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Where to put a hole?

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We give an overview over the field of shape and topology optimization and show an application from electrical engineering. We conclude with an extended outlook about further aspects to be considered for a practical optimization tool.

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1 Introduction

In many engineering applications one is faced with the task of determining a design for an object which minimizes or maximizes a given performance criterion. Examples include applications from mechanical engineering where one wants to find the design of a mechanical structure with maximal stiffness, fluid dynamics, e.g. the optimal layout of an airplane wing, or electromagnetics. Mathematically, this task can be formulated as an optimization problem where the unknown quantity is a shape, i.e. an (admissible) subset Ω of $\mathbb{R}^d$. Then, on the one hand, one is interested in how a smooth perturbation of the boundary of the set Ω affects the performance of the design and how this sensitivity information can be used to obtain better or even optimal designs. On the other hand, one may even be interested in designs having a different topology compared to the initial one. Then, the task is to find the optimal distribution of material and void within a given design domain. Or, as the French engineer and architect Robert Le Ricolais (1894–1977) phrased it [10]:

"The art of structure is how and where to put the holes."

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In a large class of problems, the objective function depends on the shape of the domain via the solution of a boundary value problem posed on this domain Ω or on a superset $D \supset \Omega$. A generic PDE-constrained shape optimization problem can be formulated as

$$\min_{\Omega \in \mathcal{A}} J(u, \Omega) \tag{1a}$$

$$\text{subject to } u \in V : a_\Omega(u, v) = \langle F, v \rangle \quad \forall v \in V, \tag{1b}$$

with a suitable set of admissible designs $\mathcal{A}$, and the weak formulation of a partial differential equation (PDE) involving the domain Ω on a function space V.

The different approaches to tackling a problem of the form (1) are usually classified into two groups of methods, namely *shape optimization* and *topology optimization* methods. While in the former class of methods, the shape Ω is solely modified by means of smooth perturbations of its boundary, the latter class of methods allows for more general designs as also the introduction of holes or new components is possible. We mention that no clear border between these two classes of methods exists and that the terms are sometimes used interchangably in the literature.

2 Shape Optimization

In shape optimization methods, a given design Ω is successively updated by a smooth modification of its boundary in such a way that the objective function decreases. A boundary perturbation which guarantees a descent of the objective function can be determined by means of the *shape derivative* of the problem.

Given a smooth vector field V defined on the computational domain D (which contains the design Ω), and a family of deformation maps $\{T_t^V\}_t$ where $t \in (0, t_{\max})$, e.g. $T_t^V(x) = x + tV(x)$, let $\Omega_t := T_t^V(\Omega)$ denote the perturbed shape. The shape derivative gives information about the sensitivity of the objective function with respect to a perturbation of Ω by a given deformation map T_t^V. Denoting $\mathcal{J}(\Omega) := J(u(\Omega), \Omega)$ where $u(\Omega)$ denotes the solution to the constraining PDE at Ω, the shape derivative is defined as

$$d\mathcal{J}(\Omega; V) = \lim_{t \searrow 0} \frac{\mathcal{J}(\Omega_t) - \mathcal{J}(\Omega)}{t},$$

if this limit exists and the mapping $V \mapsto d\mathcal{J}(\Omega, V)$ is linear. Historically, its first appearance was in the early work of Hadamard who derived the shape derivative for the first eigenvalue of the clamped plate in 1907 [15]. In this work, Hadamard showed that the shape sensitivity only depends on the normal component of the shape perturbation on the boundary. Later, Zolésio generalized this observation

to general shape functions and showed that, for domains with smooth enough boundaries, the shape derivative can always be written as

$$d\mathcal{J}(\Omega;V) = \int_{\partial\Omega} g_\Gamma V \cdot n \,\mathrm{d}s, \tag{2}$$

with an integrable function $g_\Gamma \in L^1(\Gamma)$, see [9, Theorem 3.6], where n denotes the outward unit normal vector on $\partial\Omega$. This form is often called the Hadamard form of the shape derivative, but it is actually due to J.-P. Zolésio.

