathcal{V}}\restriction_{\mathrm{Dom} A_N}$ has dense range in $\mathcal{W}$.*

Proof. Let $\psi \in \mathcal{W}$. Using the right-inverse we obtain $f = \mathcal{E}^{\mathcal{V}}\psi \in \mathcal{V}$ with $\gamma_D^{\mathcal{V}} f = \psi$. Theorem 2.7.2 states that $\mathrm{Dom} A_N$ is dense in $\mathcal{V}$, so there exists $(f_n)_n \subset \mathrm{Dom} A_N$ with $f_n \xrightarrow{n\to\infty} f$ in $\mathcal{V}$.

Due to the boundedness of the Dirichlet trace we see that

$$\|\gamma_D^{\mathcal{V}} f_n - \psi\|_{\mathcal{W}} = \|\gamma_D^{\mathcal{V}}(f_n - f)\|_{\mathcal{W}} \le \|\gamma_D^{\mathcal{V}}\|_{\mathcal{V}\to\mathcal{W}}\|f_n - f\|_{\mathcal{V}} \xrightarrow{n\to\infty} 0, \tag{2.122}$$

which concludes the proof since $(\gamma_D^{\mathcal{V}} f_n)_n \subset \mathrm{Ran}\gamma_D^{\mathcal{V}}\restriction_{\mathrm{Dom} A_N}$. □

2.8 Surjectivity of the Neumann Trace γ_N^q

Lemma 2.8.1. *The Neumann trace* $\gamma_N^q : \mathrm{Dom}T \to \mathcal{W}^*$ *from (2.70) is surjective.*

Proof. Let $\varphi \in \mathcal{W}^*$. We need to find $f \in \mathrm{Dom}T$ such that $\gamma_N^q f = \varphi$. To this end, consider

$$F(v) = (\varphi, \gamma_D^{\mathcal{V}} v)_{\mathcal{W}^*\times\mathcal{W}}, \quad v \in \mathcal{V}. \tag{2.123}$$

This is clearly anti-linear and since the Dirichlet trace $\gamma_D^{\mathcal{V}}$ is bounded this yields

$$|F(v)| \leq \|\varphi\|_{\mathcal{W}^*} \|\gamma_D^{\mathcal{V}} v\|_{\mathcal{W}} \leq \|\gamma_D^{\mathcal{V}}\| \|\varphi\|_{\mathcal{W}^*} \|v\|_{\mathcal{V}}, \quad v \in \mathcal{V}, \tag{2.124}$$

which shows that $F \in \mathcal{V}^*$.

Corollary 2.7.1 states that $\hat{A}$ is surjective, therefore we can find $u \in \mathcal{V}$ such that $\hat{A}u = F$, i.e.

$$(\hat{A}u, v)_{\mathcal{V}^*\times\mathcal{V}} = F(v) = (\varphi, \gamma_D^{\mathcal{V}} v)_{\mathcal{W}^*\times\mathcal{W}}, \quad v \in \mathcal{V}. \tag{2.125}$$

By the definition of $\hat{A}$ it thus holds that

$$\langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega q u \overline{v} \mathrm{d}x = (\varphi, \gamma_D^{\mathcal{V}} v)_{\mathcal{W}^*\times\mathcal{W}}, \quad v \in \mathcal{V}. \tag{2.126}$$

Since we have $\mathcal{D}(\Omega) \subset \mathcal{V}$, see Lemma 2.2.1, this yields

$$\langle \nabla u, \nabla \phi \rangle_{L^2} + \int_\Omega q u \overline{\phi} \mathrm{d}x = 0, \quad \phi \in \mathcal{D}(\Omega), \tag{2.127}$$

and hence $(-\Delta + q)u = 0 \in L^2(\Omega)$, which shows $u \in \mathrm{Dom}T$. Furthermore we have that

$$\begin{aligned}(\gamma_N^q u, \psi)_{\mathcal{W}^*\times\mathcal{W}} &= (\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} - \langle (-\Delta + q)u, \mathcal{E}^{\mathcal{V}}\psi \rangle_{L^2} \\ &= (\hat{A}u, \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{V}^*\times\mathcal{V}} = (\varphi, \gamma_D^{\mathcal{V}} \mathcal{E}^{\mathcal{V}}\psi)_{\mathcal{W}^*\times\mathcal{W}} \\ &= (\varphi, \psi)_{\mathcal{W}^*\times\mathcal{W}}, \qquad \psi \in \mathcal{W},\end{aligned} \tag{2.128}$$

which proves that $\gamma_N^q u = \varphi$ and thus concludes the proof. □

2.9 Quasi Boundary Triple for Sectorial and Almog-Helffer Coercive Schrödinger Operators

We now want to find a quasi boundary triple for the operators T and $\tilde{T}$ as defined in (2.63).

Note that there exists a dense and continuous embedding of $\mathcal{W}$ into $L^2(\partial\Omega)$, see Lemma 2.3.6, and thus by Lemma 1.11.3 we find isometric isomorphisms $\iota_+ : \mathcal{W} \to L^2(\Omega)$, $\iota_- : \mathcal{W}^* \to L^2(\partial\Omega)$, which satisfy

$$\langle \iota_- u, \iota_+ v \rangle_{L^2} = (u, v)_{\mathcal{W}^*\times\mathcal{W}}, \quad u \in \mathcal{W}^*, \, v \in \mathcal{W}. \tag{2.129}$$

We define mappings

$$
\begin{aligned}
(\Gamma_0, \Gamma_1)^T &: \mathrm{Dom} T \to L^2(\partial\Omega) \oplus L^2(\partial\Omega), \quad (\Gamma_0, \Gamma_1)^T f = (\iota_- \gamma_N^q f, \iota_+ \gamma_D^{\mathcal{V}} f)^T, \\
(\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T &: \mathrm{Dom}\tilde{T} \to L^2(\partial\Omega) \oplus L^2(\partial\Omega), \quad (\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T f = (\iota_- \gamma_N^{\overline{q}} f, \iota_+ \gamma_D^{\mathcal{V}} f)^T.
\end{aligned}
\tag{2.130}
$$

In order to construct a quasi boundary triple, we need a few more Lemmas.

Lemma 2.9.1. *Let $f \in \mathrm{Dom} T$ or $f \in \mathrm{Dom}\tilde{T}$ and $\eta \in C_0^\infty(\Omega)$. Then it holds true that $\eta f \in \ker \gamma_N^q \cap \ker \gamma_D^{\mathcal{V}}$ or $\eta f \in \ker \gamma_N^{\overline{q}} \cap \ker \gamma_D^{\mathcal{V}}$, respectively.*

Proof. Let $f \in \mathrm{Dom} T$. From $f \in H^1(\Omega)$ we see by 1.2.10 that $\eta f \in H^1(\Omega)$. Additionally we have that

$$\||q|^{\frac{1}{2}} \eta f\|_{L^2} \leq \|\eta\|_{L^\infty} \||q|^{\frac{1}{2}} f\|_{L^2}, \tag{2.131}$$

so it follows that $f \in \mathcal{V}$. We can use the product rule in the distributional sense to see that

$$
\begin{aligned}
(-\Delta + q)(\eta f) &= -\Delta(\eta f) + q\eta f = (-\Delta\eta) f - \nabla\eta\nabla f - \eta\Delta f + q\eta f \\
&= (-\Delta\eta) f - \nabla\eta\nabla f + \eta(-\Delta + q) f.
\end{aligned}
\tag{2.132}
$$

Since by $f \in \mathrm{Dom} T$ we have that $f \in H^1(\Omega)$ and $(-\Delta + q) f \in L^2(\Omega)$, it follows that

$$(-\Delta + q)\eta f \in L^2(\Omega), \tag{2.133}$$

so $\eta f \in \mathrm{Dom} T$.

The density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in $H^1(\Omega)$ shows that there exists $(\phi_n)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ which converges in H^1 to f. But this yields that $(\eta\phi_n)_n \subset C_0^\infty(\Omega)$ converges to ηf, and thus $\eta f \in H_0^1(\Omega)$, which shows that it has zero Dirichlet trace.

It remains to show is that $\gamma_N^q(\eta f) = 0$. To this end, note that for every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ it holds that $\overline{\eta}\phi \in C_0^\infty(\Omega)$, and thus we obtain from the definition of the Neumann trace that

$$
\begin{aligned}
(\gamma_N^q(\eta f), \gamma_D\phi)_{\mathcal{W}^* \times \mathcal{W}} &= (\hat{A}(\eta f), \phi)_{\mathcal{V}^* \times \mathcal{V}} - \langle(-\Delta + q)(\eta f), \phi\rangle_{L^2} \\
&= \langle\nabla(\eta f), \nabla\phi\rangle_{L^2} + (qf, \overline{\eta}\phi)_{\mathcal{D}' \times \mathcal{D}} - ((-\Delta + q) f, \overline{\eta}\phi)_{\mathcal{D}' \times \mathcal{D}} \\
&\quad + 2\langle\nabla f, \phi\nabla\overline{\eta}\rangle_{L^2} + \langle f, \phi\Delta\overline{\eta}\rangle_{L^2} \\
&= (f, \nabla\overline{\eta}\nabla\phi)_{\mathcal{D}' \times \mathcal{D}} - (f, \nabla(\overline{\eta}\nabla\phi))_{\mathcal{D}' \times \mathcal{D}} + (f, \Delta(\overline{\eta}\phi))_{\mathcal{D}' \times \mathcal{D}} \\
&\quad - 2(f, \nabla(\phi\nabla\overline{\eta}))_{\mathcal{D}' \times \mathcal{D}} + (f, \phi\Delta\overline{\eta})_{\mathcal{D}' \times \mathcal{D}} = 0.
\end{aligned}
\tag{2.134}
$$

Thus the density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in $\mathcal{V}$, see Lemma 2.2.2, this yields that $\gamma_N^q(\eta f) = 0$.

The statement for $f \in \mathrm{Dom}\tilde{T}$ follows analogously. □

Lemma 2.9.2. *Let $\mathcal{G}, \mathcal{H}_0, \mathcal{H}_0$ be inner product spaces and*

$$\Sigma_0 : \mathcal{G} \to \mathcal{H}_0, \quad \Sigma_1 : \mathcal{G} \to \mathcal{H}_1, \tag{2.135}$$

such that Σ_0 and $\Sigma_1 \restriction_{\ker \Sigma_0}$ have dense range. Then it holds that $(\Sigma_0, \Sigma_1)^T$ has dense range in $\mathcal{H}_0 \oplus \mathcal{H}_1$.

Proof. Let $(u, v)^T \in \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\epsilon > 0$. By density of $\mathrm{Ran}\Sigma_0$ we can find $f_u \in \mathcal{G}$ such that

$$\|\Sigma_0 f_u - u\|_{\mathcal{H}_0} < \epsilon. \tag{2.136}$$

Next by density of $\mathrm{Ran}\Sigma_1 \restriction_{\ker \Sigma_0}$ we can find $f_v \in \mathcal{G}$ with $\Sigma_0 f_v = 0$ such that

$$\|\Sigma_1 f_v - (v - \Sigma_1 f_u)\|_{\mathcal{H}_1} < \epsilon. \tag{2.137}$$

This now shows that $f = f_u + f_v \in \mathcal{G}$ satisfies

$$\begin{aligned} \|(\Sigma_0 f, \Sigma_1 f)^T - (u, v)^T\|^2_{\mathcal{H}_0 \times \mathcal{H}_1} &= \|\Sigma_0(f_u + f_v) - u\|^2_{\mathcal{H}_0} + \|\Sigma_1(f_u + f_v) - v\|^2_{\mathcal{H}_1} \\ &= \|\Sigma_0 f_u - u\|^2_{\mathcal{H}_0} + \|\Sigma_1 f_u - (u - \Sigma_1 f_v)\|^2_{\mathcal{H}_1} < 2\epsilon^2, \end{aligned} \tag{2.138}$$

which concludes the proof. □

With these preparations we can now state our final result.

Theorem 2.9.3. *Let $T, \tilde{T}$ be as in (2.63) and $(L^2(\partial\Omega), (\Gamma_0, \Gamma_1)^T, (\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T)$ as in (2.130). Then $(L^2(\partial\Omega), (\Gamma_0, \Gamma_1)^T, (\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T)$ is a quasi boundary triple for the dual pair*

$$S = \tilde{T} \restriction_{\ker \tilde{\Gamma}_0 \cap \ker \tilde{\Gamma}_1}, \quad \tilde{S} = T \restriction_{\ker \Gamma_0 \cap \ker \Gamma_1} . \tag{2.139}$$

Proof. We will use Theorem 1.13.3.

We start by showing property (M) in the definition of quasi boundary triples. Since ι_- is by definition an isometry, it holds true that

$$\ker \Gamma_0 = \ker(\iota_- \gamma_N^q) = \ker \gamma_N^q, \tag{2.140}$$

so it is easy to see that A_N as defined in (2.105) satisfies

$$A_N = T \restriction_{\ker \gamma_N^q} = T \restriction_{\ker \Gamma_0}, \tag{2.141}$$

Analogously it follows with Theorem 2.7.2 that

$$A_N^* = \tilde{T} \restriction_{\ker \gamma_N^{\overline{q}}} = \tilde{T} \restriction_{\ker \tilde{\Gamma}_0} . \tag{2.142}$$

Note that Theorem 2.7.2 also states that A_N is closed in $L^2(\Omega)$, therefore

$$A_N^{**} = A_N = T\restriction_{\ker\Gamma_0}. \tag{2.143}$$

All together this know shows that

$$(T\restriction_{\ker\Gamma_0})^* = (A_N)^* = \tilde{T}\restriction_{\ker\tilde{\Gamma}_0}, \quad (\tilde{T}\restriction_{\ker\tilde{\Gamma}_0})^* = A_N^{**} = T\restriction_{\ker\Gamma_0}, \tag{2.144}$$

which is property (M).

Next we want to show property (DD). We will show the density of $\operatorname{Ran}(\Gamma_0, \Gamma_1)^T$ in $L^2(\partial\Omega) \oplus L^2(\partial\Omega)$, the density of $\operatorname{Ran}(\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T$ follows then analogously.

Let $f \in L^2(\partial\Omega)$. Then it holds that $\iota_+^{-1} f \in \mathcal{W}$, and since $\operatorname{Ran}\gamma_D^{\mathcal{V}}\restriction_{\operatorname{Dom}A_N}$ has dense range in $\mathcal{W}$, see Corollary 2.7.3, there exists a sequence $(u_n) \subset \operatorname{Ran}\gamma_D^{\mathcal{V}}\restriction_{\operatorname{Dom}A_N}$ such that

$$u_n \xrightarrow{n\to\infty} \iota_+^{-1} f \quad \text{in } \mathcal{W}. \tag{2.145}$$

But ι_+ is an isometric isomorphism, so this shows that

$$\|\iota_+ u_n - f\|_{L^2(\partial\Omega)} = \|u_n - \iota_+^{-1} f\| \xrightarrow{n\to\infty} 0, \tag{2.146}$$

which shows that $\Gamma_1\restriction_{\operatorname{Dom}A_N} = \Gamma_1 \restriction\restriction_{\ker\Gamma_0}$ has dense range in $L^2(\Omega)$. Since γ_N^q is surjective, see Lemma 2.8.1, and ι_- is an isomorphism, Γ_0 is surjective. Lemma 2.9.2 now states that $(\Gamma_0, \Gamma_1)^T$ has dense range, so (DD) is fulfilled.

Next we use the definition of the Neumann traces to find that

$$\begin{aligned}
\langle Tf, g\rangle_{L^2(\Omega)} - \langle f, \tilde{T}g\rangle_{L^2(\Omega)} &= \langle Tf, g\rangle_{L^2(\Omega)} - \overline{\langle \tilde{T}g, f\rangle_{L^2(\Omega)}} \\
&= (\hat{A}f, g)_{\mathcal{V}^*\times\mathcal{V}} - (\gamma_N^q f, \gamma_D^{\mathcal{V}} g)_{c\mathcal{W}^*\times\mathcal{W}} - \overline{(\hat{A}^C g, f)_{\mathcal{V}^*\times\mathcal{V}}} + \overline{(\gamma_N^{\overline{q}} g, \gamma_D^{\mathcal{V}} f)_{c\mathcal{W}^*\times\mathcal{W}}} \\
&= -\langle \Gamma_0 f, \tilde{\Gamma}_1 g\rangle_{L^2(\partial\Omega)} + \overline{\langle \tilde{\Gamma}_0 g, \Gamma_1 f\rangle_{L^2(\partial\Omega)}} + \mathbf{a}(f,g) - \overline{\overline{\mathbf{a}(f,g)}} \\
&= \langle \Gamma_1 f, \tilde{\Gamma}_0 g\rangle_{L^2(\partial\Omega)} - \langle \Gamma_0 f, \tilde{\Gamma}_1 g\rangle_{L^2(\partial\Omega)},
\end{aligned} \tag{2.147}$$

which shows (G).

What remains is to prove that $\ker\Gamma_0 \cap \ker\Gamma_1$ and $\ker\tilde{\Gamma}_0 \cap \ker\tilde{\Gamma}_1$ are dense in $L^2(\Omega)$. We will show the density of the first, the later follows analogously.

It holds true that

$$\ker\Gamma_0 \cap \ker\Gamma_1 = \ker\gamma_D^{\mathcal{V}} \cap \ker\gamma_N^q. \tag{2.148}$$

From Theorem 2.7.2 we see that $\operatorname{Dom}A_N$ is dense in $L^2(\Omega)$, so it suffices if we can show that $\ker\Gamma_0 \cap \ker\Gamma_1$ is dense in $\operatorname{Dom}A_N$ with respect to the L^2-norm.

Let $f \in \operatorname{Dom}A_N \subset \operatorname{Dom}T$. Define cut-off functions $(\eta_n)_n \subset C_0^\infty(\Omega)$ with $0 \le \eta_n \le 1$ and

$$\eta_n(x) = \begin{cases} 1 & \text{for } |x| < n \text{ and } \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}, \\ 0 & \text{for } |x| > 2n \text{ and } \operatorname{dist}(x, \partial\Omega) < \frac{1}{2n}, \end{cases} \quad n \in \mathbb{N}. \tag{2.149}$$

Lemma 2.9.1 shows that

$$\eta_n f \in \ker \gamma_D^{\mathcal{V}} \cap \ker \gamma_N^q = \ker \Gamma_0 \cap \ker \Gamma_1, \quad n \in \mathbb{N}. \tag{2.150}$$

Furthermore we have by definition that $(\eta_n f)_n$ converges pointwise to f and also $|\eta_n f| \leq |f|$, so dominated convergence gives

$$\eta_n f \xrightarrow{n\to\infty} f \quad \text{in } L^2(\Omega), \tag{2.151}$$

so we proved the density of $\ker \Gamma_0 \cap \ker \Gamma_1$ in $L^2(\Omega)$, which concludes the proof. □

3 Kato's Inequality in the Neumann Case

3.1 Motivation and Standard Version of Kato's Inequality

We recall the standard version of Kato's inequality.

