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Spectral asymptotics of Landau Hamiltonians with δ-perturbations supported on curves in $\mathbb{R}^2$

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Abstract

In this master's thesis the spectrum of the two-dimensional Landau Hamiltonian with a singular δ-potential supported on a $\mathcal{C}^{1,1}$ curve Γ in $\mathbb{R}^2$ is analyzed. It is well-known that the essential spectrum of the Landau Hamiltonian is stable under perturbations by a singular potential $\alpha\delta_\Gamma$ with real-valued interaction strength $\alpha \in L^\infty(\Gamma)$, hence the discrete eigenvalues of the perturbed Landau Hamiltonian must accumulate at the so-called *Landau levels* Λ_q, $q \in \mathbb{N}_0$, which are the isolated eigenvalues of infinite multiplicity of the Landau Hamiltonian. It turns out, that the rate of accumulation towards the Landau levels is closely related to the rate at which the singular values of the compact and self-adjoint *Toeplitz-type operators* $P_q\delta_\Gamma P_q$ tend to zero, where $P_q : L^2(\mathbb{R}^2) \to \ker(\mathsf{A}_0 - \Lambda_q)$, $q \in \mathbb{N}_0$, denotes the projections into the eigenspaces of the Landau Hamiltonian. The main focus of this thesis is to extend the spectral theory for these Toeplitz-type operators, which exists for the case where Γ is a smooth curve, to the slightly more general situation where Γ is a $\mathcal{C}^{1,1}$ curve in $\mathbb{R}^2$.

Kurzfassung

In dieser Arbeit wird untersucht, wie sich das Spektrum des Landau-Operators bei einer Störung durch ein auf einer $\mathcal{C}^{1,1}$-Kurve Γ getragenes δ-Potential verhält. Es ist weithin bekannt, dass das wesentliche Spektrum des Landau-Operators bei einer Störung durch ein singuläres Potential $\alpha\delta_\Gamma$ mit reeller Interaktionsstärke $\alpha \in L^\infty(\Gamma)$ erhalten bleibt, womit sich die diskreten Eigenwerte des gestörten Landau-Operators um die sogenannten *Landau-Level* Λ_q, $q \in \mathbb{N}_0$, häufen müssen. Die Landau-Level sind dabei die isolierten Eigenwerte unendlicher Vielfachheit des Landau-Operators, die das gesamte Spektrum bilden. Es stellt sich heraus, dass die Konvergenzgeschwindigkeit mit der eine solche Häufung auftritt im direkten Zusammenhang zu den Singulärwerten der kompakten und selbstadjungierten *Toeplitz-Operatoren* $P_q\delta_\Gamma P_q$ steht. Hierbei steht $P_q : L^2(\mathbb{R}^2) \to \ker(\mathsf{A}_0 - \Lambda_q), q \in \mathbb{N}_0$, für die orthogonalen Projektionen in die Eigenräume des Landau-Operators. Zur Zeit existieren spektrale Abschätzungen der Singulärwerte für Toeplitz-Operatoren für den Fall, dass Γ eine glatte Kurve in $\mathbb{R}^2$ ist. Das Ziel dieser Arbeit ist es, dieses Resultat für die leicht verallgemeinerte Situation auszudehnen, in der Γ eine $\mathcal{C}^{1,1}$-Kurve in $\mathbb{R}^2$ ist.

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Contents

1 Introduction

Quantum motions of charged particles in wires are often modelled by networks of leaky quantum wires, which from a mathematical standpoint can be modelled by Schrödinger operators with singular potentials supported on families of curves (see e.g. [6, 8, 10, 26, 39] for references). Nowadays, even though the spectral properties of Schrödinger operators with singular potentials are a well-studied field, there still exist only a handful of mathematical contributions that consider the influence of magnetic fields (see [3, 5, 11–14, 18, 29]), despite their importance in modern physics.