Besides this boundary integral form, the shape derivative can also be represented as a volume integral over the whole domain,

$$d\mathcal{J}(\Omega;V) = \int_D g(V, \mathrm{D}V) \,\mathrm{d}x, \tag{3}$$

for some function g depending on the vector field V and its Jacobian $\mathrm{D}V$. This representation has the advantage that it requires less regularity of the solutions to the PDE constraint as well as to the adjoint equation. For an overview over different ways to derive the shape derivative, see [23].

We call V a descent direction at Ω if $d\mathcal{J}(\Omega;V) < 0$. In the boundary-based form (2), given a shape Ω, a descent direction $V = -g_\Gamma n$ is readily available whereas the extraction of a descent direction is not obvious in the volume-based form (3). However, we remark that, in many applications $V = -g_\Gamma n$ might not be regular enough or one might be interested in an extension of V from $\partial\Omega$ to D. In these cases, the volume-based form (3) may be a better choice [16]. In the case of (3), a descent direction V can be extracted by solving an auxiliary boundary value problem of the form

$$\text{Find } V \in X : b(V,W) = -d\mathcal{J}(\Omega;W) \quad \forall W \in X,$$

with a suitable function space X and a positive bilinear form $b(\cdot,\cdot)$, since then $d\mathcal{J}(\Omega;V) = -b(V,V) < 0$.

Given the shape derivative of the problem and a way to extract a descent vector field V, a standard way to optimize a shape consists in a gradient descent method, i.e., in moving the boundary of Ω a certain distance $\tau > 0$ in the direction given by V. Since V is a descent direction, choosing τ small enough will always yield a descent of the objective unless we have reached a local optimum, in which case the shape derivative vanishes. It is clear that by proceeding like this, the topology of the design is most likely not to be changed. Even though it may happen that different parts of the shape Ω merge, no information about when introducing a hole in the interior of Ω would be beneficial is encoded in the shape derivative, thus, holes do not suddenly appear.

Beside an explicit representation of the boundary of the shape Ω (e.g. by polygonal, polynomial or spline curves), the interface between Ω and its complement

can also be represented in an implicit way, as the zero level set of a continuous function $\psi : D \to \mathbb{R}$, called a level set function, which has positive values inside Ω and negative values in $D \setminus \overline{\Omega}$ [18]. In what became well-known as *the level set method*, the evolution of the function ψ is guided by the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}\psi + V \cdot \nabla\psi = 0,$$

where t is a pseudo-time parameter and V determines the direction of the evolution and is chosen as a descent direction for the shape derivative [19].

The implicit geometry representation naturally allows for an easy and flexible handling of topological changes, and it happens in practice that components of the design merge, yielding a topologically much different structure[1]. However, also this approach does not have a nucleation mechanism to introduce holes in the interior of Ω when it is beneficial.

Thus, the question of how and where to put the holes persists.

3 Topology Optimization

The first approaches to variable-topology design optimization came up in the late 1980s with the homogenization method [7] and the density-based approach [6] which is now better known as the SIMP (Solid Isotropic Material with Penalization) method. While the former suffers from a large number of degrees of freedom and is not widely used in practice, density-based methods have been applied to a vast range of problems, see e.g. [1] for the topology optimization of an aircraft wing.

The idea of density-based topology optimization is to represent a design by a function $\rho : D \to \mathbb{R}$ where $\rho(x) = 1$ if point x is occupied by material and $\rho(x) = 0$ if it is not. Then, in order to avoid a discrete optimization problem, during the optimization one allows the function ρ to attain any values between 0 and 1, and calls it a density function. In order to eventually end up with a "black-and-white"-picture where ρ has the value 0 or 1 almost everywhere in the design region, the idea of the SIMP method is to penalize intermediate density values. This however leaves an optimization problem which lacks a solution, a fact which can be observed from numerical experiments in the phenomena of mesh dependency (choosing a smaller discretization size yields a different, finer structure as an "optimal" solution) or so-called checkerboard patterns. In order to treat this ill-posed problem