Theorem 3.1.1. *Let $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$. Then it holds that*

$$\Delta|u| \geq \operatorname{Re}\left[\operatorname{sgn}(\overline{u})\Delta u\right] \quad \text{in } \mathcal{D}'(\Omega), \tag{3.1}$$

so more explicitly

$$\int_\Omega |u|\Delta\phi \mathrm{d}x \geq \int_\Omega \operatorname{Re}\left[\operatorname{sgn}(\overline{u})\Delta u\right]\phi \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega),\ \phi \geq 0, \tag{3.2}$$

where

$$\operatorname{sgn}(z) := \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0, \\ 0, & \text{else.} \end{cases} \tag{3.3}$$

Proof. See [21]. □

Kato's inequality as stated above only tests with $\mathcal{D}(\Omega)$, which makes it useful for functions with zero Dirichlet trace. In detail, consider $\Omega \subset \mathbb{R}^d$ open and non-empty, $u \in H^1_0(\Omega)$ with $\Delta u = Vu$ for some potential $V \in L^1_{\text{loc}}(\Omega)$ with $\operatorname{Re} V \geq 1$ such that also $\Delta u = Vu \in L^1_{\text{loc}}(\Omega)$. Then from Theorem 3.1.1 we see that

$$\int_\Omega |u|\Delta\phi \mathrm{d}x \geq \int_\Omega \operatorname{Re}\left[\operatorname{sgn}(\overline{u})Vu\right]\phi \mathrm{d}x \geq \int_\Omega |u|\phi \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega),\ \phi \geq 0. \tag{3.4}$$

Since $u \in H^1_0(\Omega)$, we have also that $|u| \in H^1_0(\Omega)$, and thus we can find a sequence $(u_n)_n \subset \mathcal{D}(\Omega)$ with $u_n \geq 0$ and $\lim_{n\to\infty} u_n = u$ in $H^1(\Omega)$. We then see that

$$\int_\Omega |u|\Delta u_n \mathrm{d}x = -\int_\Omega \nabla|u|\nabla u_n \mathrm{d}x \geq \int_\Omega |u|u_n \mathrm{d}x, \quad n \in \mathbb{N}, \tag{3.5}$$

and thus

$$\begin{aligned} \||u|\|_{H^1}^2 &= \int_\Omega \nabla|u|\nabla|u|\mathrm{d}x + \int_\Omega |u|^2\mathrm{d}x \\ &= \lim_{n\to\infty}\left[\int_\Omega \nabla|u|\nabla u_n\mathrm{d}x + \int_\Omega |u|u_n\mathrm{d}x\right] \le 0, \end{aligned} \tag{3.6}$$

which shows $u = 0$.

The important point here is that we can approximate u in $H_0^1(\Omega)$ with test functions. If we assume only u in $H^1(\Omega)$, then this will no longer be possible.

3.2 Kato's Inequality in the Neumann Case

In this chapter we want to give a new version of Kato's inequality for the case where we have zero Neumann trace in the sense of (3.7) (a rigorous definition of a Neumann trace is not needed here). More specifically, we want to consider a scenario where the following assumptions are fulfilled.

Assumption 3.2.1. Let $\Omega \subset \mathbb{R}^d$ be a C^∞-domain as in Definition 1.4.2, $u \in H^1(\Omega)$, $\Delta u \in L^p_{\mathrm{loc}}(\overline{\Omega})$ for $p \in (1, 2]$ and assume that

$$\langle \nabla u, \nabla \phi\rangle_{L^2} + \int_\Omega \Delta u\overline{\phi}\mathrm{d}x = 0, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega. \tag{3.7}$$

In this setting we can prove the following theorem.

Theorem 3.2.2. *Let Ω and u satisfy Assumption 3.2.1. Then it holds true that*

$$-\int_\Omega \nabla|u|\nabla\phi\mathrm{d}x \ge \int_\Omega \mathrm{Re}\left[\mathrm{sgn}(\overline{u})\Delta u\right]\phi\mathrm{d}x, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi \ge 0, \tag{3.8}$$

where

$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{|z|} & \textit{if } z \neq 0, \\ 0 & \textit{if } z = 0. \end{cases} \tag{3.9}$$

3.3 Local Approximation of H^1-functions with weak Laplacian

As a technical tool for the proof of Theorem 3.2.2 we will need to approximate functions which satisfy Assumptions 3.2.1 by smooth functions with vanishing normal derivatives locally. The aim of this section is to prove the following theorem.

Theorem 3.3.1. *Let Assumptions 3.2.1 be satisfied and $G \subset \Omega$ be open and bounded. Then $u\restriction_G \in W^{2,p}(G)$ and there exists a sequence $(\phi_n)_n \subset C_0^\infty(\mathbb{R}^d)$ with*

$$\nabla\phi_n(x) \cdot \nu(x) = 0, \quad x \in \partial\Omega,\, n \in \mathbb{N}, \tag{3.10}$$

and

$$\phi_n\restriction_G \xrightarrow{n\to\infty} u\restriction_G \ \text{ in } H^1(G), \quad \phi_n\restriction_G \xrightarrow{n\to\infty} u\restriction_G \ \text{ in } W^{2,p}(G). \tag{3.11}$$

Since Ω is a C^∞-domain, the boundary can locally be described with C^∞-hypographs. In order to prove Theorem 3.3.1, we will construct the approximations locally and glue them together via a suitable partition of unity. The following lemma shows that we can extend the function locally near the boundary.

Lemma 3.3.2. *Let $U \in \mathbb{R}^d$ be open, non-empty and bounded, divided into three parts U', U'' and Γ by a C^∞-function ξ as in Definition 1.7.1. Let $u \in H^1(U')$ be such that $\Delta u \in L^p(U')$ for some $p \in (1, 2]$ and*

$$\langle \nabla u, \nabla \phi \rangle_{L^2(U')} + \int_{U'} \Delta u \overline{\phi} \mathrm{d}x = 0, \quad \phi \in C_0^\infty(U), \tag{3.12}$$

and let $\Gamma_C \subset \Gamma$ be compact. Then there exists a open neighbourhood $W \Subset U$ of Γ_C such that $\mathcal{R}\restriction_W : W \to W$ is bijective and there exists a function $\tilde{u} \in H_0^1(\mathbb{R}^d) \cap W^{2,p}(\mathbb{R}^d)$ such that

$$\tilde{u} = \begin{cases} u & \text{in } W \cap U', \\ u \circ \mathcal{R} & \text{in } W \cap U'', \end{cases} \tag{3.13}$$

where $\mathcal{R}$ is the reflection across the neighbourhood corresponding to the C^∞-graph given by ξ as in Definition 1.8.4.

Proof. The plan to construct the function $\tilde{u}$ is to extend u to U'' by projecting the function values on U' across the normal vector in a tubular neighbourhood and then use a suitable cut-off.

In order to make the proof better understandable, Figure 3.1 illustrates the setting.

To this end, we will use the reflection across the normal vector as defined in Definition 1.8.4. We can now use Lemma 1.8.6 to obtain a neighbourhood V of Γ_C for which it holds true that

- $V \Subset \mathrm{Dom}\mathcal{R} \cap U$,
- the Jacobi matrix $D\mathcal{R}$ satisfies

$$|\det(D\mathcal{R})| \big((D\mathcal{R})^T D\mathcal{R}\big)^{-1}\restriction_V \in W^{1,\infty}(V), \tag{3.14}$$

- $v \cdot \Big(|\det(D\mathcal{R}(x))| \big((D\mathcal{R}(x))^T D\mathcal{R}(x)\big)^{-1} v\Big) \geq \frac{1}{2}|v|^2 \quad v \in \mathbb{R}^d, x \in V,$

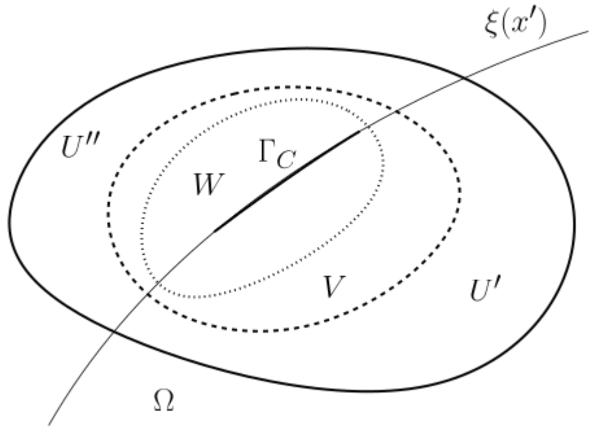

Figure 3.1: Setting in the proof of Lemma 3.3.2.

and Lemma 1.8.7 shows that it can even be chosen such that additionally

- $x \in V \Leftrightarrow \mathcal{R}(x) \in V$,
- $\mathcal{R} : V \to V$ is a bijection and a smooth diffeomorphism.

We now define an extended function on V via

$$\hat{u} = \begin{cases} u & \text{in } U' \cap V, \\ u \circ \mathcal{R} & \text{in } U'' \cap V. \end{cases} \tag{3.15}$$

Note that due to $x \in V \Leftrightarrow \mathcal{R}(x) \in V$ and the fact that $\mathcal{R}$ by its definition maps points above the boundary to points below the boundary (and vice versa), so

$$\begin{aligned} \mathcal{R}\restriction_{U'\cap V} &: U' \cap V \to U'' \cap V, \\ \mathcal{R}\restriction_{U''\cap V} &: U'' \cap V \to U' \cap V, \end{aligned} \tag{3.16}$$

are smooth diffeomorphisms. Thus $\hat{u}$ is well-defined.

Next we need to show that $\hat{u} \in H^1(V)$. Since $u \in H^1(U')$, it is clear that $\hat{u}\restriction_{U'\cap V} = u\restriction_V \in H^1(U'\cap V)$ and from Lemma 1.6.2 we see that $\hat{u}\restriction_{U''\cap V} = u \circ \mathcal{R}\restriction_{U''\cap V} \in H^1(U''\cap V)$ with

$$\nabla(\hat{u}\restriction_{U''\cap V}) = (D\mathcal{R})^T (\nabla u) \circ \mathcal{R}\restriction_{U''\cap V} . \tag{3.17}$$

Lemma 1.8.5 states that $\mathcal{R} = I$ on Γ. Note that due to (3.16) $\mathcal{R}$ is structure preserving for U in the sense of Definition 1.7.6. Therefore we can apply Lemma 1.7.7 to see that on every compact set $\Gamma_K \subset \Gamma$ it holds true that

$$\gamma_{D,\Gamma_K}(\hat{u}\restriction_{U''}) = \gamma_{D,\Gamma_K}(u \circ \mathcal{R}) = (\gamma_{D,\mathrm{Ran}\mathcal{R}\restriction_{\Gamma_K}} u) \circ \mathcal{R} = \gamma_{D,\Gamma_K} u = \gamma_{D,\Gamma_K}(\hat{u}\restriction_{U'}), \tag{3.18}$$

where γ_{D,Γ_K} denotes the Dirichlet trace on compact subsets of the boundary as defined in (1.177). By Lemma 1.7.9 this shows that $\hat{u} \in H^1(V)$.

Next we want to obtain a result for the weak second derivatives. To this end consider $\phi \in \mathcal{D}(V)$. Since $V \Subset U$ we know that for the zero extension $\tilde{\phi}$ it holds true that $\tilde{\phi} \in \mathcal{D}(U)$. From (3.12) we have that

$$\langle \nabla \hat{u}, \nabla \phi \rangle_{L^2(U' \cap V)} = \langle \nabla u, \nabla \tilde{\phi} \rangle_{L^2(U')} = -\int_{U'} \Delta u \overline{\tilde{\phi}} \mathrm{d}x = -\int_{U' \cap V} \Delta u \overline{\phi} \mathrm{d}x. \tag{3.19}$$

Since by definition of V we know that $\mathcal{R} : V \to V$ is bijective, we can define

$$\hat{\phi}(x) = \phi \circ \mathcal{R}^{-1}(x), \quad x \in V. \tag{3.20}$$

As $\mathcal{R}$ is smooth and hence maps compact sets to compact sets, we see that also $\hat{\phi} \in \mathcal{D}(V)$, so we can repeat the arguments from above to obtain that

$$\begin{aligned} \langle \nabla \hat{u}, \nabla (\phi \circ \mathcal{R}^{-1}) \rangle_{L^2(U' \cap V)} = \langle \nabla u, \nabla \hat{\phi} \rangle_{L^2(U' \cap V)} &= -\int_{U' \cap V} \Delta u \overline{\hat{\phi}} \mathrm{d}x \\ &= -\int_{U' \cap V} \Delta u \overline{\phi \circ \mathcal{R}^{-1}} \mathrm{d}x. \end{aligned} \tag{3.21}$$

The coordinate transformation $y = \mathcal{R}^{-1}(x)$ due to Lemma 1.6.5 gives

$$\begin{aligned} &\int_{U'' \cap V} \nabla \hat{u}(y) \cdot \Big(|\det(D\mathcal{R}(y))| \big((D\mathcal{R}(y))^T D\mathcal{R}(y) \big)^{-1} \nabla \overline{\phi}(y) \Big) \, \mathrm{d}y \\ &\quad = \int_{U'' \cap V} \nabla (u \circ \mathcal{R})(y) \cdot \Big(|\det(D\mathcal{R}(y))| \big((D\mathcal{R}(y))^T D\mathcal{R}(y) \big)^{-1} \nabla \overline{\phi}(y) \Big) \, \mathrm{d}y \\ &\quad = -\int_{U'' \cap V} |\det(D\mathcal{R})| \Delta u(\mathcal{R}(y)) \overline{\phi} \mathrm{d}y. \end{aligned} \tag{3.22}$$

Adding (3.19) and (3.22) and the fact that $\phi \in \mathcal{D}(V)$ was arbitrary then show that

$$\int_V \nabla \hat{u}(x) \cdot \big(A(x) \nabla \overline{\phi}(x) \big) \, \mathrm{d}x = -\int_V w(x) \overline{\phi}(x), \quad \phi \in \mathcal{D}(V), \tag{3.23}$$

where

$$A(x) = \begin{cases} I & \text{for } x \in U' \cap V, \\ |\det(D\mathcal{R}(x))| \big((D\mathcal{R}(x))^T D\mathcal{R}(x) \big)^{-1} & \text{for } x \in U'' \cap V, \end{cases} \tag{3.24}$$

and

$$w = \begin{cases} \Delta u & \text{on } U' \cap V, \\ |\det(D\mathcal{R})| (\Delta u) \circ \mathcal{R} & \text{on } U'' \cap V. \end{cases} \tag{3.25}$$

Since $\mathcal{R}$ is smooth, all its derivatives are bounded on $V \Subset \mathrm{Dom}\mathcal{R}$, so we have that $w \in L^p(V)$, and consequently

$$\operatorname{div}(A \nabla \hat{u}) = w \in L^p(V). \tag{3.26}$$

The goal is now to apply Lemma 1.9.1 to conclude that all weak second derivatives are in $L^p(V)$. This requires an extension of both $\hat{u}$ and A to $\mathbb{R}^d$. To this end, choose $W \Subset W_1 \Subset W_2 \Subset V$ with $\Gamma_C \subset W$ and W such that $\mathcal{R}\restriction_W \colon W \to W$ is bijective, which is possible due to Lemma 1.8.7.

Define $\eta_1, \eta_2 \in C_0^\infty(\mathbb{R}^d)$ with

$$\eta_1(x) = \begin{cases} 1 & \text{for } x \in W, \\ 0 & \text{for } x \notin W_1, \end{cases}, \qquad \eta_2(x) = \begin{cases} 1 & \text{for } x \in W_1, \\ 0 & \text{for } x \notin W_2. \end{cases} \tag{3.27}$$

Now we can define the extensions to $\mathbb{R}^d$

$$\tilde{u} = \begin{cases} \eta_1 \hat{u} & \text{on } V, \\ 0 & \text{else}, \end{cases}, \qquad \tilde{A} = \begin{cases} \eta_2 A + (1-\eta_2) I & \text{on } V, \\ I & \text{else}. \end{cases} \tag{3.28}$$

Multiplication with functions from $C_0^\infty(\mathbb{R}^d)$ does not change the regularity, see Lemma 1.2.10, and since $\eta_1 \hat{u}$ is compactly supported in V it holds true that $\tilde{u} \in H_0^1(\mathbb{R}^d)$. Furthermore, since $\tilde{A} = A$ on W_1 and $\tilde{u}$ is only supported on W_1, we also see that $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$ and

$$\begin{aligned} |\operatorname{div}(\tilde{A}\nabla\tilde{u})| &= |\operatorname{div}(A\nabla\tilde{u})| \leq |\operatorname{div}(\eta_1 A\nabla u)| + |\operatorname{div}(uA\nabla\eta_1)| \\ &\leq |\nabla\eta_1 \cdot (A\nabla u)| + |\eta_1 \operatorname{div}(A\nabla u)| + |\nabla u \cdot (A\nabla\eta)| + |u \operatorname{div}(A\nabla\eta_1)|, \end{aligned} \tag{3.29}$$

which shows with (3.26) that $\operatorname{div}(A\nabla\tilde{u}) \in L^p(\mathbb{R}^d)$ with

$$\begin{aligned} &\|\operatorname{div}(A\nabla\tilde{u})\|_{L^p} \\ &\leq 2\|A\nabla\eta_1\|_{L^\infty}\|\nabla u\|_{L^p} + \|\eta_1\|_{L^\infty}\|\operatorname{div}(A\nabla\hat{u})\|_{L^p} + \|\operatorname{div}(A\nabla\eta_1)\|_{L^\infty}\|u\|_{L^p}. \end{aligned} \tag{3.30}$$

Since $\eta_1 = 1$ on W we also see that we still have

$$\tilde{u}\restriction_W = \hat{u}\restriction_W = \begin{cases} u & \text{in } W \cap U', \\ u \circ \mathcal{R} & \text{in } W \cap U''. \end{cases} \tag{3.31}$$

By assumption on V we had that

$$v \cdot \Big(|\det(D\mathcal{R}(x))| \big((D\mathcal{R}(x))^T D\mathcal{R}(x) \big)^{-1} v \Big) \geq \frac{1}{2}|v|^2 \quad v \in \mathbb{R}^d, x \in V, \tag{3.32}$$

and therefore

$$\begin{aligned} v \cdot (\tilde{A}v) &= v \cdot \Big(\Big[\eta_2(x) |\det(D\mathcal{R}(x))| \big((D\mathcal{R}(x))^T D\mathcal{R}(x) \big)^{-1} + (1-\eta_2(x)) I \Big] v \Big) \\ &\geq \big(\tfrac{1}{2}\eta_2(x) + 1 - \eta_2(x) \big) |v|^2 \geq \frac{1}{2}|v|^2 \quad v \in \mathbb{R}^d, x \in V, \end{aligned} \tag{3.33}$$

and outside of V we have that $A = I$, so $\tilde{A}$ is uniformly elliptic.