In this master's thesis Schrödinger operators with a constant magnetic field and a singular δ-potential should be considered. In order to explain the focus and results of this master's thesis, let us start by introducing the Landau Hamiltonian, which is a special case of the magnetic Schrödinger operator under a constant magnetic field. Assuming that the strength of the magnetic field is given by some real-valued constant $B > 0$ with corresponding vector potential $\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1)^\top$, $(x_1, x_2) \in \mathbb{R}^2$, in symmetric gauge, the two-dimensional Landau Hamiltonian is defined as the operator

$$\mathsf{A}_0 = \nabla_{\mathbf{A}}^2, \quad \mathrm{dom}(\mathsf{A}_0) = \{f \in \mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2) : \nabla_{\mathbf{A}}^2 f \in L^2(\mathbb{R}^2)\}, \tag{1.1}$$

where $\mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : |\nabla_{\mathbf{A}} f| \in L^2(\mathbb{R}^2)\}$ is the magnetic Sobolev space of first order, and

$$\nabla_{\mathbf{A}} := i\nabla + \mathbf{A} \tag{1.2}$$

is the magnetic gradient. It is a well-known fact the unperturbed Landau Hamiltonian is self-adjoint in $L^2(\mathbb{R}^2)$ and that its spectrum is given by

$$\sigma(\mathsf{A}_0) = \sigma_{ess}(\mathsf{A}_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}, \tag{1.3}$$

where the so-called *Landau levels* $\Lambda_q = (2q + 1)B$ are eigenvalues of infinite multiplicity. Historically, magnetic Schrödinger operators were first studied from a physical point of view in 1928 by Fock in [16] and two years later by Landau in [23], who was the first to investigate the spectrum of the Landau Hamiltonian A_0. In 1962, first mathematical descriptions followed by Ikebe and Kato in [19]. Around 1970 the field around magnetic Schrödinger operators began to expand, in particular, under the efforts of Kato [20] and Simon [37]. A lot of references about the Landau Hamiltonian and magnetic Sobolev spaces in general can also be found in Nicolas Raymonds *Little Magnetic Book* [34].

One topic that has been of historical interest, is the behaviour of the spectrum of the Landau Hamiltonian under perturbation by a regular electric potential $V : \mathbb{R}^2 \to \mathbb{R}$. From a physical point of view, such a potential V can be interpreted as an electric potential that is interfering with the magnetic field induced by the vector potential $\mathbf{A}$. It is well-known that perturbations of the Landau Hamiltonian by a decreasing electric field can generate an accumulation of discrete eigenvalues of the perturbed Landau Hamiltonian $\mathsf{A}_0 + V$ at the Landau levels, which was first described by Raikov in [32] (see also [15, 22, 28, 30, 33, 35, 36]). Under the additional assumption that $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ is non-negative with $V \not\equiv 0$ and $\|V\|_{L^\infty(\mathbb{R}^2;\mathbb{R})} < 2B$ it was shown in [31] that

$$\ker(\mathsf{A}_0 \pm V - \Lambda_q) = \{0\}, \quad q \in \mathbb{N}_0,$$

i.e. the Landau levels are not eigenvalues of $\mathsf{A}_0 \pm V$ anymore, and the discrete eigenvalues of $\mathsf{A}_0 \pm V$ must accumulate towards the Landau levels from above or below, respectively. The assumption that V is sign-definite is essential here, as it was shown in the same paper that for each $q \in \mathbb{N}_0$ there exists a compactly supported potential $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ with $\|V\|_{L^\infty(\mathbb{R}^2;\mathbb{R})} < B$ such that

$$\dim \ker(\mathsf{A}_0 \pm V - \Lambda_q) = \infty.$$

Furthermore, it turns out that the rate of accumulation of the discrete eigenvalues of $\mathsf{A}_0 + V$ towards the Landau levels is closely related to the rate at which the singular values of the compact and self-adjoint Toeplitz-type operator $P_q V P_q$ tend to zero (see [15, 18, 28, 33, 36]), where $P_q : L^2(\mathbb{R}^2) \to \ker(\mathsf{A}_0 - \Lambda_q), q \in \mathbb{N}_0$, denotes the projections onto the infinite-dimensional eigenspaces of each Landau level. Hence, it makes sense to study the spectral behaviour of regular Toeplitz operators of the form $P_q V P_q$ to analyze the spectrum of the Landau Hamiltonian with an electric potential.

