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Calculus of variations

Aufgabenblatt 1, 24.04.2012

Aufgabe 1: Show that the minimization problem

$$F(u) := \int_{-1}^{1} \big(xu'(x)\big)^2 dx = \min!, \quad u \in C^1([-1,1]), \quad u(-1) = 0, u(1) = 1,$$

has no solutions. [1] HINT: Use the sequence of functions $u_n(x) = \frac{1}{2} + \frac{\arctan(nx)}{2\arctan(n)}$, $n \in \mathbb{N}$ and look at the behavior of $F(u_n)$.

Aufgabe 2: Show the following variant of the fundamental lemma of variational calculus. Given an open interval $I \subseteq \mathbb{R}$ show that for $u \in L^2(I)$ satisfying

$$\int_I u\varphi' dx = 0, \quad \forall \varphi \in C_0^\infty(I),$$

there exists a constant c such that $u = c$ almost everywhere.

Aufgabe 3: Let $u \in \mathring{W}_1^2(a,b)$, where (a,b) is a finite interval. Show that there exists a unique continuous function $v\colon [a,b] \to \mathbb{R}$ satisfying $u(x) = v(x)$ for almost all $x \in (a,b)$ and $v(a) = v(b) = 0$. Moreover, show the estimate

$$\|v\|_\infty \le (b-a)^{1/2}\|u'\|_{L^2(a,b)} \le (b-a)^{1/2}\|u\|_1.$$

Aufgabe 4: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. By Poincare inequality there exists a constant $c > 0$ such that

$$\|u\|^2_{L^2(\Omega)} \le c\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

for all $u \in \mathring{W}_1^2(\Omega)$. Estimate the constant c in terms of geometry of Ω as sharp as possible.

Aufgabe 5: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given $f \in L^2(\Omega)$ and $g \in W_1^2(\Omega)$. Let $u \in W_1^2(\Omega)$ be the unique solution of the boundary value problem

$$\int_\Omega \sum_{j=1}^n \partial_j u \partial_j v dx = \int_\Omega f v dx, \quad \text{for all } v \in \mathring{W}_1^2(\Omega),$$

with the boundary condition $u - g \in \mathring{W}_1^2(\Omega)$. Show that u is also the unique solution of the minimization problem

$$W_1^2(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega \sum_{j=1}^n (\partial_j u)^2 dx - \int_\Omega f u dx = \min!$$

with the same boundary condition $u - g \in \mathring{W}_1^2(\Omega)$.

[1] This example of a non-solvable minimization problem belongs Weierstrass from 1870.