It remains to show that $\tilde{A}$ is $W^{1,\infty}(\mathbb{R}^d)$. By assumption on V we have already that $|\det(D\mathcal{R}(\cdot))|\big((D\mathcal{R}(\cdot))^T D\mathcal{R}(\cdot)\big)^{-1} \in W^{1,\infty}(V \cap U'')$, therefore we obtain also

$$\begin{aligned} A\restriction_{V\cap U'} &= I \in W^{1,\infty}(V\cap U'), \\ A\restriction_{V\cap U''} &= |\det(D\mathcal{R}(\cdot))|\big((D\mathcal{R}(\cdot))^T D\mathcal{R}(\cdot)\big)^{-1} \in W^{1,\infty}(V\cap U''). \end{aligned} \tag{3.34}$$

Since $D\mathcal{R}$ is orthogonal on the boundary Γ, see Lemma 1.8.5, we obtain

$$A\restriction_{V\cap U''} = |\det(D\mathcal{R})|\big((D\mathcal{R})^T D\mathcal{R}\big)^{-1} = I = A\restriction_{V\cap U'} \quad \text{on } \Gamma, \tag{3.35}$$

and therefore, by Lemma 1.7.9 we see that $A \in W^{1,\infty}(V)$.

Since $\operatorname{supp}(\eta_2) \subset V$ we know that $\tilde{A} = I$ in some neighbourhood of ∂V and consequently we also have that $\tilde{A} \in W^{1,\infty}(\mathbb{R}^d)$.

Now we satisfy all the conditions of Lemma 1.9.1, which yields the desired result $\tilde{u} \in W^{2,p}(\mathbb{R}^d)$, and together with $\tilde{u} \in H^1_0(\mathbb{R}^d)$ and (3.31) this concludes the proof. □

Corollary 3.3.3. *In the setting of Lemma 3.3.2 there exists an open neighbourhood $V \Subset U$ of Γ_C and there exists a sequence $(\phi_n)_n \subset C^\infty(\overline{V})$ with*

$$\nabla\phi_n(x)\cdot\nu(x) = 0, \quad x \in \Gamma\cap V,\, n \in \mathbb{N}, \tag{3.36}$$

and

$$\phi_n \xrightarrow{n\to\infty} \tilde{u} \text{ in } H^1(V), \quad \phi_n \xrightarrow{n\to\infty} \tilde{u} \text{ in } W^{2,p}(V). \tag{3.37}$$

In particular, since $\tilde{u} = u$ in $V \cap U'$, it holds that

$$\phi_n \xrightarrow{n\to\infty} u \text{ in } H^1(V\cap U'), \quad \phi_n \xrightarrow{n\to\infty} u \text{ in } W^{2,p}(V\cap U'). \tag{3.38}$$

Proof. Usually one obtains smooth approximations in Sobolev norms via mollification. In our case the problem is that we also want zero normal derivative. We defined $\tilde{u}$ in a way such that it is symmetric across the normal vector, i.e. (3.31) holds true in W, where W is as in Lemma 3.3.2, but the standard mollifiers do not satisfy this.

To circumvent this problem, we will make a change of coordinates based on the tubular mapping, where the boundary is given by $\mathbb{R}^d \times \{0\}$. Here the mollifiers are symmetric around the boundary, and thus we can mollify and then transform back.

First note that in Lemma 3.3.2 we had that $W \subset \operatorname{Dom}\mathcal{R} = \operatorname{Ran}\mathcal{T}$, where $\mathcal{T}$ is the tubular mapping as defined in (1.237), so consequently the set

$$T_W := \mathcal{T}^{-1}(W) \tag{3.39}$$

is well-defined and open. We now pick an open neighbourhood $V \Subset W$ of Γ_C. Since $\mathcal{T}$ is smooth it maps compact sets to compact sets, thus

$$T_V := \mathcal{T}^{-1}(V) \Subset T_W, \tag{3.40}$$

and

$$\rho_0 = \operatorname{dist}(\partial T_W, T_V) > 0. \tag{3.41}$$

The function

$$v = \tilde{u} \circ \mathcal{T} \tag{3.42}$$

is well-defined on T_W and since $\mathcal{T}$ is smooth we see by Lemma 1.6.2 and 1.6.4 that

$$v \in H^1(T_W), \quad v \in W^{2,p}(T_W). \tag{3.43}$$

Now let

$$v^{(\rho)} := v * \omega_\rho, \quad \rho \in (0, \rho_0), \tag{3.44}$$

where $(\omega_\rho)_{\rho>0}$ is the family of standard mollifiers as defined in (1.36).

These functions are well-defined on T_V as $\rho_0 = \operatorname{dist}(\partial T_W, T_V)$. Theorem 1.3.5 shows that

$$\partial^\alpha v^{(\rho)} = (\partial^\alpha v)^{(\rho)} = (\partial^\alpha v) * \omega_\rho, \quad |\alpha| \leq 2,\ \rho \in (0, \rho_0), \tag{3.45}$$

and hence by Theorem 1.3.3 we see that

$$v^{(\rho)} \xrightarrow{\rho \to 0^+} v \text{ in } H^1(T_V), \quad v^{(\rho)} \xrightarrow{\rho \to 0^+} v \text{ in } W^{2,p}(T_V). \tag{3.46}$$

Next we want to show that the normal derivatives at the boundary vanish. By the definition of $\mathcal{T}$ in (1.237) we see that

$$\mathcal{T}^{-1}(W \cap \Gamma) = \{(x_1, ..., x_{d-1}, 0) \in \mathbb{R}^d : (x_1, ..., x_{d-1}, \xi(x_1, ..., x_{d-1})) \in W\}, \tag{3.47}$$

so the normal derivative at the boundary is just the x_d-derivative.

The standard mollifiers from (1.36) are defined in a way such that their value only depends on the modulus of the argument, therefore it is clear that

$$\omega_\rho(x', x_d) = \omega_\rho(x', -x_d), \quad \rho > 0,\ x' \in \mathbb{R}^{d-1},\ x_d \in \mathbb{R}. \tag{3.48}$$

From Lemma 3.3.2 we know that $\tilde{u} = \tilde{u} \circ \mathcal{R}$ on W and $\operatorname{Ran}\mathcal{R}\restriction_W = W$, therefore it holds true that

$$\begin{aligned} v = \tilde{u} \circ \mathcal{T} = \tilde{u} \circ \mathcal{R} \circ \mathcal{T} &= \tilde{u} \circ (\mathcal{T} \circ \operatorname{diag}(1, ..., 1, -1) \circ \mathcal{T}^{-1}) \circ \mathcal{T} \\ &= \tilde{u} \circ \mathcal{T} \circ \operatorname{diag}(1, ..., 1, -1) = v \circ \operatorname{diag}(1, ..., 1, -1), \end{aligned} \tag{3.49}$$

and so with these symmetries and one change of variables we obtain

$$\begin{aligned} v^{(\rho)}(x', x_d) &= \int_{\mathcal{B}_\rho(0)} v(x' - y', x_d - y_d)\omega_\rho(y', y_d)\mathrm{d}(y', y_d) \\ &= \int_{\mathcal{B}_\rho(0)} v(x' - y', -x_d + y_d)\omega_\rho(y', y_d)\mathrm{d}(y', y_d) \\ &= \int_{\mathcal{B}_\rho(0)} v(x' - y', -x_d - \hat{y}_d)\omega_\rho(y', -\hat{y}_d)\mathrm{d}(y', \hat{y}_d) \\ &= \int_{\mathcal{B}_\rho(0)} v(x' - y', -x_d - \hat{y}_d)\omega_\rho(y', \hat{y}_d)\mathrm{d}(y', \hat{y}_d) \\ &= v^{(\rho)}(x', -x_d), \qquad (x', x_d) \in T_V,\ (x', -x_d) \in T_V,\ \rho \in (0, \rho_0). \end{aligned} \tag{3.50}$$

Lemma 1.3.2 shows that $v^{(\rho)} \in C^\infty(\overline{T_V})$, so it holds true that

$$\partial_d v^{(\rho)}(x',0) = \lim_{h\to 0} \frac{v^{(\rho)}(x',h) - v^{(\rho)}(x',-h)}{2h} = 0, \quad (x',0) \in T_V,\ \rho \in (0,\rho_0). \tag{3.51}$$

Now choose any zero sequence $(\rho_n)_n \subset (0,\rho_0)$, and define

$$v_n := v^{(\rho_n)}. \tag{3.52}$$

Then it is clear by the previous observations that $(v_n)_n \subset C^\infty(\overline{T_V})$,

$$v_n \xrightarrow{n\to\infty} v \text{ in } H^1(T_V), \quad v_n \xrightarrow{n\to\infty} v \text{ in } W^{2,p}(T_V), \tag{3.53}$$

and

$$\partial_d v_n(x',0) = 0, \qquad (x',0) \in T_V. \tag{3.54}$$

Now we transform these function back to our original domain V, i.e. let

$$\phi_n(x) := v_n(\mathcal{T}^{-1}(x)), \quad x \in V. \tag{3.55}$$

Since $\mathcal{T}$ is a smooth diffeomorphism we have that $(\phi_n)_n \subset C^\infty(\overline{V})$, so by Lemma 1.6.2 and Lemma 1.6.4 with $v = \tilde{u} \circ \mathcal{T}$ we also see that

$$\phi_n \xrightarrow{n\to\infty} \tilde{u} \text{ in } H^1(V), \quad \phi_n \xrightarrow{n\to\infty} \tilde{u} \text{ in } W^{2,p}(V). \tag{3.56}$$

Note that

$$\begin{aligned} (D\mathcal{T}^{-1}(x))\nu(x) &= \lim_{h\to 0} \frac{\mathcal{T}^{-1}(x+h\nu(x)) - \mathcal{T}^{-1}(x-h\nu(x))}{2h} \\ &= \lim_{h\to 0} \frac{(x',h) - (x',-h)\nu(x))}{2h} \\ &= (0,...,0,1)^T, \qquad x = (x',\xi(x')) \in \Gamma, \end{aligned} \tag{3.57}$$

so finally from the chain rule we obtain that

$$\begin{aligned} \nabla\phi_n(x)\cdot\nu(x) &= \left[(D\mathcal{T}^{-1}(x))^T \nabla v_n(\mathcal{T}^{-1}(x))\right]\cdot\nu(x) \\ &= \left[(D\mathcal{T}^{-1}(x))\nu(x)\right]\cdot\nabla v_n(x) = (0,...,0,1)^T \cdot \nabla v_n(\mathcal{T}^{-1}(x)) \\ &= \partial_d v_n(\mathcal{T}^{-1}(x)) = \partial_d v_n(x_1,...,x_{d-1},0) = 0, \qquad x \in \Gamma\cap V,\ n\in\mathbb{N}, \end{aligned} \tag{3.58}$$

which concludes the proof. □

Lemma 3.3.4. *In the setting of Corollary 3.3.3 there exists an open neighbourhood* $W \Subset V$ *of* Γ_C *and a cut-off function* $\eta \in C_0^\infty(U)$ *with* $0 \le \eta \le 1$ *and*

$$\eta(x) = \begin{cases} 1 & \text{for } x \in W, \\ 0 & \text{for } x \notin V, \end{cases} \tag{3.59}$$

such that

$$\nabla\eta(x)\cdot\nu(x) = 0, \qquad x \in \Gamma. \tag{3.60}$$

Proof. From Lemma 1.8.7 we see that there exists an open neighbourhood $\tilde{V} \subset V$ of Γ_C such that $\mathcal{R}\restriction_{\tilde{V}}: \tilde{V} \to \tilde{V}$ is bijective, w.l.o.g. assume that $V = \tilde{V}$.

Also due to Lemma 1.8.7 we can find an open neighbourhood $W \Subset V$ of Γ_C such that $\mathcal{R}\restriction_W: W \to W$ is bijective. Then we can choose a function $\phi \in C_0^\infty(U)$ with $0 \le \phi \le 1$ such that

$$\phi(x) = \begin{cases} 1 & \text{for } x \in W, \\ 0 & \text{for } x \notin V. \end{cases} \tag{3.61}$$

Since $\mathcal{R}\restriction_V: V \to V$ is a smooth diffeomorphism we see that

$$\hat{\phi}(x) = \begin{cases} \phi \circ \mathcal{R} & \text{for } x \in V, \\ 0 & \text{else,} \end{cases} \tag{3.62}$$

also satisfies $\hat{\phi} \in C_0^\infty(U)$, $0 \le \hat{\phi} \le 1$ and

$$\hat{\phi}(x) = \begin{cases} 1 & \text{for } x \in \operatorname{Ran}\mathcal{R}\restriction_W = W, \\ 0 & \text{for } x \notin V. \end{cases} \tag{3.63}$$

Therefore if we now define $\eta \in C_0^\infty(U)$ as

$$\eta(x) = \frac{1}{2}(\phi(x) + \hat{\phi}(x)), \quad x \in U, \tag{3.64}$$

we see that $0 \le \eta \le 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for } x \in W, \\ 0 & \text{for } x \notin V. \end{cases} \tag{3.65}$$

It remains to show that the normal derivatives vanish. To this end, first note that by the construction of $\mathcal{R}$ in Definition 1.8.4 there exists for every $x \in \Gamma$ a $h_0 > 0$ such that

$$\begin{aligned} \mathcal{R}(x + h\nu(x)) &= \mathcal{T} \circ \operatorname{diag}(1, ..., 1, -1) \circ \mathcal{T}(x + h\nu(x)) \\ &= \mathcal{T} \circ \operatorname{diag}(1, ..., 1, -1)(x_1, ..., x_{d-1}, h) \\ &= \mathcal{T}(x_1, ..., x_{d-1}, -h) = x - h\nu(x), \qquad |h| < h_0, \end{aligned} \tag{3.66}$$

and consequently

$$\begin{aligned} &\eta(x + h\nu(x)) = \phi(x + h\nu(x)) + \hat{\phi}(x + h\nu(x)) \\ &\quad = \phi(x + h\nu(x)) + \phi(\mathcal{R}(x + h\nu(x))) = \phi(x + h\nu(x)) + \phi(x - h\nu(x)) \\ &\quad = \eta(x - h\nu(x)), \qquad |h| < h_0. \end{aligned} \tag{3.67}$$

Therefore the definition of the normal derivative for smooth functions shows that for every $x \in \Gamma \cap V$

$$\nabla\eta(x) \cdot \nu(x) = \lim_{h \to 0} \frac{\eta(x + h\nu(x)) - \eta(x - h\nu(x))}{2h} = 0, \tag{3.68}$$

which concludes the proof. □

Proof of Theorem 3.3.1. In order to make the proof better understandable we will divide it into steps.

Step 1: Dividing into local sub problems

Let $\epsilon > 0$ as in Definition 1.4.2 Since Ω is a C^∞ domain we can choose a regular C^∞-atlas $((U_i, \xi_i, \kappa_i))_{i\in\mathbb{N}}$ of Ω as in Definition 1.4.4. By the definition of regular atlases we have that the $(U_j)_{j\in\mathbb{N}}$ are all bounded and include at least some ϵ-ball.

There is a global limit for how many U_i may intersect at any given point, so only finitely many U_i can intersect with the compact set $\overline{G}$. W.l.o.g. let those sets be $(U_i)_{i=1}^N$.

Now define compact parts of the boundary

$$\Gamma_{i,C} := \left\{ x \in \partial\Omega \cap U_i : \operatorname{dist}(x, \partial U_i) \geq \frac{\epsilon}{2} \right\}, \quad i \in \{1, ..., N\}. \tag{3.69}$$

In order to show that these parts cover $\partial\Omega \cap \overline{G}$, let $x \in \partial\Omega \cap \overline{G}$. Then by the definition of C^∞-domains there exists a $j \in \mathbb{N}$ such that

$$\mathbb{B}_\epsilon(x) \subset U_j. \tag{3.70}$$

Since $x \in \overline{G}$ and only $(U_i)_{i=1}^N$ intersect with $\overline{G}$ it is clear that $j \in \{1, ..., N\}$. Furthermore we see that

$$\operatorname{dist}(x, \partial U_j) \geq \epsilon, \tag{3.71}$$

and therefore $x \in \Gamma_{j,C}$. Consequently we obtain that

$$\partial\Omega \cap \overline{G} \subset \bigcup_{i=1}^N \Gamma_{i,C}. \tag{3.72}$$

We now define for every $i \in \{1, ..., N\}$ sets

$$\tilde{U}_i = \kappa_i(U_i), \quad U_i' = \kappa_i(U_i \cap \Omega), \quad \Gamma_i = \kappa_i(U_i \cap \partial\Omega), \quad U_i'' = \kappa_i(U_i \cap (\mathbb{R}^d \setminus \overline{\Omega})), \tag{3.73}$$

where $(\kappa_i)_{i\in\{1,...,N\}}$ are the rigid body motions from the definition of C^∞-domains.

Step 2: Provide local approximations

In the following let $i \in 1, ..., N$ be fixed. By the definition of C^∞ domains $\tilde{U}_i$ is divided into U_i', U_i'', Γ_i by the C^∞-function ξ_i according to Definition 1.7.1. We define the function

$$\hat{u}_i = u \circ \kappa_i^{-1} \text{ on } U_i' \tag{3.74}$$

and conclude by Lemma 1.6.2 and Corollary 1.6.6 that

$$\hat{u}_i \in H^1(U_i'), \quad \Delta\hat{u}_i \in L^p(U_i'). \tag{3.75}$$

Let $\phi \in C_0^\infty(\tilde{U}_i)$. Then $\phi \circ \kappa_i \in C_0^\infty(U_i)$, which shows that its zero extension satisfies $\phi \circ \kappa_i \in C_0^\infty(\mathbb{R}^d)$. Thus by Corollary 1.6.6 and Assumptions 3.2.1 we see that

$$\begin{aligned}\langle \nabla \hat{u}_i, \nabla \phi \rangle_{L^2(U_i')} &+ \int_{U_i'} \Delta \hat{u}_i \overline{\phi} \mathrm{d}x \\ &= \langle \nabla u, \nabla(\phi \circ \kappa_i) \rangle_{L^2(U_i)} + \int_{U_i} \Delta u \overline{\phi \circ \kappa_i} \mathrm{d}x = 0,\end{aligned} \tag{3.76}$$

and since this holds true for all $\phi \in C_0^\infty(\tilde{U}_i)$ we are in the situation of Lemma 3.3.2.

Together with Corollary 3.3.3 we see that there exists an open neighbourhood $\tilde{V}_i \subset \tilde{U}_i$ of $\kappa_i(\Gamma_{i,C})$ and a sequence $(\phi_n^{(i)})_n \subset C^\infty(\overline{\tilde{V}_i})$ such that

$$\nabla \phi_n^{(i)} \cdot \nu_i(x) = 0, \quad x \in \Gamma_i \cap \tilde{V}_i,\, n \in \mathbb{N}, \tag{3.77}$$

where ν_i denotes the unit outwards normal vector at Γ and

$$\phi_n^{(i)} \xrightarrow{n\to\infty} \hat{u}_i \text{ in } H^1(\tilde{V}_i \cap U_i'), \quad \phi_n^{(i)} \xrightarrow{n\to\infty} \hat{u}_i \text{ in } W^{2,p}(\tilde{V}_i \cap U_i'). \tag{3.78}$$

Furthermore by Lemma 3.3.4 we find an open neighbourhood $\tilde{W}_i \Subset \tilde{V}_i$ of $\kappa_i(\Gamma_{i,C})$ and a cut-off function $\tilde{\eta}_i \in C_0^\infty(\mathbb{R}^d)$ with

$$\tilde{\eta}_i(x) = \begin{cases} 1 & \text{for } x \in \tilde{W}_i, \\ 0 & \text{for } x \notin \tilde{V}_i, \end{cases} \tag{3.79}$$

and

$$\nabla \tilde{\eta}_i \cdot \nu_i(x) = 0, \quad x \in \Gamma_i \cap \tilde{V}_i,\, n \in \mathbb{N}. \tag{3.80}$$

Step 3: Transforming back

Now we reverse the rigid body motion κ_i again. Let

$$V_i := \kappa_i^{-1}(\tilde{V}_i), \qquad W_i := \kappa_i^{-1}(\tilde{W}_i). \tag{3.81}$$

Since $\tilde{W}_i \Subset \tilde{V}_i$ is an open neighbourhood of $\kappa_i(\Gamma_{i,C})$ and κ_i is a rigid body motion it is clear that $W_i \Subset V_i$ is an open neighbourhood of $\Gamma_{i,C}$.