[1] This is exploited in the rather popular approach of [2] where the authors start with a perforated design containing many holes and end up with the optimal topology by only using shape sensitivity information.

one introduces a "length scale", e.g. by filtering the density variable ρ, i.e. by averaging its values over a certain filter radius $R > 0$. Density-based approaches are very flexible and can handle variable topologies without any problems. They have been applied to a wide variety of problems and are used in commercial software. Nevertheless, we remark that in the final design some "gray" areas may remain and one usually does not get a crisp interface between the two materials. For a thorough overview of density-based topology optimization methods, we refer the interested reader to [21].

A different way to deal with topological changes in the course of a shape optimization procedure was first introduced in [11] as the so-called bubble method. Here, the authors determined a criterion which gives information about where the introduction of a hole is most beneficial. The idea of the bubble method is to successively introduce holes based on this criterion and perform shape optimization on all boundaries, also considering the newly created boundaries of the holes.

This positioning criterion for a hole was later extended and rigorously introduced as the *topological derivative* in [22]. Given a shape functional $\mathcal{J} = \mathcal{J}(\Omega)$, its topological derivative at a spatial point $x_0 \in \Omega$ is defined as the quantity $G(x_0)$ which satisfies a topological asymptotic expansion of the form

$$\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega) = \varepsilon^d \, G(x_0) + o(\varepsilon^d),$$

as ε tends to zero, where $\Omega_\varepsilon = \Omega \setminus \overline{B(x_0, \varepsilon)}$ denotes the perturbed domain where a hole of radius ε is introduced around the point x_0 and d denotes the space dimension. From this expansion, it can be seen that, whenever $G(x_0) < 0$, then $\mathcal{J}(\Omega_\varepsilon) < \mathcal{J}(\Omega)$ for ε small enough. Thus, introducing a small enough hole around x_0 will yield a decrease of the objective function $\mathcal{J}$, which finally answers the question in the title of this article.

For the derivation of the topological derivative of a large class of problems of the type (1) with linear PDE constraints (1b), we refer to [3]. In the case of a quasilinear PDE constraint, the literature is rather scarce, see e.g. [5].

Of course, this topological sensitivity information can be incorporated into shape optimiztion algorithms such as the level set method in order to also allow for topological changes as it was done in [8]. Another way of exploiting the flexibility of a level set representation of a design together with the topological sensitivity information provided by the topological derivative was introduced in [4]. In this algorithm, the evolution of the level set function is purely guided by the topological derivative, which enables topological changes whenever it is beneficial.

In the next section we will apply this algorithm to a model problem from electromagnetics.

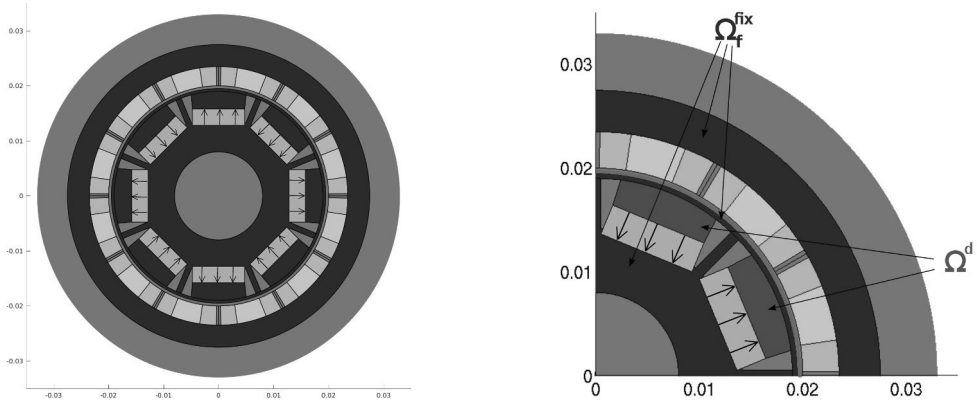

Figure 1: Left: Computational domain D representing electric motor with different subdomains. Right: Zoom on upper right quarter (for a different rotor-to-stator constellation) with design region Ω^d and fixed ferromagnetic set Ω_f^{fix}. For a given design $\Omega \subset \Omega^d$, we have $\overline{\Omega_f} = \overline{\Omega_f^{fix}} \cup \overline{\Omega}$. Figures taken from [13].