In this master's thesis we are considering an analogous problem, where the perturbation of the Landau Hamiltonian is given by a singular potential $\alpha\delta_\Sigma$, that is supported on a compact $\mathcal{C}^{1,1}$ curve $\Sigma \subset \mathbb{R}^2$ with interaction strength $\alpha \in L^\infty(\Sigma;\mathbb{R})$. Similar to the above problem with a regular potential V, one can verify that the addition of such a singular potential can generate an accumulation of discrete eigenvalues of the perturbed Landau Hamiltonian $\mathsf{A}_0 + \alpha\delta_\Sigma$ towards the Landau levels. Moreover, it was shown in [3] that the rate of accumulation at the Landau levels Λ_q can be estimated in terms of the singular values of the compact and self-adjoint Toeplitz-type operators $P_q \delta_\Gamma P_q$, where $\Gamma = \operatorname{supp} \alpha$ denotes the essential support of the interaction strength α. Resorting to the spectral analysis of the Toeplitz operators $P_q \delta_\Gamma P_q$ in [31], which was done for a simple $\mathcal{C}^\infty$ curve in $\mathbb{R}^2$, sharp estimates for the rate of accumulation at the Landau levels were derived in [3]. The main goal of this master's thesis is to extend the asymptotic estimates for the singular values of the Toeplitz-type operators $P_q \delta_\Gamma P_q$ provided in [31] to the case, where Γ is a slightly less regular $\mathcal{C}^{1,1}$ curve in $\mathbb{R}^2$. After that we are going to employ the derived results in the spectral analysis of the Landau Hamiltonian with a δ-potential, following the lines of [3].

In order to explain the results of this master's thesis more precisely, let Σ be the boundary of a compact $\mathcal{C}^{1,1}$ domain $\Omega_i \subset \mathbb{R}^2$ and let $\alpha \in L^\infty(\Sigma;\mathbb{R})$. Formally, the Landau Hamiltonian with a δ-potential is given by the expression

$$\mathsf{A}_\alpha = \nabla_{\mathbf{A}}^2 + \alpha\delta_\Sigma = \mathsf{A}_0 + \alpha\delta_\Sigma, \tag{1.4}$$

where δ_Σ denotes the δ-interaction supported on Σ and α functions as the interaction strength of our singular perturbation. In order to model the Landau Hamiltonian A_α with a δ-potential in a mathematically rigorous way, we make use of the sesquilinear form

$$\mathfrak{a}_\alpha[f,g] = (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\mathbb{R}^2)} + (\alpha f|_\Sigma, g|_\Sigma)_{L^2(\Sigma)}, \quad \mathrm{dom}(\mathfrak{a}_\alpha) = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2), \tag{1.5}$$

which is densely defined, closed and symmetric and hence induces a self-adjoint operator A_α in $L^2(\mathbb{R}^2)$. If we denote the unit normal vector field pointing outward of Ω_i by ν and set $\Omega_e = \mathbb{R}^2 \backslash \overline{\Omega}_i$ for the unbounded exterior domain, we will be able to use interface conditions on the boundary Σ in Theorem 4.6 to show that the Landau Hamiltonian with a singular potential is explicitly given by

$$\begin{aligned} &\mathsf{A}_\alpha f := \nabla^2_{\mathbf{A}} f\restriction_{\Omega_i} \oplus \nabla^2_{\mathbf{A}} f\restriction_{\Omega_e}, \\ &\mathrm{dom}(\mathsf{A}_\alpha) := \{ f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) : \nabla^2_{\mathbf{A}} f\restriction_{\Omega_{i/e}} \in L^2(\Omega_{i/e}),\ \partial_\nu f_e - \partial_\nu f_i = \alpha f|_\Sigma \}. \end{aligned} \tag{1.6}$$

In particular, the first representation theorem for sesquilinear forms now implies the self-adjointness of A_α.