We also define

$$\hat{\eta}_i = \tilde{\eta}_i \circ \kappa_i, \qquad \psi_n^{(i)} = \phi_n^{(i)} \circ \kappa_i, \quad n \in \mathbb{N}. \tag{3.82}$$

Clearly we have $\hat{\eta}_i \in C_0^\infty(\mathbb{R}^d)$ with

$$\hat{\eta}_i(x) = \begin{cases} 1 & \text{for } x \in W_i, \\ 0 & \text{for } x \notin V_i, \end{cases} \tag{3.83}$$

and $(\psi_n^{(i)})_n \subset C^\infty(\overline{V_i})$. Lemma 1.6.2 and Lemma 1.6.4 give

$$\psi_n^{(i)} \xrightarrow{n\to\infty} u \text{ in } H^1(V_i \cap \Omega), \quad \psi_n^{(i)} \xrightarrow{n\to\infty} u \text{ in } W^{2,p}(V_i \cap \Omega). \tag{3.84}$$

Since the rigid body motion $\kappa_i : \mathbb{R}^d \to \mathbb{R}^d$ acts as

$$\kappa_i(x) = R_i x + t_i, \qquad x \in \mathbb{R}^d, \tag{3.85}$$

with some orthogonal matrix $R_i \in \mathbb{R}^{d\times d}$ and vector $t_i \in \mathbb{R}^d$, see Definition 1.4.1, we see that

$$D\kappa_i^{-1} = R_i^{-1}. \tag{3.86}$$

This shows that

$$\begin{aligned}\nabla\psi_n^{(i)} \cdot \nu(x) &= \left(R_i^T \nabla\phi_n^{(i)}(\kappa_i(x))\right) \cdot \left(R_i^{-1}\nu_i(\kappa_i(x))\right) \\ &= \nabla\phi_n^{(i)}(\kappa_i(x)) \cdot \left(R_i(R_i)^{-1}\nu_i(\kappa_i(x))\right) \\ &= \nabla\phi_n^{(i)}(\kappa_i(x)) \cdot \nu_i(\kappa_i(x)) = 0, \qquad x \in \partial\Omega \cap V_i,\ n \in \mathbb{N},\end{aligned} \tag{3.87}$$

where ν denotes the unit outwards normal vector of Ω. Analogously we obtain

$$\nabla\hat{\eta}_i \cdot \nu(x) = 0, \qquad x \in \partial\Omega \cap V_i. \tag{3.88}$$

These results hold true for every $i \in \{1, ..., N\}$.

Step 4: Approximation in the interior

Next we consider

$$G_0 := G \setminus \left(\bigcup_{i=1}^{N} \overline{W_i}\right). \tag{3.89}$$

We need to show that $G_0 \Subset \Omega$. To this end note that $G_0 \subset G$ is bounded since G is bounded, and G_0 is open as the difference of an open set and closed sets.

Let $x \in \partial\Omega$. We need to show that $x \notin \overline{G_0}$. By the definition of C^∞ domains it holds true that $\mathbb{B}_\epsilon(x) \subset U_i$ for some $i \in \mathbb{N}$. If we would have $x \in \overline{G_0} \subset \overline{G}$, then U_i intersects with $\overline{G}$ and this means that $i \in \{1, ..., N\}$. This would also show that

$$\operatorname{dist}(x, \partial U_i) \geq \frac{\epsilon}{2} \tag{3.90}$$

and so $x \in \Gamma_{i,C}$. Since W_i is an open neighbourhood of $\Gamma_{i,C}$ we would have $x \notin \overline{G_0}$, which is a contradiction. In summary, we showed that $G_0 \Subset \Omega$.

Consequently we can find an open set G_1 with $G_0 \Subset G_1 \Subset \Omega$ and a function $\hat{\eta}_0 \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq \hat{\eta}_0 \leq 1$ and

$$\hat{\eta}_0(x) = \begin{cases} 1 & \text{for } x \in G_0, \\ 0 & \text{for } x \notin G_1, \end{cases} \tag{3.91}$$

which clearly satisfies

$$\nabla\hat{\eta}_0 \cdot \nu(x) = 0, \quad x \in \partial\Omega, \tag{3.92}$$

since the function is constant zero in some open neighbourhood of the boundary.

Now choose G_2 such that $G_1 \Subset G_2 \Subset \Omega$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ with

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in G_1, \\ 0 & \text{for } x \notin G_2. \end{cases} \tag{3.93}$$

Then the function

$$\tilde{u}_0 = \varphi u, \tag{3.94}$$

implicitly with a zero extension to the whole $\mathbb{R}^d$, satisfies $\tilde{u}_0 \in H^1(\mathbb{R}^d)$ since multiplication with $C_0^\infty(\mathbb{R}^d)$-function does not change the regularity, see Lemma 1.2.10. Furthermore it holds true that $\Delta u \in L^p(G_2)$, and thus by the same lemma we see that $\Delta \tilde{u} \in L^p(\mathbb{R}^d)$, since $\tilde{u}$ is zero outside G_2.

Now define

$$\psi_\rho^{(0)} := \tilde{u}_0 * \omega_\rho, \quad \rho \in (0, \infty). \tag{3.95}$$

where $(\omega_\rho)_{\rho>0}$ is the family of standard mollifiers as in (1.36). Due to Lemma 1.3.2 we have $\psi_\rho^{(0)} \in C_0^\infty(\mathbb{R}^d)$ for every $\rho \in (0, \infty)$ and Theorem 1.3.5 shows that

$$\partial^\alpha \psi_\rho^{(0)} = (\partial^\alpha \tilde{u}_0) * \omega_\rho, \quad |\alpha| \le 2. \tag{3.96}$$

Therefore Theorem 1.3.3 yields

$$\psi_\rho^{(0)} \xrightarrow{\rho \to 0^+} \tilde{u}_0 \text{ in } H^1(\mathbb{R}^d), \quad \psi_\rho^{(0)} \xrightarrow{\rho \to 0^+} \tilde{u}_0 \text{ in } W^{2,p}(\mathbb{R}^d). \tag{3.97}$$

Now choose a zero-sequence $(\rho_n)_n \subset (0, \rho_0)$ and define

$$\psi_n^{(0)}(x) := \psi_{\rho_n}^{(0)}(x), \quad x \in G_1. \tag{3.98}$$

Since $\tilde{u}_0 = u$ in G_1 we obtain that

$$\psi_n^{(0)} \xrightarrow{n \to \infty} u \text{ in } H^1(G_1), \quad \psi_n^{(0)} \xrightarrow{n \to \infty} u \text{ in } W^{2,p}(G_1). \tag{3.99}$$

Step 5: Constructing a suitable partition of unity

We now want to construct a partition of unity of G where all functions have zero normal derivative. To this end define $(\eta_i)_{i=0}^N$ with

$$\eta_0(x) = \hat{\eta}_0(x), \quad \eta_i = \hat{\eta}_i(x) \prod_{j=0}^{i-1} (1 - \hat{\eta}_j(x)), \quad i \in \{1, ..., N\}, \, x \in \mathbb{R}^d, \tag{3.100}$$

with $(\hat{\eta}_i)_{i=0}^N$ from (3.83) and (3.91).

By construction we have that $0 \le \eta_i \le 1$ and $\eta_i \in C_0^\infty(\mathbb{R}^d)$ for $i \in \{0, ..., N\}$.

Since these are all products of functions with zero normal derivative, the product rule shows that also

$$\nabla \eta_i \cdot \nu(x) = 0, \qquad x \in \partial\Omega, \, i \in \{0, ..., N\}. \tag{3.101}$$

Next we want to show that

$$\sum_{i=0}^{M}\eta_i(x) = 1-\prod_{j=0}^{M}(1-\hat{\eta}_j(x)), \qquad M\in\{0,...,N\}, \tag{3.102}$$

via induction. For $M=0$ we have

$$\sum_{i=0}^{0}\eta_i(x) = \eta_0(x) = \hat{\eta_0}(x) = 1-(1-\hat{\eta_0}(x)) = 1-\prod_{j=0}^{0}(1-\hat{\eta}_i(x)), \quad x\in\mathbb{R}^d. \tag{3.103}$$

Now assume the statement holds already true for $M-1$ for some $M\in\{1,...,N\}$. Then

$$\begin{aligned}\sum_{i=0}^{M}\eta_i(x) &= \sum_{i=0}^{M-1}\eta_i(x)+\eta_M(x) = 1-\prod_{j=0}^{M-1}(1-\hat{\eta}_j(x))+\hat{\eta}_i(x)\prod_{j=0}^{M-1}(1-\hat{\eta}_j(x))\\ &= 1-(1-\hat{\eta}_M(x))\prod_{j=0}^{M-1}(1-\hat{\eta}_j(x)) = 1-\prod_{j=0}^{M}(1-\hat{\eta}_j(x)), \quad x\in\mathbb{R}^d.\end{aligned} \tag{3.104}$$

Note that by construction $\hat{\eta}_i=1$ on $\overline{W_i}$ for $i\in\{1,...,N\}$, and $\hat{\eta_0}=1$ on $G\backslash\bigcup_{i=1}^N\overline{W_i}$, so we see that

$$\sum_{i=0}^{N}\eta_i(x) = 1-\prod_{j=0}^{N}(1-\hat{\eta}_j(x)) = 1, \quad x\in G. \tag{3.105}$$

Finally we define an sequence

$$\psi_n = \sum_{i=0}^{N}\eta_i\psi_n^{(i)}, \qquad n\in\mathbb{N}. \tag{3.106}$$

Note that the only function on the right-hand side of (3.106) which does not necessarily have zero normal derivative is $\psi_n^{(0)}$ for $n\in\mathbb{N}$. Since η_0 has compact support in Ω it nevertheless holds true that $\eta_0\psi_n^{(0)}$ has zero normal derivative for every $n\in\mathbb{N}$, hence the product rule shows that

$$\nabla\psi_n(x)\cdot\nu(x) = 0, \quad x\in\partial\Omega,\, n\in\mathbb{N}. \tag{3.107}$$

Furthermore, from Lemma 1.2.10 with $\eta_i\in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\eta_i\Subset V_i$ for $i\in\{1,...,N\}$ and $\operatorname{supp}\eta_0\subset G_1$ we obtain that there exists a $C>0$ such that

$$\begin{aligned}\|\psi_n-u\|_{H^1(G)} &= \left\|\sum_{i=0}^{N}\eta_i\psi_n^{(i)}-\sum_{i=0}^{N}\eta_i u\right\|_{H^1(G)} = \left\|\sum_{i=0}^{N}\eta_i(\psi_n^{(i)}-u)\right\|_{H^1(G)}\\ &\le \sum_{i=0}^{N}\|\eta_i(\psi_n^{(i)}-u)\|_{H^1(V_i\cap\Omega)} \le C\sum_{i=0}^{N}\|\psi_n^{(i)}-u\|_{H^1(V_i\cap\Omega)}.\end{aligned} \tag{3.108}$$

Thus

$$\psi_n \xrightarrow{n\to\infty} u \quad \text{in } H^1(G), \tag{3.109}$$

and in the same way we obtain that

$$\psi_n \xrightarrow{n\to\infty} u \quad \text{in } W^{2,p}(G), \tag{3.110}$$

which concludes the proof. □

3.4 Proof of Kato's Inequality for the Neumann Case

In order to prove Theorem 3.2.2 we need the following Lemma.

Lemma 3.4.1. *Let Ω and u satisfy Assumption 3.2.1, and define*

$$u_\epsilon = (|u|^2 + \epsilon^2)^{\frac{1}{2}}, \quad \epsilon > 0. \tag{3.111}$$

Then it holds true that $u_\epsilon \in H^1_{\mathrm{loc}}(\Omega)$ and

$$-\int_\Omega \nabla u_\epsilon \nabla \phi \mathrm{d}x \geq \int_\Omega \mathrm{Re}\left[\frac{\overline{u}}{u_\epsilon}\Delta u\right]\phi \mathrm{d}x, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi \geq 0. \tag{3.112}$$

Proof. First focus on sufficiently regular functions. To this end, let $v \in C^2(\overline{\Omega})$ with

$$\nabla v(x) \cdot \nu(x) = 0, \qquad x \in \partial\Omega, \tag{3.113}$$

and

$$v_\epsilon = (|v|^2 + \epsilon^2)^{\frac{1}{2}}, \quad \epsilon > 0. \tag{3.114}$$

Since v is twice continuously differentiable we see that

$$\partial_j v_\epsilon = \frac{1}{2(|v|^2+\epsilon^2)^{\frac{1}{2}}}\partial_j |v|^2 = \frac{1}{2(|v|^2+\epsilon^2)^{\frac{1}{2}}}(v\partial_j\overline{v} + \overline{v}\partial_j v) = \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\partial_j v\right], \tag{3.115}$$

and

$$\begin{aligned}
\partial_j^2 v_\epsilon &= \partial_j \,\mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\partial_j v\right] = -\frac{1}{v_\epsilon^2}(\partial_j v_\epsilon)\,\mathrm{Re}(\overline{v}\partial_j v) + \frac{1}{2v_\epsilon}\partial_j\left(v\partial_j\overline{v} + \overline{v}\partial_j v\right) \\
&= -\frac{1}{v_\epsilon^2}\,\mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\partial_j v\right]\mathrm{Re}(\overline{v}\partial_j v) + \frac{1}{2v_\epsilon}\left(v\partial_j^2\overline{v} + 2\partial_j\overline{v}\partial_j v + \overline{v}\partial_j^2 v\right) \\
&= -\frac{1}{v_\epsilon}\,\mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\partial_j v\right]^2 + \frac{1}{v_\epsilon}\,\mathrm{Re}\left[|\partial_j v|^2 + \overline{v}\partial_j^2 v\right],
\end{aligned} \tag{3.116}$$

so we obtain that

$$\begin{aligned}\nabla v_\epsilon &= \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\nabla v\right]\\ \Delta v_\epsilon &= \sum_{j=1}^{d}\partial_j^2 v_\epsilon = -\frac{1}{v_\epsilon}|\nabla v_\epsilon|^2 + \frac{1}{v_\epsilon}|\nabla v|^2 + \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\Delta v\right].\end{aligned} \tag{3.117}$$

Since by definition $|v| < v_\epsilon$, we see that

$$|\nabla v_\epsilon| \le \frac{|v|}{v_\epsilon}|\nabla v| \le |\nabla v|, \tag{3.118}$$

and therefore with (3.117) we obtain

$$\Delta v_\epsilon \ge \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\Delta v\right] \tag{3.119}$$

pointwise for $v \in C^2(\overline{\Omega})$. Notice that this yields

$$\int_\Omega \Delta v_\epsilon \phi \mathrm{d}x \ge \int_\Omega \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\Delta v\right]\phi \mathrm{d}x, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi \ge 0, \tag{3.120}$$

since every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ has bounded support and $v, v_\epsilon \in C^2(\overline{\Omega})$, so the integrals are finite.

We can use Green's first identity for $\mathcal{C}^2$-functions in order to see that for every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ with $\phi \ge 0$ it holds true that

$$\int_{\partial\Omega} \phi \nabla v_\epsilon \cdot \nu \mathrm{d}\sigma(x') \ge \int_\Omega \nabla v_\epsilon \nabla \phi \mathrm{d}x + \int_\Omega \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\Delta v\right]\phi \mathrm{d}x. \tag{3.121}$$

Note that

$$\begin{aligned}\nabla v_\epsilon(x)\cdot\nu(x) &= \mathrm{Re}\left[\frac{\overline{v}(x)}{v_\epsilon(x)}\nabla v(x)\right]\cdot\nu(x)\\ &= \mathrm{Re}\left[\frac{\overline{v}(x)}{v_\epsilon(x)}(\nabla v(x)\cdot\nu(x))\right] = 0, \qquad x \in \partial\Omega,\end{aligned} \tag{3.122}$$

where we used (3.113) in the last step. Therefore it follows that

$$0 \ge \int_\Omega \nabla v_\epsilon \nabla\phi \mathrm{d}x + \int_\Omega \mathrm{Re}\left[\frac{\overline{v}}{v_\epsilon}\Delta v\right]\phi \mathrm{d}x, \quad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi \ge 0. \tag{3.123}$$

Now let u satisfy Assumptions 3.2.1, $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$, $\phi \ge 0$, and let $G \subset \Omega$ be a bounded set such that $\operatorname{supp}\phi \subset \overline{G}$.

By Theorem 3.3.1 there exists a sequence $(\psi_n)_n \subset C^2(\overline{\Omega})$ such that

$$\nabla\psi_n(x)\cdot\nu(x) = 0, \qquad x \in \partial\Omega,\, n \in \mathbb{N}, \tag{3.124}$$

and

$$\begin{aligned} \psi_n &\xrightarrow{n\to\infty} u \quad \text{in } H^1(G), \\ \Delta\psi_n &\xrightarrow{n\to\infty} \Delta u \quad \text{in } L^1(G). \end{aligned} \tag{3.125}$$

Since the ψ_n are sufficiently regular, we see by the previous step of the proof that

$$0 \geq \int_\Omega \nabla(\psi_n)_\epsilon \nabla\phi \mathrm{d}x + \int_\Omega \mathrm{Re}\left[\frac{\overline{\psi_n}}{(\psi_n)_\epsilon}\Delta(\psi_n)\right]\phi \mathrm{d}x, \quad n \in \mathbb{N}. \tag{3.126}$$

The goal is now to conclude the proof with the limit for $n \to \infty$ of the above inequality.

First, since $\psi_n \xrightarrow{n\to\infty} u$ in $L^2(G)$, there exists a subsequence which converges pointwise a.e., w.l.o.g. assume that the whole sequence $(\psi_n)_n$ does.