4 Application to optimal electrical machine design

4.1 Model Problem

In this section, we consider a practical problem from electrical engineering. We are given an electric motor and consider its two-dimensional cross-section, see Figure 1. The motor contains coils, where electric current is induced, and permanent magnets which are magnetized in the directions indicated in Figure 1. Furthermore, there are regions of ferromagnetic material, and the rest of the computational domain is filled with air. Given these sources, the magnetic flux density $\mathbf{B}$ can be computed from the magnetostatic approximation of Maxwell's equations via the two-dimensional vector potential ansatz $\mathbf{B} = \operatorname{curl}\left((0,0,u)^\top\right)$, where the scalar function u solves the boundary value problem

$$\begin{aligned} -\operatorname{div}(\nu_\Omega(x,|\nabla u|)\nabla u) &= I_z - \nu_0 \operatorname{div}\left(\mathbf{M}^\perp\right) && \text{in } D, \\ u &= 0 && \text{on } \partial D. \end{aligned} \tag{4}$$

Here, the magnetic reluctivity ν_Ω is a nonlinear function only in the ferromagnetic parts of the motor, and piecewise constant elsewhere; I_z represents the third component of the impressed currents in the coils and $\mathbf{M}^\perp = (-M_2, M_1)^\top$ is the perpendicular of the first two components of the magnetization $\mathbf{M}$. These sources vanish outside the coil or the magnet areas, respectively.

For this model problem, we are interested in determining the optimal distribution of ferromagnetic material inside the design areas Ω^d (the highlighted regions in

the right picture of Figure 1) such that a given objective function $J = J(u)$ is minimized. In other words, we are looking for the optimal subset $\Omega \subset \Omega^d$ (usually out of a set $\mathcal{A}$ of sufficiently "nice", admissible subsets of Ω^d) to be occupied with ferromagnetic material. The shape of Ω influences the objective function via the solution u to the PDE constraint (4). Note that Ω enters the PDE constraint via the magnetic reluctivity

$$\nu_\Omega(x, |\nabla u|) = \chi_{\Omega_f^{fix} \cup \Omega}(x)\hat{\nu}(|\nabla u|) + \chi_{(D \setminus \Omega_f^{fix}) \cup (\Omega^d \setminus \Omega)}(x)\nu_0,$$

where χ_A denotes the characteristic function of a set A, $\hat{\nu}$ is a scalar function and ν_0 is constant, $\nu_0 = 10^7/(4\pi)$. Here, Ω_f^{fix} denotes the fixed ferromagnetic part of the motor which is not subject to the optimization, see also Figure 1 (right), and Ω is the part of the design region Ω^d that is currently occupied by ferromagnetic material. We remark that the magnetic reluctivity represents the relation between the magnetic flux density $\mathbf{B}$ and the magnetic field intensity $\mathbf{H}$ and that its physical properties yield conditions which allow to show existence of a unique solution (in suitable function spaces) of (4), see [12].

For the optimization criterion, we consider a functional $J = J(u)$ which – when minimized – yields a smooth rotation of the motor. For more details on the functional, see [12, Section 8.3].

5 Design Optimization

We approach the design optimization described in Section 4.1 by means of a two-stage algorithm.

In a first stage, we employ the level set algorithm of [4], which is based on the topological derivative. In order to be able to do so, the main challenge was to derive the topological derivative for the model problem including the quasilinear PDE constraint (4). The final formula and the detailed derivation can be found in [12].

We remark that the topology optimization algorithm of [4] is based on the topological derivative, a quantity that is only defined in the interior of Ω or in the interior of its complement, but not on the material interface. In the algorithm, the topological derivative is interpolated at the interface. In a second step of the overall algorithm, shape optimization based on the shape derivative is performed as a post-processing. For the derivation of the shape derivative of the problem at hand, see [14].