We see that

$$\begin{aligned} |(\psi_n)_\epsilon - u_\epsilon| &= \left|\left(|\psi_n|^2+\epsilon^2\right)^{\frac{1}{2}} - \left(|u|^2+\epsilon^2\right)^{\frac{1}{2}}\right| = \left|\frac{|\psi_n|^2 - |u|^2}{(|\psi_n|^2+\epsilon^2)^{\frac{1}{2}} + (|u|^2+\epsilon^2)^{\frac{1}{2}}}\right| \\ &\leq \left|\frac{|\psi_n|^2 - |u|^2}{|\psi_n| + |u|}\right| = ||\psi_n| - |u|| \leq |\psi_n - u|\,, \end{aligned} \tag{3.127}$$

so we also have that $(\psi_n)_\epsilon \xrightarrow{n\to\infty} u_\epsilon$ pointwise a.e.. Since by definition

$$\left|\frac{\overline{\psi_n}}{(\psi_n)_\epsilon}\right| \leq 1, \quad \left|\frac{\overline{u}}{u_\epsilon}\right| \leq 1, \quad (\psi_n)_\epsilon \geq \epsilon, \quad u_\epsilon \geq \epsilon, \tag{3.128}$$

we also have that

$$\frac{\overline{\psi_n}}{(\psi_n)_\epsilon} - \frac{\overline{u}}{u_\epsilon} \xrightarrow{n\to\infty} 0 \text{ pointwise a.e.}, \qquad \left|\left(\frac{\overline{\psi_n}}{(\psi_n)_\epsilon} - \frac{\overline{u}}{u_\epsilon}\right)(\Delta u)\phi\right| \leq 2|(\Delta u)\phi|. \tag{3.129}$$

Thus via dominated convergence

$$\int_\Omega \left(\frac{\overline{\psi_n}}{(\psi_n)_\epsilon} - \frac{\overline{u}}{u_\epsilon}\right)\Delta u\phi \mathrm{d}x \xrightarrow{n\to\infty} 0. \tag{3.130}$$

Additionally it holds true that

$$\left|\int_\Omega \frac{\overline{\psi_n}}{(\psi_n)_\epsilon}\left(\Delta\psi_n - \Delta u\right)\phi \mathrm{d}x\right| \leq \|\phi\|_{L^\infty(G)} \int_G |\Delta\psi_n - \Delta u| \mathrm{d}x \xrightarrow{n\to\infty} 0, \tag{3.131}$$

and together from (3.130) and (3.131) we conclude that

$$\begin{aligned}\lim_{n\to\infty}&\left|\int_\Omega\left(\operatorname{Re}\left[\frac{\overline{\psi_n}}{(\psi_n)_\epsilon}\Delta(\psi_n)\right]-\operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\Delta u\right]\right)\phi\mathrm{d}x\right|\\&\leq\lim_{n\to\infty}\left|\int_\Omega\operatorname{Re}\left[\frac{\overline{\psi_n}}{(\psi_n)_\epsilon}(\Delta\psi_n-\Delta u)\right]\phi\mathrm{d}x\right|\\&\quad+\lim_{n\to\infty}\left|\int_\Omega\operatorname{Re}\left[\left(\frac{\overline{\psi_n}}{(\psi_n)_\epsilon}-\frac{\overline{u}}{u_\epsilon}\right)\Delta u\right]\phi\mathrm{d}x\right|=0.\end{aligned}\tag{3.132}$$

Now we need to calculate the gradient of u_ϵ. From Lemma 1.6.10 know that $|u|\in H^1(\Omega)$ and

$$\partial_j|u|=\operatorname{Re}(\operatorname{sgn}(\overline{u})\partial_j v),\quad j\in\{1,...,d\}.\tag{3.133}$$

Since $|u|$ is real-valued we can now use the chain rule from Lemma 1.6.9 to obtain

$$\partial_j u_\epsilon=\frac{1}{2(|u|^2+\epsilon^2)^{\frac{1}{2}}}\partial_j|u|^2=\operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\partial_j v\right].\tag{3.134}$$

This shows that

$$\nabla u_\epsilon=\operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\nabla u\right],\tag{3.135}$$

and thus

$$\begin{aligned}\lim_{n\to\infty}&\left|\int_\Omega(\nabla(\psi_n)_\epsilon-\nabla u_\epsilon)\nabla\phi\mathrm{d}x\right|\\&=\lim_{n\to\infty}\left|\int_\Omega\left(\operatorname{Re}\left[\frac{\overline{\psi_n}}{(\psi_n)_\epsilon}\nabla\psi_n\right]-\operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\nabla u\right]\right)\nabla\phi\mathrm{d}x\right|=0,\end{aligned}\tag{3.136}$$

follows with the analogous arguments as (3.132).

Finally we obtain from (3.126) together with the limits from (3.132) and (3.136) that

$$-\int_\Omega\nabla u_\epsilon\nabla\phi\mathrm{d}x\geq\int_\Omega\operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\Delta u\right]\phi\mathrm{d}x,\tag{3.137}$$

and since $\phi\in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$, $\phi\geq 0$ was arbitrary, this ends the proof. □

With these preparations we are finally ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. We will show this with the help of Lemma 3.4.1, where we calculate the limit for $\epsilon\to 0^+$.

First recall that by Lemma 1.6.10 we have $|u|\in H^1(\Omega)$ and

$$\nabla|u|=\operatorname{Re}(\operatorname{sgn}(\overline{u})\nabla u).\tag{3.138}$$

It holds true that

$$|u_\epsilon - |u|| = \left|(|u|^2+\epsilon^2)^{\frac{1}{2}} - |u|\right| = \left|\frac{\epsilon^2}{(|u|^2+\epsilon^2)^{\frac{1}{2}}+|u|}\right| \le \epsilon, \quad \epsilon > 0, \tag{3.139}$$

so we see that $u_\epsilon \xrightarrow{\epsilon\to 0^+} |u|$ pointwise and hence

$$\frac{\overline{u}}{u_\epsilon} \xrightarrow{\epsilon\to 0^+} \operatorname{sgn}(\overline{u}) \ \text{ pointwise}, \qquad \left|\frac{\overline{u}}{u_\epsilon}\Delta u\right| \le |\Delta u|, \qquad \left|\frac{\overline{u}}{u_\epsilon}\nabla u\right| \le |\nabla u|. \tag{3.140}$$

Dominated convergence therefore yields for a fixed $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$, $\phi \ge 0$, that

$$\lim_{\epsilon\to 0^+}\int_\Omega \operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\Delta u\right]\phi \mathrm{d}x = \int_\Omega \operatorname{Re}\left[\operatorname{sgn}(\overline{u})\Delta u\right]\phi \mathrm{d}x, \tag{3.141}$$

and

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_\Omega \nabla u_\epsilon \nabla\phi \mathrm{d}x = \lim_{\epsilon\to 0^+}\int_\Omega \operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\nabla u\right]\nabla\phi \mathrm{d}x = \int_\Omega \operatorname{Re}\left[\operatorname{sgn}(\overline{u})\nabla u\right]\nabla\phi \mathrm{d}x \\ = \int_\Omega \nabla|u|\nabla\phi \mathrm{d}x.\end{aligned} \tag{3.142}$$

By Lemma 3.4.1 we have that

$$-\int_\Omega \nabla u_\epsilon \nabla\phi \mathrm{d}x \ge \int_\Omega \operatorname{Re}\left[\frac{\overline{u}}{u_\epsilon}\Delta u\right]\phi \mathrm{d}x, \quad \epsilon > 0, \tag{3.143}$$

so we can conclude that

$$-\int_\Omega \nabla|u|\nabla\phi \mathrm{d}x \ge \int_\Omega \operatorname{Re}\left[\operatorname{sgn}(\overline{u})\Delta u\right]\phi \mathrm{d}x. \tag{3.144}$$

Since $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$, $\phi \ge 0$, was arbitrary we can conclude the proof. □

4 Accretive Schrödinger Operators

4.1 Assumptions

In this chapter we will consider general accretive Schrödinger operators with potentials in $L^p(\overline{\Omega})$ for large enough p. More specific, we want the potential q to satisfy the following assumptions.

Assumption 4.1.1. Let $\Omega \subset \mathbb{R}^d$ be a C^∞ domain as in Definition 1.4.2, $q \in L^p_{\text{loc}}(\overline{\Omega})$ and $\operatorname{Re} q \geq 1$ a.e., where

$$p \begin{cases} > 1 & \text{if } d = 2, \\ > \frac{2d}{d+2} & \text{if } d \geq 3. \end{cases} \tag{4.1}$$

4.2 Mapping Properties of the Potential q

In this chapter we will work on a form domain $\mathcal{G} \subset H^1(\Omega)$ which is the largest domain such that for every $u \in \mathcal{G}$

$$\langle \nabla u, \nabla \cdot \rangle_{L^2} + \int_\Omega q u \overline{\cdot} \, \mathrm{d}x \tag{4.2}$$

can be interpreted as an element from $(H^1(\Omega))^*$. Consequently there will be no explicit definition of the form domain. This requires careful consideration since it is ad hoc not clear if this domain is even non-empty. The following two Lemmas will be useful in this regard.

Theorem 4.2.1 (Sobolev, Gagliardo, Nirenberg)**.** *Let $1 \leq t < d$. Then it holds true that $W^{1,t}(\mathbb{R}^d) \hookrightarrow L^{t'}(\mathbb{R}^d)$ with*

$$\frac{1}{t'} = \frac{1}{t} - \frac{1}{d} \tag{4.3}$$

and there exists $C(t,d) > 0$ such that

$$\|u\|_{L^{t'}} \leq C(t,d) \|\nabla u\|_{L^t}, \quad u \in W^{1,t}(\mathbb{R}^d). \tag{4.4}$$

Furthermore we have that $W^{1,t}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ for $r \in [t, t']$, and for the limiting case $t = d$ it holds that $W^{1,t}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ for $r \in [t, \infty)$ and there exists $C(r) > 0$ with

$$\|u\|_{L^r} \leq C(r) \|u\|_{W^{1,t}}, \quad u \in W^{1,t}(\mathbb{R}^d), \tag{4.5}$$

with $C(r) \xrightarrow{r \to \infty} \infty$.

Proof. See [8, Theorem 9.9, Corollary 9.10, Corollary 9.11]. □

Lemma 4.2.2. *Let Assumptions 4.1.1 be satisfied. Then there exists $r > 1$ such that for every $u \in H^1(\Omega)$ it holds true that $qu \in L^r_{\mathrm{loc}}(\overline{\Omega})$, and for every compact set $K \subset \overline{\Omega}$ there exists a $C(K) > 0$ such that*

$$\int_K |qu| \mathrm{d}x \leq C(K) \|u\|_{H^1} \left(\int_K |q|^p \mathrm{d}x \right)^{\frac{1}{p}}, \quad u \in H^1(\Omega). \tag{4.6}$$

Proof. As Ω is a C^∞ domain there exists a bounded extension operator $E_\Omega : H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that

$$\|E_\Omega u\|_{H^1(\mathbb{R}^d)} \leq \|E_\Omega\| \|u\|_{H^1(\Omega)}, \quad E_\Omega u = u \text{ in } \Omega, \tag{4.7}$$

see Lemma 1.5.3.

For $d = 2$ Theorem 4.2.1 shows that $H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d) \hookrightarrow L^{t'}(\mathbb{R}^d)$ for all $t' \in [2, \infty)$, and there exists a $C_{t'} > 0$ such that

$$\|u\|_{L^{t'}(\Omega)} \leq \|E_\Omega u\|_{L^{t'}(\mathbb{R}^d)} \leq C_{t'} \|E_\Omega u\|_{H^1(\mathbb{R}^d)} \leq C_{t'} \|E_\Omega\| \|u\|_{H^1(\Omega)}, \quad u \in H^1(\Omega), \tag{4.8}$$

so in particular we see that $u \in L^{t'}(\Omega)$ for all $u \in H^1(\Omega)$ and $t' \in [2, \infty)$.

For $p \in (1, \infty)$ there exist $t' \in [2, \infty)$, $r \in (1, \infty)$ such that

$$\frac{1}{p} + \frac{1}{t'} < 1, \quad \frac{r}{p} + \frac{r}{t'} = 1, \tag{4.9}$$

and now Hölder's inequality shows that on every compact $K \subset \overline{\Omega}$

$$\begin{aligned} \int_K |qu|^r \mathrm{d}x &\leq \left(\int_K |q|^p \mathrm{d}x \right)^{\frac{r}{p}} \left(\int_K |u|^{t'} \mathrm{d}x \right)^{\frac{r}{t'}} = \left(\int_K |q|^p \mathrm{d}x \right)^{\frac{r}{p}} \|u\|^r_{L^{t'}(\Omega)} \\ &\leq C^r_{t'} \|E_\Omega\|^r \left(\int_K |q|^p \mathrm{d}x \right)^{\frac{r}{p}} \|u\|^r_{H^1(\Omega)}, \qquad u \in H^1(\Omega), \end{aligned} \tag{4.10}$$

in particular we see that $qu \in L^r_{\mathrm{loc}}(\overline{\Omega})$ for every $u \in H^1(\Omega)$.

Also with Hölder's inequality we obtain

$$\int_K |qu| \mathrm{d}x \leq \left(\int_K |qu|^r \mathrm{d}x \right)^{\frac{1}{r}} \left(\int_K 1 \mathrm{d}x \right)^{\frac{r-1}{r}}, \quad u \in H^1(\Omega), \tag{4.11}$$

so we can define a constant

$$C(K) = C_{t'} \|E_\Omega\| \left(\int_K 1 \mathrm{d}x \right)^{\frac{r-1}{r}} < \infty, \tag{4.12}$$

and obtain from (4.10) and (4.11) that

$$\int_K |qu|\mathrm{d}x \leq C(K) \left(\int_K |q|^p \mathrm{d}x \right)^{\frac{1}{p}} \|u\|_{H^1(\Omega)}, \quad u \in H^1(\Omega). \tag{4.13}$$

For $d \geq 3$ Theorem 4.2.1 shows that $H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d) \hookrightarrow L^{t'}(\mathbb{R}^d)$ with $t' = \frac{2d}{d-2} > 1$ and thus $u = E_\Omega u\restriction_\Omega \in L^{t'}(\Omega)$ for every $u \in H^1(\Omega)$. Furthermore the theorem states that there exists $C_{t'} > 0$ such that

$$\|u\|_{L^{t'}(\Omega)} \leq \|E_\Omega u\|_{L^{t'}(\mathbb{R}^d)} \leq C_{t'} \|E_\Omega u\|_{H^1(\mathbb{R}^d)} \leq C_{t'} \|E_\Omega\| \|u\|_{H^1(\Omega)}, \quad u \in H^1(\Omega). \tag{4.14}$$

Since $p > \frac{2d}{d+2}$ it holds true that

$$\frac{1}{p} + \frac{1}{t'} < \frac{d+2}{2d} + \frac{d-2}{2d} = 1, \tag{4.15}$$

so there exists $r > 1$ with

$$\frac{r}{p} + \frac{r}{t'} = 1. \tag{4.16}$$

The rest follows now analogously to the previous case with $d = 2$. □

4.3 Operator $\hat{A}$

We can now define the sesquilinear forms

$$\begin{aligned} \tilde{\mathbf{a}}(u,v) &= \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega qu\overline{v}\mathrm{d}x, \qquad u \in H^1(\Omega),\, v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \\ \tilde{\mathbf{a}}^C(u,v) &= \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega \overline{q}u\overline{v}\mathrm{d}x, \qquad u \in H^1(\Omega),\, v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \end{aligned} \tag{4.17}$$

which are well-defined since by Lemma 4.2.2 it holds true that

$$\left| \int_\Omega |q|u\overline{v}\mathrm{d}x \right| \leq \|qu\|_{L^1(\operatorname{supp} v)} \|v\|_{L^\infty(\Omega)} < \infty, \quad u \in H^1(\Omega),\, v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega. \tag{4.18}$$

We start with $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$-functions in the second argument, since we need the boundedness and compact support for the existence of the integral.

However, if for some particular $u \in H^1(\Omega)$ there exists a $C > 0$ such that

$$|\tilde{\mathbf{a}}(u,v)| \leq C\|v\|_{H^1}, \quad v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \tag{4.19}$$

we can use the density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in $H^1(\Omega)$ from Lemma 1.5.2 in order to see that there exists a unique functional $F_u \in (H^1(\Omega))^*$ with

$$(F_u, v)_{(H^1)^* \times H^1} = \tilde{\mathbf{a}}(u,v), \quad v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega. \tag{4.20}$$

This observation motivates the definition of operators

$$
\begin{aligned}
&\hat{A} : \mathrm{Dom}\hat{A} \subset H^1(\Omega) \to (H^1(\Omega))^*, \\
&\mathrm{Dom}\hat{A} = \{u \in H^1(\Omega) : \exists \eta \in (H^1(\Omega))^* \text{ s.t.} \\
&\qquad\qquad (\eta, v)_{(H^1)^* \times H^1} = \tilde{\mathbf{a}}(u, v),\ v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega\}, \\
&\hat{A}u = \eta, \quad u \in \mathrm{Dom}\hat{A},
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
&\hat{A}^C : \mathrm{Dom}\hat{A}^C \subset H^1(\Omega) \to (H^1(\Omega))^*, \\
&\mathrm{Dom}\hat{A}^C = \{u \in H^1(\Omega) : \exists \eta \in (H^1(\Omega))^* \text{ s.t.} \\
&\qquad\qquad (\eta, v)_{(H^1)^* \times H^1} = \tilde{\mathbf{a}}^C(u, v),\ v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega\}, \\
&\hat{A}^C u = \eta, \quad u \in \mathrm{Dom}\hat{A}^C.
\end{aligned}
\tag{4.22}
$$

The following Lemma shows that these operator domains are dense in $H^1(\Omega)$.

Lemma 4.3.1. *Let* $\mathrm{Dom}\hat{A}$ *and* $\mathrm{Dom}\hat{A}^C$ *be as in (4.21) and (4.22), respectively. It holds true that* $C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathrm{Dom}\hat{A}$ *and* $C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathrm{Dom}\hat{A}^C$*, so* $\mathrm{Dom}\hat{A}$ *and* $\mathrm{Dom}\hat{A}^C$ *are dense in* $H^1(\Omega)$*.*

Proof. Let $u \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$. Then u is bounded and with $q \in L^p_{\mathrm{loc}}(\overline{\Omega})$ this implies $qu \in L^p_{\mathrm{loc}}(\overline{\Omega})$. Since $\operatorname{supp} u$ is compact we can use Lemma 4.2.2 to see that there exists a constant $C > 0$ such that

$$
\left| \int_\Omega qu\bar{v}\mathrm{d}x \right| = \left| \int_{\operatorname{supp} u} qu\bar{v}\mathrm{d}x \right| = C \left(\int_{\operatorname{supp} u} |qu|^p \right)^{\frac{1}{p}} \|v\|_{H^1}, \quad v \in H^1(\Omega). \tag{4.23}
$$

Consequently

$$
v \mapsto \int_\Omega qu\bar{v}\mathrm{d}x, \qquad v \in H^1(\Omega), \tag{4.24}
$$

defines a bounded, anti-linear functional and thus there exists $\eta \in (H^1(\Omega))^*$ such that

$$
(\eta, v)_{(H^1)^* \times H^1} = \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega qu\bar{v}\mathrm{d}x, \quad v \in H^1(\Omega). \tag{4.25}
$$

In particular, since $C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset H^1(\Omega)$, we see that

$$
(\eta, v)_{(H^1)^* \times H^1} = \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega qu\bar{v}\mathrm{d}x = \tilde{\mathbf{a}}(u, v), \quad v \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \tag{4.26}
$$

which proves that $u \in \mathrm{Dom}\hat{A}$. The density of $\mathrm{Dom}\hat{A}$ in $H^1(\Omega)$ follows now from the density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in $H^1(\Omega)$, see Lemma 1.5.2.

The assertion for $\mathrm{Dom}\hat{A}^C$ follows analogously. □

Lemma 4.3.1 motivates the definition of the restrictions

$$\hat{A}_0 := \hat{A}\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}, \quad \hat{A}_0^C := \hat{A}^C\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}. \tag{4.27}$$

We can use them in order to analyse the operators $\hat{A}$ and $\hat{A}^C$ further.

Lemma 4.3.2. *It holds true that*

$$\hat{A} = (\hat{A}_0^C)^* J, \quad \hat{A}^C = (\hat{A}_0)^* J, \tag{4.28}$$

where $J : H^1(\Omega) \to (H^1(\Omega))^{**}$ *denotes the natural embedding of* $H^1(\Omega)$ *into its bidual space from Definition 1.10.5. In particular,* $\hat{A}$ *and* $\hat{A}^C$ *are closed.*

Proof. By definition of the adjoint we have

$$\begin{aligned} \mathrm{Dom}\hat{A}_0^* = \big\{&u^{**} \in (H^1(\Omega))^{**} : \exists \eta \in (H^1(\Omega))^* \text{ s.t. } \forall \varphi \in \mathrm{Dom}\hat{A}_0 : \\ &(\eta, \varphi)_{(H^1(\Omega))^* \times H^1(\Omega)} = (u^{**}, \hat{A}_0\varphi)_{(H^1(\Omega))^{**} \times (H^1(\Omega))^*}\big\}, \\ \hat{A}_0^* u^{**} = \eta.& \end{aligned} \tag{4.29}$$

From Lemma 1.10.8 we see that the embedding J is bijective, so for every $u^{**} \in (H^1(\Omega))^{**}$ there exists $u \in H^1(\Omega)$ such that

$$Ju = u^{**}, \tag{4.30}$$

and by the definition of the bidual mapping we have that in this case

$$(u^{**}, \hat{A}_0\varphi)_{(H^1(\Omega))^{**} \times (H^1(\Omega))^*} = \overline{(\hat{A}_0\varphi, u)_{(H^1(\Omega))^* \times H^1(\Omega)}}, \quad \varphi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega. \tag{4.31}$$

With this we can rewrite (4.29) as

$$\begin{aligned} \mathrm{Dom}\hat{A}_0^* = J\big\{&u \in H^1(\Omega) : \exists \eta \in (H^1(\Omega))^* \text{ s.t. } \forall \varphi \in \mathrm{Dom}\hat{A}_0 : \\ &(\eta, \varphi)_{(H^1(\Omega))^* \times H^1(\Omega)} = \overline{(\hat{A}_0\varphi, u)_{(H^1(\Omega))^* \times H^1(\Omega)}}\big\}, \\ \hat{A}_0^* Ju = \eta.& \end{aligned} \tag{4.32}$$

Now note that $\mathrm{Dom}\hat{A}_0 = C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ and

$$\begin{aligned} \overline{(\hat{A}_0\varphi, u)_{(H^1(\Omega))^* \times H^1(\Omega)}} &= \overline{\langle \nabla\varphi, \nabla u\rangle_{L^2}} + \overline{\int_\Omega q\varphi\overline{u}\mathrm{d}x} = \langle \nabla u, \nabla\varphi\rangle_{L^2} + \int_\Omega \overline{q}u\overline{\varphi}\mathrm{d}x \\ &= \tilde{\mathbf{a}}^C(u, \varphi), \qquad u \in H^1(\Omega),\, \varphi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \end{aligned} \tag{4.33}$$

so we see that

$$\begin{aligned} \mathrm{Dom}\hat{A}_0^* = J\big\{&u \in H^1(\Omega) : \exists \eta \in (H^1(\Omega))^* \text{ s.t. } \forall \varphi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega : \\ &(\eta, \varphi)_{(H^1(\Omega))^* \times H^1(\Omega)} = \tilde{\mathbf{a}}^C(u, \varphi)\big\}, \\ &= J\mathrm{Dom}\hat{A}^C, \end{aligned} \tag{4.34}$$

and also

$$\hat{A}_0^* J = \hat{A}^C. \tag{4.35}$$

Let now $(u_n)_n \subset \mathrm{Dom}\hat{A}^C$, $u \in H^1(\Omega)$ and $\eta \in (H^1(\Omega))^*$ such that

$$u_n \xrightarrow{n\to\infty} u \text{ in } H^1(\Omega), \qquad \hat{A}^c u_n \xrightarrow{n\to\infty} \eta \text{ in } (H^1(\Omega))^*. \tag{4.36}$$

If we show that $u \in \mathrm{Dom}\hat{A}^C$ and $\hat{A}^C u = \eta$, then $\hat{A}^C$ is closed. Since the embedding J is bijective and bounded, see Lemma 1.10.6 and Lemma 1.10.8, we see that

$$Ju_n \xrightarrow{n\to\infty} Ju \text{ in } (H^1(\Omega))^{**}, \qquad (\hat{A}_0)^* Ju_n = \hat{A}^c u_n \xrightarrow{n\to\infty} \eta \text{ in } (H^1(\Omega))^*. \tag{4.37}$$

The adjoint $(\hat{A}_0)^*$ is closed, hence we obtain that $Ju \in \mathrm{Dom}(\hat{A}_0)^*$, and $(\hat{A}_0)^* Ju = \eta$, and consequently $u \in \mathrm{Dom}\hat{A}^C$ and $\hat{A}^C u = (\hat{A}_0)^* Ju = \eta$.

The statement for $\hat{A}$ follows analogously. □

4.4 Form Domain $\mathcal{G}$

Lemma 4.3.2 shows that $\hat{A}$ and $\hat{A}^C$ are closed, which means that

$$\begin{gathered}\big(\mathrm{Dom}\hat{A}, \|\cdot\|_{\hat{A}}^2 = \|\cdot\|_{H^1}^2 + \|\hat{A}\cdot\|_{(H^1)^*}^2\big),\\ \big(\mathrm{Dom}\hat{A}^C, \|\cdot\|_{\hat{A}^C}^2 = \|\cdot\|_{H^1}^2 + \|\hat{A}^C\cdot\|_{(H^1)^*}^2\big),\end{gathered} \tag{4.38}$$

are Hilbert spaces, and consequently $\hat{A}_0$ and $\hat{A}_0^C$ are closable. This motivates the definition of operators

$$\hat{B} = \overline{\hat{A}_0}, \qquad \hat{B}^C = \overline{\hat{A}_0^C}. \tag{4.39}$$

We will later see that $\hat{A} = \hat{B}$ and $\hat{A}^C = \hat{B}^C$. For now it is easier to show useful properties of $\hat{B}$ and $\hat{B}^C$. To this end consider the spaces

$$\begin{aligned}\mathcal{G} &= \big(\mathrm{Dom}\hat{B}, \|\cdot\|_{\hat{B}}^2 = \|\cdot\|_{H^1}^2 + \|\hat{B}\cdot\|_{(H^1)^*}^2\big),\\ \mathcal{G}^C &= \big(\mathrm{Dom}\hat{B}^C, \|\cdot\|_{\hat{B}^C}^2 = \|\cdot\|_{H^1}^2 + \|\hat{B}^C\cdot\|_{(H^1)^*}^2\big),\end{aligned} \tag{4.40}$$

which are also Hilbert spaces since $\hat{B}$ and $\hat{B}^C$ are closed by definition.

4.5 Invertibility of $\hat{A}$

As mentioned we will later on show that $\hat{A} = \hat{B}$ and $\hat{A}^C = \hat{B}^C$. Now we want to prove that $\hat{B}$ and $\hat{B}^C$ are boundedly invertible.

Lemma 4.5.1. *The operators $\hat{B} : \mathcal{G} \to (H^1(\Omega))^*$ and $\hat{B}^C : \mathcal{G}^C \to (H^1(\Omega))^*$ from (4.39) are bounded, bijective and boundedly invertible.*

Proof. By the definition of the spaces $\mathcal{G}$ and $\mathcal{G}^C$ in (4.40) the operators are clearly bounded.

We will now consider the bounded sesquilinear form associated with the operator $\hat{B}$, i.e.

$$\mathbf{b} : \mathcal{G} \times H^1(\Omega) \to \mathbb{C}, \quad \mathbf{b}(u,v) = (\hat{B}u, v)_{(H^1)^* \times H^1}, \quad u \in \mathcal{G},\, v \in H^1(\Omega). \tag{4.41}$$

Due to Lemma 1.12.6 we know that $\hat{B}$ is bijective and boundedly invertible if and only if (rf1) and (rf2) hold true.

We will start with (rf1). Since $\hat{B} = \overline{\hat{A}_0}$ we know that $\mathrm{Dom}\hat{A}_0 = C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ is dense in $\mathrm{Dom}\hat{B}$, and thus

$$\inf_{0 \neq u \in \mathcal{G}} \sup_{0 \neq v \in H^1(\Omega)} \frac{|\mathbf{b}(u,v)|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}} = \inf_{0 \neq u \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega} \sup_{0 \neq v \in H^1(\Omega)} \frac{|\mathbf{b}(u,v)|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}}. \tag{4.42}$$

Furthermore we see that for $u \in \mathrm{Dom}\hat{A}_0$ and $v \in H^1(\Omega)$

$$\begin{aligned} \mathbf{b}(u,v) &= (\hat{B}u, v)_{(H^1)^* \times H^1} = (\hat{A}_0 u, v)_{(H^1)^* \times H^1} = \tilde{\mathbf{a}}(u,v) \\ &= \langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega q u \overline{v} \mathrm{d}x, \end{aligned} \tag{4.43}$$

and consequently we have that

$$\begin{aligned} \inf_{0 \neq u \in \mathcal{G}} \sup_{0 \neq v \in H^1(\Omega)} \frac{|\mathbf{b}(u,v)|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}} &= \inf_{0 \neq u \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega} \sup_{0 \neq v \in H^1(\Omega)} \frac{|\langle \nabla u, \nabla v \rangle_{L^2} + \int_\Omega q u \overline{v} \mathrm{d}x|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}} \\ &\geq \inf_{0 \neq u \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega} \frac{|\operatorname{Re}\left(\langle \nabla u, \nabla u \rangle_{L^2} + \int_\Omega q u \overline{u} \mathrm{d}x\right)|}{\|u\|_{\mathcal{G}} \|u\|_{H^1(\Omega)}} \\ &\geq \inf_{0 \neq u \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega} \frac{\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2}{\|u\|_{\mathcal{G}} \|u\|_{H^1(\Omega)}} \\ &= \inf_{0 \neq u \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega} \frac{\|u\|_{H^1}}{\|u\|_{\mathcal{G}}} = \inf_{0 \neq u \in \mathcal{G}} \frac{\|u\|_{H^1}}{\|u\|_{\mathcal{G}}}, \end{aligned} \tag{4.44}$$

where we used that $\operatorname{Re} q \geq 1$.

On the other hand we can use the definition of the dual norm in (1.262) to see that

$$\sup_{0 \neq v \in H^1(\Omega)} \frac{|\mathbf{b}(u,v)|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}} = \sup_{0 \neq v \in H^1(\Omega)} \frac{|(\hat{B}u, v)_{(H^1)^* \times H^1}|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}} = \frac{\|\hat{B}u\|_{(H^1)^*}}{\|u\|_{\mathcal{G}}}, \quad u \in \mathcal{G}. \tag{4.45}$$

Together (4.44) and (4.45) show that

$$\begin{aligned} \inf_{0 \neq u \in \mathcal{G}} \sup_{0 \neq v \in H^1(\Omega)} \frac{|\mathbf{b}(u,v)|}{\|u\|_{\mathcal{G}} \|v\|_{H^1(\Omega)}} &\geq \frac{1}{2} \inf_{0 \neq u \in \mathcal{G}} \frac{\|u\|_{H^1} + \|\hat{B}u\|_{(H^1)^*}}{\|u\|_{\mathcal{G}}} \\ &\geq \frac{1}{2} \inf_{0 \neq u \in \mathcal{G}} \frac{\left(\|u\|_{H^1}^2 + \|\hat{B}u\|_{(H^1)^*}^2\right)^{\frac{1}{2}}}{\|u\|_{\mathcal{G}}} = \frac{1}{2}, \end{aligned} \tag{4.46}$$

and thus (rf1) is satisfied.

Now we want to show (rf2). Assume that

$$\mathbf{b}(u,v)=0, \qquad u\in\mathcal{G}, \tag{4.47}$$

for some $v\in H^1(\Omega)$. We need to show that $v=0$ in order to satisfy (rf2).

Using $C_0^\infty(\mathbb{R}^d)\restriction_\Omega=\mathrm{Dom}\hat{A}_0\subset\mathcal{G}$ we see that

$$0=\mathbf{b}(u,v)=\tilde{\mathbf{a}}(u,v)=\langle\nabla u,\nabla v\rangle_{L^2}+\int_\Omega qu\overline{v}\mathrm{d}x, \quad u\in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \tag{4.48}$$

and the complex conjugation of this equation then gives

$$\langle\nabla v,\nabla u\rangle_{L^2}+\int_\Omega\overline{q}v\overline{u}\mathrm{d}x=0, \quad u\in C_0^\infty(\mathbb{R}^d)\restriction_\Omega. \tag{4.49}$$

Lemma 4.2.2 shows that $\overline{q}v\in L^r_{\mathrm{loc}}(\overline{\Omega})$ for some $r>1$, so in particular we have that $\overline{q}v\in L^1_{\mathrm{loc}}(\Omega)\subset\mathcal{D}'(\Omega)$. Since $\mathcal{D}(\Omega)\subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ we obtain from (4.49) that

$$\sum_{j=1}^d(\partial_j v,\partial_j\phi)_{\mathcal{D}'\times\mathcal{D}}=(-\overline{q}v,\phi)_{\mathcal{D}'\times\mathcal{D}}, \qquad \phi\in\mathcal{D}(\Omega), \tag{4.50}$$

and therefore

$$\Delta v=\overline{q}v\in L^r_{\mathrm{loc}}(\overline{\Omega}). \tag{4.51}$$

Furthermore from (4.49) we obtain that

$$\langle\nabla v,\nabla u\rangle_{L^2}+\int_\Omega\Delta v\overline{u}\mathrm{d}x=0, \quad u\in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \tag{4.52}$$

and thus $v\in H^1(\Omega)$ satisfies Assumptions 3.2.1. Therefore we can apply Theorem 3.2.2 to see that

$$0\geq\int_\Omega\nabla|v|\nabla\phi\mathrm{d}x+\int_\Omega\mathrm{Re}\left[\mathrm{sgn}(\overline{v})\Delta v\right]\phi\mathrm{d}x, \quad \phi\in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi\geq0. \tag{4.53}$$

Plugging in $\Delta v=\overline{q}v$ and using $\mathrm{Re}\,q\geq1$ shows that

$$\begin{aligned}0&\geq\int_\Omega\nabla|v|\nabla\phi\mathrm{d}x+\int_\Omega\mathrm{Re}\left[\mathrm{sgn}(\overline{v})\overline{q}v\right]\phi\mathrm{d}x\\&=\int_\Omega\nabla|v|\nabla\phi\mathrm{d}x+\int_\Omega\mathrm{Re}\left[\overline{q}|v|\right]\phi\mathrm{d}x\\&\geq\int_\Omega\nabla|v|\nabla\phi\mathrm{d}x+\int_\Omega|v|\phi\mathrm{d}x, \quad \phi\in C_0^\infty(\mathbb{R}^d)\restriction_\Omega,\ \phi\geq0.\end{aligned} \tag{4.54}$$

Now we can use Lemma 1.6.11 to find a sequence $(\phi_n)_n \subset C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ with $\phi_n \geq 0$, $n \in \mathbb{N}$, such that

$$\phi_n \xrightarrow{n\to\infty} |v| \text{ in } H^1(\Omega), \tag{4.55}$$

and consequently we have that

$$\begin{aligned} 0 &\geq \lim_{n\to\infty} \left(\int_\Omega \nabla |v| \nabla \phi_n \mathrm{d}x + \int_\Omega \mathrm{Re}\,[\mathrm{sgn}(\overline{v})\overline{q}v]\, \phi_n \mathrm{d}x \right) \\ &= \int_\Omega |\nabla |v||^2 \mathrm{d}x + \int_\Omega |v|^2 \mathrm{d}x = \||v|\|_{H^1}^2. \end{aligned} \tag{4.56}$$

This proves that $|v| = 0$, and thus we see that $v = 0$ and (rf2) is satisfied.

Analogously one shows that $\hat{B}^C$ is boundedly invertible. □

Lemma 4.5.2. *It holds true that $\hat{A} = \hat{B}$ and $\hat{A}^C = \hat{B}^C$, in particular $\hat{A} : \mathcal{G} \to (H^1)^*$ and $\hat{A}^C : \mathcal{G}^C \to (H^1)^*$ are bounded, bijective and boundedly invertible.*

Proof. From Lemma 4.5.1 we see that $\hat{B}$ and $\hat{B}^C$ are bijective and boundedly invertible. Furthermore, they are bounded by definition.

First note that the fact that $\hat{A}$ is closed, see Lemma 4.3.2, together with $\hat{A}_0 \subset \hat{A}$, implies that

$$\hat{B} = \overline{\hat{A}_0} \subset \hat{A}. \tag{4.57}$$

Therefore as $\hat{B}$ is surjective, so is $\hat{A}$.

Let now $J : H^1(\Omega) \to (H^1(\Omega))^{**}$ and $\tilde{J} : H^1(\Omega)^* \to (H^1(\Omega))^{***}$ be the bidual mappings from Definition 1.10.5. Since both are bijective, see Lemma 1.10.6 and Lemma 1.10.8, we find that

$$\tilde{J}\hat{B}^C J^{-1} \tag{4.58}$$

is bijective, in particular surjective.

From the definition of $\hat{B}^C$ in (4.39) we find together with Lemma 1.10.11 and Lemma 4.3.2 that

$$\tilde{J}\hat{B}^C J^{-1} = \tilde{J}\overline{\hat{A}_0^C} J^{-1} = (\hat{A}_0^C)^{**} = (\hat{A}J^{-1})^* \tag{4.59}$$

is surjective. Therefore by Lemma 1.10.10 we see that $\hat{A}J^{-1}$ is injective, and since J is bijective we finally see that $\hat{A}$ is injective.

It remains to show that $\hat{A} \subset \hat{B}$. To this end, let $u \in \mathrm{Dom}\hat{A}$. Since $\hat{B}$ is surjective, there exists $v \in \mathrm{Dom}\hat{B}$ such that

$$\hat{A}u = \hat{B}v. \tag{4.60}$$

But we already know that $\hat{B} \subset \hat{A}$, and so we see that

$$0 = \hat{B}v - \hat{A}u = \hat{A}(v - u), \tag{4.61}$$

which together with the injectivity of $\hat{A}$ proves $u = v \in \mathrm{Dom}\hat{B}$.