Figure 2 shows the final design of the two-stage algorithm. The objective value could be reduced from 0.455 to 0.036, yielding a much smoother rotation of the rotor and less vibrations and noise during operation of the machine.

Figure 2: Final design of two-stage algorithm together with magnetic field lines. Figures taken from [13].

6 Outlook

We saw that the shape and topological derivative allow for an elegant way of finding improved and optimized designs. However, when dealing with real-world applications, this is not yet enough for providing an unconditionally satisfactory optimization tool. In order to achieve this, the work presented here should be extended into several directions.

First of all, we only performed optimization with respect to one electromagnetic performance criterion (smoothness of rotation) and didn't take into consideration other important criteria (e.g. high average torque). Of course, it is possible to minimize a weighted average of different objectives by the presented algorithms, however then the choice of the weights becomes crucial. In practical applications, engineers are interested in a solution which gives a good compromise between several, in general conflicting, objectives. For that purpose, one is interested in finding the set of Pareto-optimal solutions to the multi-objective optimization problem, i.e. the set of designs which cannot be improved with respect to one criterion without deteriorating the performance with respect to one of the others. Multi-objective optimization problems are often approached by means of derivative-free evolutionary algorithms, which have the additional advantage that they are not prone to getting stuck in local minima. However, on the other hand they are computationally much more costly compared to gradient-based optimization algorithms. By a smart coupling of these two worlds, it might be possible to minimize the drawbacks of both kinds of methods and obtain the set of Pareto-optimal solutions using less computational effort.

As a second aspect where there is room for improvement, we note from the op-

timized design in Figure 2 that, even though the design is (locally) optimal with respect to the electromagnetic performance criterion given by J, its mechanical stability might not be sufficient to sustain a rotation at a high speed. Thus, the optimal design process should be considered subject to a constraint on the maximum arising mechanical stress. Moreover, in order to ensure manufacturability of the optimized design, it would be desirable to have a way to impose a minimum thickness of all components of the design. This can be done in a similar way as the density filtering in density-based topology optimization methods, see [17].

A further aspect that is important in practice is the sensitivity of the optimized design with respect to manufacturing imperfections or uncertainty in measured input data due to measurement errors. In order to avoid designs whose actual performance after production is far off from the simulated performance due to these influences, the design optimization should be performed in a robust way, taking into account these sources of uncertainty.

Robust optimization algorithms typically involve a much larger number of function evaluations (and therefore of PDE solves) than deterministic optimization approaches. Therefore, fast solution strategies for the forward problem are crucial. On the one hand, this can be achieved by exploiting parallelism, on the other hand also model order reduction techniques can be very helpful in this respect.

Finally, we remark that the magnetostatic approximation to Maxwell's equations (4) can be used to model an electric motor which rotates at a constant speed. When considering the starting phase of the motor, an additional time derivative $\sigma(x)\frac{\partial u}{\partial t}$ has to be considered. Here, the coefficient σ vanishes in non-conducting regions, yielding a mixed elliptic-parabolic boundary value problem, called the eddy current problem. Moreover, the eddy currents, which are modeled by $\sigma(x)\frac{\partial u}{\partial t}$, produce heat. On the other hand, temperature influences the properties of permanent magnets (which again has an impact on the eddy currents). In order to consider temperature-dependent behavior of the permanent magnets, the electromagnetic eddy current problem has to be coupled with a heat equation. Here, an efficient solution strategy (e.g. parallelism in time) is an important aspect before thinking about performing optimization for this setting.

Conclusion As we could see, the field of optimal electrical machine design can be seen as a kind of mathematical playground where many disciplines from mathematical and numerical analysis as well as optimization meet. We saw that there remain many challenges to be tackled in order to obtain a generic practical optimization tool. Most of these challenges are interesting from both a mathematical and an engineering perspective and the practical problem can be solved best by interdisciplinary cooperations.

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