The assertion $\hat{A}^C = \hat{B}^C$ follows analogously. □

Corollary 4.5.3. *Let* $f \in \mathrm{Dom}\hat{A}$ *and* $g \in \mathrm{Dom}\hat{A}^C$. *Then*

$$(\hat{A}f, g)_{(H^1)^* \times H^1} = \overline{(\hat{A}^C g, f)_{(H^1)^* \times H^1}}. \tag{4.62}$$

Proof. Lemma 4.3.2 shows that $\hat{A} = (\hat{A}_0^C)^* J$, where J is the natural embedding of $H^1(\Omega)$ into its bidual space from Definition 1.10.5. Therefore we see that

$$\begin{aligned}(\hat{A}f, g)_{(H^1)^* \times H^1} &= ((\hat{A}_0^C)^* Jf, g)_{(H^1)^* \times H^1} = (Jf, \hat{A}_0^C g)_{(H^1)^{**} \times (H^1)^*} \\ &= \overline{(\hat{A}_0^C g, f)_{(H^1)^* \times H^1}}, \qquad g \in \mathrm{Dom}\hat{A}_0^C,\ f \in \mathrm{Dom}\hat{A}.\end{aligned} \tag{4.63}$$

We know from Lemma 4.5.2 that $\hat{A}^C = \hat{B}^C = \overline{\hat{A}_0^C}$, and consequently it follows from (4.63) that

$$(\hat{A}f, g)_{(H^1)^* \times H^1} = \overline{(\hat{A}^C g, f)_{(H^1)^* \times H^1}}, \qquad g \in \mathrm{Dom}\hat{A}^C,\ f \in \mathrm{Dom}\hat{A}. \tag{4.64}$$

□

4.6 Neumann Trace γ_N

We define the operators

$$\begin{aligned}&T : \mathrm{Dom}T \subset L^2(\Omega) \to L^2(\Omega), \\ &\mathrm{Dom}T = \{u \in \mathrm{Dom}\hat{A} :\ (-\Delta + q)u \in L^2(\Omega)\}, \\ &Tu = (-\Delta + q)u, \quad u \in \mathrm{Dom}T,\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}&\tilde{T} : \mathrm{Dom}\tilde{T} \subset L^2(\Omega) \to L^2(\Omega), \\ &\mathrm{Dom}\tilde{T} = \{u \in \mathrm{Dom}\hat{A}^C :\ (-\Delta + \overline{q})u \in L^2(\Omega)\}, \\ &\tilde{T}u = (-\Delta + \overline{q})u, \quad u \in \mathrm{Dom}\tilde{T}.\end{aligned} \tag{4.66}$$

Next we define a new notion of Neumann trace which, as we will see, coincides with the standard definition on the intersection of their domains.

Definition 4.6.1. *Let Assumptions 4.1.1 be satisfied. The Neumann traces*

$$\begin{aligned}\gamma_N &: \mathrm{Dom}T \to H^{-\frac{1}{2}}(\partial\Omega), \\ \widetilde{\gamma_N} &: \mathrm{Dom}\tilde{T} \to H^{-\frac{1}{2}}(\partial\Omega),\end{aligned} \tag{4.67}$$

for $u \in \mathrm{Dom}T$ *and* $\tilde{u} \in \mathrm{Dom}\tilde{T}$ *are defined via*

$$\begin{aligned}(\gamma_N u, h)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} &= (\hat{A}u, \mathcal{E}h)_{(H^1)^* \times H^1} - \langle(-\Delta + q)u, \mathcal{E}h\rangle_{L^2}, \quad h \in H^{\frac{1}{2}}(\partial\Omega), \\ (\widetilde{\gamma_N}\tilde{u}, h)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} &= (\hat{A}^C\tilde{u}, \mathcal{E}h)_{(H^1)^* \times H^1} - \langle(-\Delta + \overline{q})\tilde{u}, \mathcal{E}h\rangle_{L^2}, \quad h \in H^{\frac{1}{2}}(\partial\Omega),\end{aligned} \tag{4.68}$$

where $\mathcal{E}$ *is the bounded right inverse of the Dirichlet trace from Theorem 1.5.1.*

Remark 4.6.2. *Note that the definitions above do not depend on the choice of the right inverse $\mathcal{E}$ of the Dirichlet trace.*

Assume that $u \in \mathrm{Dom}T$ and $\phi \in C_0^\infty(\Omega)$. Then it holds that

$$(\hat{A}u, \phi)_{(H^1)^* \times H^1} - \langle(-\Delta + q)u, \phi\rangle_{L^2} = \tilde{\mathbf{a}}(u, \phi) - \langle\nabla u, \nabla\phi\rangle_{L^2} - \int_\Omega qu\overline{\phi}\mathrm{d}x = 0. \quad (4.69)$$

As $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we see that the same holds for all $\phi \in H_0^1(\Omega)$, so the right-hand side of (4.68) does not change if we add $H_0^1(\Omega)$-functions to $\mathcal{E}h$.

But if there exists a second right inverses $\mathcal{E}'$ of the Dirichlet trace, then it clearly holds true that

$$\gamma_D(\mathcal{E} - \mathcal{E}')h = 0, \quad h \in H^{\frac{1}{2}}(\partial\Omega), \quad (4.70)$$

and consequently

$$(\mathcal{E} - \mathcal{E}')h \in H_0^1(\Omega), \quad h \in H^{\frac{1}{2}}(\partial\Omega), \quad (4.71)$$

which does not change the right-hand side of (4.68).

The next Lemmas will show that this notion of Neumann trace is indeed compatible with other definitions.

Lemma 4.6.3. *Let $u \in \mathrm{Dom}T \cap \mathrm{Dom}\tilde{T}$. Then $\gamma_N u = \widetilde{\gamma_N} u$.*

Proof. If $u \in \mathrm{Dom}T \cap \mathrm{Dom}\tilde{T}$ we obtain

$$(q - \overline{q})u = (-\Delta + q)u - (-\Delta + \overline{q})u \in L^2(\Omega). \quad (4.72)$$

From this we see that for every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ it holds true that

$$\begin{aligned}(\gamma_N u, \gamma_D\phi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} &= (\hat{A}u, \phi)_{(H^1)^* \times H^1} - \langle(-\Delta + q)u, \phi\rangle_{L^2} \\ &= \langle\nabla u, \nabla\phi\rangle + \int_\Omega qu\overline{\phi}\mathrm{d}x - \langle(-\Delta + q)u, \phi\rangle_{L^2} \\ &\quad - \int_\Omega (q - \overline{q})u\overline{\phi}\mathrm{d}x + \langle(q - \overline{q})u, \phi\rangle_{L^2} \\ &= \langle\nabla u, \nabla\phi\rangle + \int_\Omega \overline{q}u\overline{\phi}\mathrm{d}x - \langle(-\Delta + \overline{q})u, \phi\rangle_{L^2} \\ &= (\hat{A}^C u, \phi)_{(H^1)^* \times H^1} - \langle(-\Delta + \overline{q})u, \phi\rangle_{L^2} = (\tilde{\gamma}_N u, \gamma_D\phi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}.\end{aligned} \quad (4.73)$$

Since $\mathrm{Ran}\gamma_D\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ is dense in $H^{\frac{1}{2}}(\Omega)$, see Lemma 1.5.2, this concludes the proof. □

Lemma 4.6.4. *Let $u \in \mathrm{Dom}T$ such that $\Delta u \in L^2(\Omega)$. Then $\gamma_N u = \gamma_N^\Delta u$, where γ_N^Δ is the standard definition of the Neumann trace, i.e.*

$$(\gamma_N^\Delta u, h)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} = \langle\nabla u, \nabla\mathcal{E}h\rangle_{L^2} + \langle\Delta u, \mathcal{E}h\rangle_{L^2}, \ u \in H_\Delta^1(\Omega), h \in H^{\frac{1}{2}}(\partial\Omega). \quad (4.74)$$

The analogous statement holds for $\widetilde{\gamma_N}$.

Proof. Let $u \in \mathrm{Dom}T$ and $\Delta u \in L^2(\Omega)$. Then it is clear that also

$$qu = (-\Delta + q)u + \Delta u \in L^2(\Omega), \tag{4.75}$$

so we see that

$$\begin{aligned} (\hat{A}u, \phi)_{(H^1)^* \times H^1} &= \langle \nabla u, \nabla \phi \rangle_{L^2} + \int_\Omega qu\overline{\phi}\mathrm{d}x \\ &= \langle \nabla u, \nabla \phi \rangle_{L^2} + \langle qu, \phi \rangle_{L^2}, \qquad \phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega, \end{aligned} \tag{4.76}$$

and thus we obtain from the density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in H^1, see Lemma 1.5.2, that

$$(\hat{A}u, v)_{(H^1)^* \times H^1} = \langle \nabla u, \nabla v \rangle_{L^2} + \langle qu, v \rangle_{L^2}, \quad v \in H^1(\Omega). \tag{4.77}$$

For $h \in H^{\frac{1}{2}}(\partial\Omega)$ it therefore holds that

$$\begin{aligned} &(\gamma_N^\Delta u, h)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} = \langle \nabla u, \nabla \mathcal{E}h \rangle_{L^2} + \langle \Delta u, \mathcal{E}h \rangle_{L^2} \\ &\quad= \langle \nabla u, \nabla \mathcal{E}h \rangle_{L^2} + \langle qu, \mathcal{E}h \rangle_{L^2} + \langle \Delta u, \mathcal{E}h \rangle_{L^2} - \langle qu, \mathcal{E}h \rangle_{L^2} \\ &\quad= (\hat{A}u, \mathcal{E}h)_{(H^1)^* \times H^1} - \langle (-\Delta + q)u, \mathcal{E}h \rangle_{L^2} = (\gamma_N u, h)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}. \end{aligned} \tag{4.78}$$

The statement for $\widetilde{\gamma_N}$ follows analogously. □

Recall that we also constructed a Neumann trace in Chapter 2. The following Lemma shows that γ_N is also compatible with this version.

Lemma 4.6.5. *Assume that* $u \in \mathrm{Dom}T$ *and* $|q|^{\frac{1}{2}}u \in L^2(\Omega)$. *Then it holds true that* $u \in \mathrm{Dom}\gamma_N^q$, *where* γ_N^q *is as defined in (2.70).*

Furthermore it holds true that $\gamma_N u \in \mathcal{W}^*$ *for* $\mathcal{W}$ *as defined in (2.38) and (2.50), and*

$$\gamma_N u = \gamma_N^q u \quad \text{in } \mathcal{W}^*. \tag{4.79}$$

Proof. By definition of $\mathrm{Dom}T$ we know that $u \in H^1(\Omega)$ and $(-\Delta+q)u \in L^2(\Omega)$, which together with $|q|^{\frac{1}{2}}u \in L^2(\Omega)$ shows that $u \in \mathrm{Dom}\gamma_N^q$.

Note that $\mathcal{W}$ is continuously embedded into $H^{\frac{1}{2}}(\partial\Omega)$, see Lemma 2.3.6, so consequently $\gamma_N u \in H^{-\frac{1}{2}}(\partial\Omega) \subset \mathcal{W}^*$.

Lemma 2.2.1 states that $C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathcal{V}$, where $\mathcal{V}$ is the Hilbert space defined in (2.9), so with Remark 4.6.2 we see that for $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$

$$\begin{aligned} (\gamma_N^q u, \gamma_D^\mathcal{V}\phi)_{\mathcal{W}^* \times \mathcal{W}} &= \langle \nabla u, \nabla \mathcal{E}^\mathcal{V}\gamma_D^\mathcal{V}\phi \rangle_{L^2} + \int_\Omega qu\overline{\mathcal{E}^\mathcal{V}\gamma_D^\mathcal{V}\phi}\mathrm{d}x \\ &\qquad - \langle (-\Delta + q)u, \mathcal{E}^\mathcal{V}\gamma_D^\mathcal{V}\phi \rangle_{L^2} \\ &= \langle \nabla u, \nabla \phi \rangle_{L^2} + \int_\Omega qu\overline{\phi}\mathrm{d}x - \langle (-\Delta + q)u, \phi \rangle_{L^2} \\ &= (\hat{A}u, \phi)_{(H^1)^* \times H^1} - \langle (-\Delta + q)u, \phi \rangle_{L^2} = (\gamma_N u, \gamma_D \phi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}, \end{aligned} \tag{4.80}$$

where $\gamma_D^{\mathcal{V}}$ denotes the Dirichlet trace from $\mathcal{V}$ into $\mathcal{W}$ as defined in (2.39), and $\mathcal{E}^{\mathcal{V}}$ its bounded right-inverse from (2.47).

Now since $\mathrm{Ran}\gamma_D^{\mathcal{V}}\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ is dense in $\mathcal{W}$, see Corollary 2.3.5, and

$$\gamma_D^{\mathcal{V}}\phi = \gamma_D\phi, \quad \phi \in \mathcal{V}, \tag{4.81}$$

by the definition of $\gamma_D^{\mathcal{V}}$, we see that

$$(\gamma_N^q u, \psi)_{\mathcal{W}^*\times\mathcal{W}} = (\gamma_N u, \psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}}, \quad \psi \in \mathcal{W}, \tag{4.82}$$

which concludes the proof. □

Lemma 4.6.6. *Let $u \in \mathrm{Dom}T$. Then $\gamma_N u = 0$ if and only if there exists $\eta \in L^2(\Omega)$ such that*

$$\langle \eta, v\rangle_{L^2} = (\hat{A}u, v)_{(H^1)^*\times H^1}, \quad v \in H^1(\Omega). \tag{4.83}$$

Proof. Let $\gamma_N u = 0$. Then it holds that for every $\psi \in H^{\frac{1}{2}}(\partial\Omega)$ we have

$$(\hat{A}u, \mathcal{E}\psi)_{(H^1)^*\times H^1} - \langle(-\Delta + q)u, \mathcal{E}\psi\rangle_{L^2} = 0, \tag{4.84}$$

and due to Remark 4.6.2 we see that therefore

$$(\hat{A}u, v)_{(H^1)^*\times H^1} = \langle(-\Delta + q)u, v\rangle_{L^2}, \quad v \in H^1(\Omega), \tag{4.85}$$

so we see that $\eta = (-\Delta + q)u \in L^2(\Omega)$ satisfies (4.83).

If on the other hand (4.83) is satisfied, then we see that

$$\langle \eta, \phi\rangle_{L^2} = (\hat{A}u, \phi)_{(H^1)^*\times H^1} = \langle \nabla u, \nabla\phi\rangle_{L^2} + \int_\Omega qu\overline{\phi}\mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega), \tag{4.86}$$

and thus in distributional sense

$$(-\Delta + q)u = \eta \in L^2(\Omega), \tag{4.87}$$

which then shows that

$$\begin{aligned}(\gamma_N u, \psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} &= (\hat{A}u, \mathcal{E}\psi)_{(H^1)^*\times H^1} - \langle(-\Delta + q)u, \mathcal{E}\psi\rangle_{L^2} \\ &= (\hat{A}u, \mathcal{E}\psi)_{(H^1)^*\times H^1} - \langle \eta, \mathcal{E}\psi\rangle_{L^2} = 0, \quad \psi \in H^{\frac{1}{2}}(\partial\Omega),\end{aligned} \tag{4.88}$$

and hence $\gamma_N u = 0$. □

4.7 Surjectivity of the Neumann Trace γ_N

Lemma 4.7.1. *The Neumann traces* $\gamma_N : \mathrm{Dom}T \to H^{-\frac{1}{2}}(\partial\Omega)$ *and* $\tilde{\gamma}_N : \mathrm{Dom}\tilde{T} \to H^{-\frac{1}{2}}(\partial\Omega)$ *are surjective.*

Proof. We show the statement for γ_N, it follows analogously for $\tilde{\gamma}_N$.
Let $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. We define

$$(F, v)_{(H^1)^* \times H^1} = (\psi, \gamma_D v)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}, \quad v \in H^1(\Omega). \tag{4.89}$$

It is clear that this expression is anti-linear in the second argument, and since the Dirichlet trace is bounded, see Theorem 1.5.1, we obtain that

$$\left|(F, v)_{(H^1)^* \times H^1}\right| = \|\psi\|_{H^{-\frac{1}{2}}} \|\gamma_D v\|_{H^{\frac{1}{2}}} < \|\gamma_D\| \|\psi\|_{H^{-\frac{1}{2}}} \|v\|_{H^1}, \quad v \in H^1(\Omega), \tag{4.90}$$

which proves that $F \in (H^1(\Omega))^*$.
Since $\hat{A}$ is surjective, see Lemma 4.5.2, we can find $u \in \mathrm{Dom}\hat{A}$ such that

$$(\hat{A}u, v)_{(H^1)^* \times H^1} = (F, v)_{(H^1)^* \times H^1} = (\psi, \gamma_D v)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}, \quad v \in H^1(\Omega). \tag{4.91}$$

In particular we obtain

$$\langle \nabla u, \nabla \phi \rangle_{L^2} + \int_\Omega q u \overline{\phi} \mathrm{d}x = (\hat{A}u, \phi)_{(H^1)* \times H^1} = 0, \quad \phi \in \mathcal{D}(\Omega), \tag{4.92}$$

and thus $(-\Delta + q)u = 0$ in distributional sense, which also proves that $u \in \mathrm{Dom}T$.
Finally by definition of the Neumann trace we see that

$$(\gamma_N u, \varphi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} = (\hat{A}u, \mathcal{E}\varphi)_{(H^1)^* \times H^1} = (\psi, \varphi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}, \quad \varphi \in H^{\frac{1}{2}}(\partial\Omega), \tag{4.93}$$

which proves that $\gamma_N u = \psi$, and thus γ_N is surjective. □

4.8 Neumann Realisation of the Accretive Schrödinger Operator

We now define the operators

$$\begin{aligned} \mathrm{Dom}A_N &= \{f \in \mathrm{Dom}\hat{A} : \, (-\Delta + q)u \in L^2(\Omega), \, \gamma_N f = 0\} \\ A_N &= (-\Delta + q)u, \end{aligned} \tag{4.94}$$

and

$$\begin{aligned} \mathrm{Dom}A_N^C &= \{f \in \mathrm{Dom}\hat{A}^C : \, (-\Delta + \overline{q})u \in L^2(\Omega), \, \widetilde{\gamma}_N f = 0\} \\ A_N^C &= (-\Delta + \overline{q})u. \end{aligned} \tag{4.95}$$

The following Theorem shows that these Neumann realisations of the Schrödinger operator satisfy all the properties we need to construct quasi boundary triples later on.

Theorem 4.8.1. *Let Assumptions 4.1.1 be satisfied. Then the operator A_N is closed with $0 \in \rho(A_N)$ and $\mathrm{Dom} A_N$ is dense in $\mathcal{G}$ and $L^2(\Omega)$. The analogous statements hold true for A_N^C, and*

$$A_N^* = A_N^C, \quad (A_N^C)^* = A_N. \tag{4.96}$$

Proof. We want to use Theorem 1.12.1 in order to show this result. To this end, we first need to show that there exists an injective, continuous embedding of $\mathcal{G}$ into $L^2(\Omega)$ with dense range.

We already know that $H^1(\Omega)$ is continuously embedded into $L^2(\Omega)$. Since $\mathcal{G} \subset H^1(\Omega) \subset L^2(\Omega)$ and by definition of the norm in $\mathcal{G}$ in (4.40) we have that

$$\|u\|_{L^2} \leq \|u\|_{H^1} \leq \|u\|_{\mathcal{G}}, \quad u \in \mathcal{G}, \tag{4.97}$$

hence we see the embedding of $\mathcal{G}$ into $L^2(\Omega)$ is continuous. Finally we know that $\mathcal{D}(\Omega) \subset \mathrm{Dom}\hat{A}_0 \subset \mathcal{G}$, and since the test functions are dense in $L^2(\Omega)$, the embedding has dense range.

As $H^1(\Omega)$ has a continuous embedding with dense range into $L^2(\Omega)$, Lemma 1.11.2 shows that such a embedding exists also for $L^2(\Omega)$ into $(H^1(\Omega))^*$.

Lemma 4.5.2 shows that $\hat{A} \in \mathcal{B}(\mathcal{G}, (H^1(\Omega))^*)$ is boundedly invertible, so Theorem 1.12.1 states that the operator

$$\begin{aligned} \mathrm{Dom}B &= \{u \in \mathrm{Ran}\iota_1 : \hat{A}u\iota_1^{-1}u \in \mathrm{Ran}\iota_2\} \\ B &= \iota_2^{-1}\hat{A}\iota_1^{-1}, \end{aligned} \tag{4.98}$$

where $\iota_1 : \mathcal{G} \to L^2(\Omega)$ and $\iota_2 :\to (H^1(\Omega))^*$ denote the natural embeddings, is a closed operator, $0 \in \rho(B)$, and $\mathrm{Dom}B$ is dense in both $\mathcal{G}$ and $L^2(\Omega)$.

We can write the operator more explicitly as

$$\begin{aligned} \mathrm{Dom}B &= \{u \in \mathcal{G} : \exists \eta \in L^2(\Omega) \text{ s.t. } \langle \eta, v \rangle_{L^2} = (\hat{A}u, v)_{(H^1)^* \times H^1} \, \forall v \in H^1(\Omega)\}, \\ Bu &= \eta. \end{aligned} \tag{4.99}$$

Note that if $u \in \mathrm{Dom}B$, then

$$\langle \eta, \phi \rangle_{L^2} = (\hat{A}u, \phi)_{(H^1)^* \times H^1} = \langle \nabla u, \nabla \phi \rangle_{L^2} + \int_\Omega q u \overline{\phi} \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega), \tag{4.100}$$

and thus in distributional sense

$$(-\Delta + q)u = \eta \in L^2(\Omega), \tag{4.101}$$

which shows $u \in \mathrm{Dom}T$.

Using Lemma 4.6.6 we now see that $B = A_N$, so A_N is closed, $0 \in \rho(A_N)$ and $\mathrm{Dom}A_N$ is dense in $\mathcal{G}$ and $L^2(\Omega)$.

In the same way one shows that

$$A_N^C = (\iota_2)^{-1}\hat{A}^C(\iota_1)^{-1}, \tag{4.102}$$

and that it satisfies the analogous statements.

Theorem 1.12.1 also states that the adjoint of A_N is given by

$$A_N^* = (\iota_1^*)^{-1}\hat{A}^*(\iota_2^*)^{-1}. \tag{4.103}$$

From Lemma 1.11.2 we see that $\iota_1^* = \iota_2$ and $\iota_2^* = \iota_1 J^{-1}$, where $J : H^1(\Omega) \to (H^1(\Omega))^{**}$ is the bidual embedding from Definition 1.10.5. Additionally Lemma 4.3.2 shows that

$$\hat{A}^* = \Big((\hat{A}_0^C)^* J\Big)^* = J^*(\hat{A}_0^C)^{**}. \tag{4.104}$$

Using Lemma 1.10.11 and Lemma 4.5.2 then show that

$$\hat{A}^* = J^*\tilde{J}\overline{\hat{A}_0^C}J^{-1} = J^*\tilde{J}\hat{A}^C J^{-1}, \tag{4.105}$$

where $\tilde{J} : (H^1(\Omega))^* \to (H^1(\Omega))^{***}$ also denotes a bidual mapping as in Definition 1.10.5.

Since J is everywhere defined and bounded, see Lemma 1.10.6, its adjoint is given by

$$\begin{aligned} &J^* : (H^1(\Omega))^{***} \to (H^1(\Omega))^*, \\ &(J^*u, v)_{(H^1)^*\times H^1} = (u, Jv)_{(H^1)^{***}\times(H^1)^*}, \quad u \in (H^1(\Omega))^{***},\ v \in H^1(\Omega). \end{aligned} \tag{4.106}$$

This is exactly the inverse of $\tilde{J}$, therefore we finally obtain

$$\hat{A}^* = \hat{A}^C J^{-1}, \tag{4.107}$$

and thus

$$A_N^* = (\iota_1^*)^{-1}\hat{A}^*(\iota_2^*)^{-1} = \iota_2^{-1}\hat{A}^C J^{-1}(\iota_1 J^{-1})^{-1} = \iota_2^{-1}\hat{A}^C\iota_1^{-1} = A_N^C. \tag{4.108}$$

Analogously one shows $(A_N^C)^* = A_N$. □

Corollary 4.8.2. *It holds true that* $\mathrm{Ran}\gamma_D \restriction_{\mathrm{Dom}A_N}$ *and* $\mathrm{Ran}\gamma_D \restriction_{\mathrm{Dom}A_N^C}$ *are dense in* $H^{\frac{1}{2}}(\partial\Omega)$.

Proof. Let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ and $\epsilon > 0$. Theorem 1.5.1 shows that the Dirichlet trace is surjective and bounded, so we can find a $u \in H^1(\Omega)$ such that $\gamma_D u = \varphi$.

From the density of $C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ in $H^1(\Omega)$, see Lemma 1.5.2, we see that there exists $\hat{u} \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega \subset \mathcal{G}$ with

$$\|\hat{u} - u\|_{H^1} < \epsilon. \tag{4.109}$$

$\mathrm{Dom}A_N$ is dense in $\mathcal{G}$, see Theorem 4.8.1, so we can find $\tilde{u} \in \mathrm{Dom}A_N$ such that

$$\|\tilde{u} - \hat{u}\|_{H^1} \le \|\hat{u} - \tilde{u}\|_{\mathcal{G}} < \epsilon, \tag{4.110}$$

and thus we see that

$$\|\tilde{u} - u\|_{H^1} < 2\epsilon. \tag{4.111}$$

Finally this yields that

$$\|\gamma_D \tilde{u} - \varphi\|_{H^{\frac{1}{2}}} = \|\gamma_D(\tilde{u} - u)\|_{H^{\frac{1}{2}}} < 2\|\gamma_D\|\epsilon, \tag{4.112}$$

with $\gamma_D \tilde{u} \in \mathrm{Ran}\gamma_D \restriction_{\mathrm{Dom}A_N}$, which proves the claimed density.

The statement with $\mathrm{Dom}A_N^C$ follows analogously. □

4.9 Quasi Boundary Triple for Accretive Schrödinger Operators

We will now consider the operators T and $\tilde{T}$ in $L^2(\Omega)$, as defined in (4.65) and (4.66), respectively. Let $\iota_\pm : H^{\pm\frac{1}{2}}(\partial\Omega) \to L^2(\Omega)$ be isometric isomorphisms as in Lemma 1.11.3 which satisfy

$$\langle \iota_- u, \iota_+ v\rangle_{L^2} = (u, v)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}, \quad u \in H^{-\frac{1}{2}}(\partial\Omega),\, v \in H^{\frac{1}{2}}(\partial\Omega). \tag{4.113}$$

We define mappings

$$\begin{aligned} (\Gamma_0, \Gamma_1)^T &: \mathrm{Dom}T \to L^2(\partial\Omega) \oplus L^2(\partial\Omega), \quad (\Gamma_0, \Gamma_1)^T f = (\iota_- \gamma_N f, \iota_+ \gamma_D f)^T, \\ (\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T &: \mathrm{Dom}\tilde{T} \to L^2(\partial\Omega) \oplus L^2(\partial\Omega), \quad (\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T f = (\iota_- \tilde{\gamma}_N f, \iota_+ \gamma_D f)^T. \end{aligned} \tag{4.114}$$

In order to show that the triple $(L^2(\partial\Omega), (\Gamma_0, \Gamma_1)^T, (\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T)$ really constitutes a quasi boundary triple for an appropriate dual pair, we will prove the following Lemma.

Lemma 4.9.1. *Let $u \in \mathrm{Dom}T$ and $\eta \in C^\infty(\Omega)$ such that $|\eta|, |\nabla\eta|, |\Delta\eta|$ are all bounded. Then it holds that $\eta u \in \mathrm{Dom}T$. Furthermore, if $\eta \in C_0^\infty(\Omega)$, then it additionally holds that*

$$\gamma_D(\eta u) = 0, \quad \gamma_N(\eta u) = 0. \tag{4.115}$$

Proof. As $\eta \in L^\infty(\Omega)$, $\nabla\eta \in L^\infty(\Omega, \mathbb{R}^d)$, we see that

$$\eta u \in L^2(\Omega), \quad \nabla(\eta u) = (\nabla\eta)u + \eta\nabla u \in L^2(\Omega), \tag{4.116}$$

and hence $\eta u \in H^1(\Omega)$, which together with Lemma 4.2.2 shows that

$$(-\Delta + q)(\eta u) \in \mathcal{D}'(\Omega). \tag{4.117}$$

Due to $\eta \in C^\infty(\Omega)$ and $(-\Delta+q)u \in L^2(\Omega)$ we then obtain

$$(-\Delta+q)(\eta u) = \eta(-\Delta+q)u - 2\nabla\eta\nabla u - u\Delta\eta \in L^2(\Omega). \tag{4.118}$$

It remains to show that $\eta u \in \mathrm{Dom}\hat{A}$. Since $\eta \in C^\infty(\Omega)$, it holds that $\eta\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ for every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$, and thus for every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$

$$\begin{aligned}
|\tilde{\mathbf{a}}(\eta u,\phi)| &= \left|\langle\nabla(\eta u),\nabla\phi\rangle_{L^2} + \int_\Omega q\eta u\overline{\phi}\mathrm{d}x\right| \\
&= \left|\langle\nabla(\eta u),\nabla\phi\rangle_{L^2} + \int_\Omega qu\overline{\overline{\eta}\phi}\mathrm{d}x + \langle\nabla u,\nabla(\overline{\eta}\phi)\rangle_{L^2} - \langle\nabla u,\nabla(\overline{\eta}\phi)\rangle_{L^2}\right| \\
&= \left|(\hat{A}u,\overline{\eta}\phi)_{(H^1)^*\times H^1} + \langle\nabla(\eta u),\nabla\phi\rangle_{L^2} - \langle\nabla u,\nabla(\overline{\eta}\phi)\rangle_{L^2}\right| \\
&\leq \left(\|\hat{A}u\|_{(H^1)^*} + \|\nabla u\|_{L^2}\right)\|\overline{\eta}\phi\|_{H^1} + \|\eta u\|_{H^1}\|\phi\|_{H^1}, \\
&\leq \left(\|\hat{A}u\|_{(H^1)^*} + \|\nabla u\|_{L^2}\right)(2\|\eta\|_{L^\infty} + \|\nabla\eta\|_{L^\infty})\|\phi\|_{H^1} + \|\eta u\|_{H^1}\|\phi\|_{H^1},
\end{aligned} \tag{4.119}$$

which shows that $\eta u \in \mathrm{Dom}\hat{A}$ and thus $\eta u \in \mathrm{Dom}T$.

Now assume that $\eta \in C_0^\infty(\Omega)$. Then for every $\phi \in C_0^\infty(\mathbb{R}^d)\restriction_\Omega$ it holds that $\eta\phi \in C_0^\infty(\Omega)$, and thus together with (4.118) we obtain

$$\begin{aligned}
(\gamma_N(\eta u),\gamma_D\phi)_{H^{-\frac12}\times H^{\frac12}} &= (\hat{A}(\eta u),\phi)_{(H^1)^*\times H^1} - \langle(-\Delta+q)(\eta u),\phi\rangle_{L^2} \\
&= \langle\nabla(\eta u),\nabla\phi\rangle_{L^2} + (qu,\overline{\eta}\phi)_{\mathcal{D}'\times\mathcal{D}} - ((-\Delta+q)u,\overline{\eta}\phi)_{\mathcal{D}'\times\mathcal{D}} \\
&\quad + 2\langle\nabla u,\phi\nabla\overline{\eta}\rangle_{L^2} + \langle u,\phi\Delta\overline{\eta}\rangle_{L^2} \\
&= (u,\nabla\overline{\eta}\nabla\phi)_{\mathcal{D}'\times\mathcal{D}} - (u,\nabla(\overline{\eta}\nabla\phi))_{\mathcal{D}'\times\mathcal{D}} + (u,\Delta(\overline{\eta}\phi))_{\mathcal{D}'\times\mathcal{D}} \\
&\quad - 2(u,\nabla(\phi\nabla\overline{\eta}))_{\mathcal{D}'\times\mathcal{D}} + (u,\phi\Delta\overline{\eta})_{\mathcal{D}'\times\mathcal{D}} = 0.
\end{aligned} \tag{4.120}$$

Lemma 1.5.2 shows that $\mathrm{Ran}\gamma_D\restriction_{C_0^\infty(\mathbb{R}^d)\restriction_\Omega}$ is dense in $H^{\frac12}(\partial\Omega)$, thus $\gamma_N(\eta u)=0$.

Since $\mathrm{supp}\,\eta \subset \Omega$, we see that ηu is zero outside some compact set in Ω and thus $\gamma_D(\eta u)=0$, which concludes the proof. □

Theorem 4.9.2. *Let $(L^2(\partial\Omega),(\Gamma_0,\Gamma_1)^T,(\tilde{\Gamma}_0,\tilde{\Gamma}_1)^T)$ as in (4.114) and $T,\tilde{T}$ be as in (4.65) and (4.66), respectively. Then $(L^2(\partial\Omega),(\Gamma_0,\Gamma_1)^T,(\tilde{\Gamma}_0,\tilde{\Gamma}_1)^T)$ is a quasi boundary triple for the dual pair*

$$S = \tilde{T}\restriction_{\ker\tilde{\Gamma}_0\cap\ker\tilde{\Gamma}_1}, \quad \tilde{S} = T\restriction_{\ker\Gamma_0\cap\ker\Gamma_1}. \tag{4.121}$$

Proof. We will use Theorem 1.13.3. First we need to show that $\ker\tilde{\Gamma}_0\cap\ker\tilde{\Gamma}_1$ and $\ker\Gamma_0\cap\ker\Gamma_1$ are dense in $L^2(\Omega)$. From Theorem 4.8.1 we know that $\mathrm{Dom}A_N$ is dense in $L^2(\Omega)$, thus it is sufficient to show density of $\ker\Gamma_0\cap\ker\Gamma_1$ in $\mathrm{Dom}A_N$.

For a given $f \in \mathrm{Dom}A_N \subset \mathrm{Dom}T$ define cut-off functions $\eta_n \in C_0^\infty(\Omega)$ such that

$$0 \le \eta_n \le 1, \quad \eta_n(x) = \begin{cases} 1 & \text{if } \mathrm{dist}(x, \partial\Omega) > \frac{2}{n} \text{ and } x < n, \\ 0 & \text{if } \mathrm{dist}(x, \partial\Omega) < \frac{1}{n} \text{ or } x > 2n, \end{cases} \tag{4.122}$$

and define

$$f_n = \eta_n f. \tag{4.123}$$

Since $|\eta_n| \le 1$, we see that $|f_n| \le |f|$. Furthermore, we have pointwise convergence on Ω and hence by dominated convergence

$$f_n \xrightarrow{n\to\infty} f \quad \text{in } L^2(\Omega). \tag{4.124}$$

By Lemma 4.9.1 we see that $(f_n)_n \subset \ker\Gamma_0 \cap \ker\Gamma_1$, so consequently $\ker\Gamma_0 \cap \ker\Gamma_1$ is dense in $L^2(\Omega)$. Analogously one shows that $\ker\tilde{\Gamma}_0 \cap \ker\tilde{\Gamma}_1$ is dense.

Next we see that

$$\begin{aligned}(\Gamma_1 f, \tilde{\Gamma}_0 g)_{\mathcal{G}} - (\Gamma_0 f, \tilde{\Gamma}_1 g)_{\mathcal{G}} &= \overline{(\tilde{\gamma}_N g, \gamma_D f)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}}} - (\gamma_N f, \gamma_D g)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} \\ &= \overline{(\hat{A}^C g, f)_{(H^1)^*\times H^1} - \langle(-\Delta+\overline{q})g, f\rangle_{L^2}} - (\hat{A}f, g)_{(H^1)^*\times H^1} + \langle(-\Delta+q)f, g\rangle_{L^2} \\ &= \langle(-\Delta+q)f, g\rangle_{L^2} - \overline{\langle(-\Delta+\overline{q})g, f\rangle_{L^2}} = \langle Tf, g\rangle_{L^2} - \langle f, \tilde{T}g\rangle_{L^2},\end{aligned} \tag{4.125}$$

where we used Corollary 4.5.3, so we see that the triple satisfies (G).

Next we need to show that $\mathrm{Ran}(\Gamma_0, \Gamma_1)^T$ and $\mathrm{Ran}(\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T$ are dense in $L^2(\partial\Omega) \oplus L^2(\partial\Omega)$.

Since ι_- is a bijection, it is clear that

$$\ker\Gamma_0 = \ker(\iota_-\gamma_N) = \ker\gamma_N \tag{4.126}$$

and consequently

$$A_N = T\restriction_{\ker\Gamma_0}, \quad A_N^C = T\restriction_{\ker\tilde{\Gamma}_0}. \tag{4.127}$$

From Corollary 4.8.2 we see that $\mathrm{Ran}\gamma_D\restriction_{\mathrm{Dom}A_N}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$, and so, since ι_+ is an isometric isometry, for $\psi \in L^2(\partial\Omega)$ and $\epsilon > 0$ there exists $u \in \mathrm{Dom}A_N = \ker\Gamma_0$ such that

$$\|\Gamma_1 u - \psi\|_{L^2} = \|\iota_+\gamma_D u - \psi\|_{L^2} = \|\gamma_D u - \iota_+^{-1}\psi\|_{H^{\frac{1}{2}}} < \epsilon, \tag{4.128}$$

which proves the density of $\Gamma_1\restriction_{\ker\Gamma_0}$ in $L^2(\partial\Omega)$.

Furthermore we know from Lemma 4.7.1 that $\gamma_N : \mathrm{Dom}T \to H^{-\frac{1}{2}}(\partial\Omega)$ is surjective, and thus, since ι_- is bijective, we see that also $\Gamma_1 = \iota_-\gamma_N$ is surjective.

Lemma 2.9.2 now shows that $\mathrm{Ran}(\Gamma_0, \Gamma_1)^T$ is dense in $L^2(\partial\Omega) \oplus L^2(\partial\Omega)$, and one can show the same for $\mathrm{Ran}(\tilde{\Gamma}_0, \tilde{\Gamma}_1)^T$ analogously, which shows that (DD) is satisfied.

Finally, we see that

$$A_N^* = A_N^C, \quad (A_N^C)^* = A_N, \tag{4.129}$$

from Theorem 4.8.1, so the triple also satisfies (M), thus all assumptions of Theorem 1.13.3 are fulfilled, which concludes the proof. □

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