Theorem 1. *The Landau Hamiltonian A_α with a δ-potential defined in* (1.6) *is self-adjoint in $L^2(\mathbb{R}^2)$.*

Once we have established the self-adjointness of the Landau Hamiltonian with a singular potential we will proceed with a spectral analysis of the operator A_α. Following the lines of [5] we are going to use the quadratic forms associated to A_α and A_0 to derive a compact resolvent factorization of the unperturbed Landau Hamiltonian and the Landau Hamiltonian with a δ-interaction. Applying an appropriate version of Weyl's theorem ensures the stability of the essential spectrum, which will be shown in Theorem 4.12. In the following theorem we make use of the compact operators $\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^2)$ and $M(\lambda) : L^2(\Sigma) \to L^2(\Sigma)$, which are introduced in Definition 4.11 and can be seen as integral operators with the Green function of A_0 as integral kernel.

Theorem 2. *For $\lambda < B$ sufficiently small the resolvent difference of A_0 and A_α admits the compact factorization*

$$W_\lambda = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1} = -\gamma(\lambda)(1 + \alpha M(\lambda))^{-1} \alpha \gamma(\lambda)^*, \tag{1.7}$$

In particular, there holds

$$\sigma_{ess}(\mathsf{A}_\alpha) = \sigma_{ess}(\mathsf{A}_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}, \tag{1.8}$$

where $\Lambda_q = (2q+1)B, q \in \mathbb{N}_0$, are the Landau levels.

After showing that the essential spectrum of the Landau Hamiltonian is stable under perturbations with a singular potential, we are going to continue with the main contribution of this master's thesis, which consists of the slight improvement of the spectral asymptotics for the Toeplitz-type operators $P_q \delta_\Gamma P_q$ that are provided in [31]. In order to study these Toeplitz-type operators, we are going to make use of the corresponding quadratic form

$$\mathfrak{t}_q^\Gamma[f] = \int_\Gamma |(P_q f)(x_\Gamma)|^2 \, \mathrm{d}\sigma(x_\Gamma), \quad \mathrm{dom}(\mathfrak{t}_q^\Gamma) = L^2(\mathbb{R}^2),$$

which gives rise to the compact and self-adjoint Toeplitz-type operator T_q^Γ. The sharp spectral estimates for the Toeplitz-type operator T_q^Γ in [31] were shown under the assumption that Γ is a simple smooth curve in $\mathbb{R}^2$. Following the lines of the proof of Proposition 4.1(ii) in [31], we are going to show the following asymptotic estimate in Proposition 5.8:

Theorem 3. *Let Σ be the boundary of a $\mathcal{C}^{1,1}$ domain Ω. Suppose that $\Gamma \subset \Sigma$ is a closed subarc with positive measure. Then for any $q \in \mathbb{N}_0$ the eigenvalues of the Toeplitz-type operator T_q^Γ satisfy*

$$\lim_{k\to\infty} \left(k! s_k(T_q^\Gamma) \right)^{1/k} = \frac{B}{2} \left(\mathrm{Cap}\,(\Gamma) \right)^2,$$

where $\mathrm{Cap}\,(\Gamma)$ *denotes the logarithmic capacity of* Γ (*see Def. 2.15*).

In the final section of this master's thesis we are going to use the spectral asymptotics that we derived for the Toeplitz-type operators T_q^Γ and follow the lines of [3], in order to provide sharp spectral estimates on the eigenvalue clustering of A_α at the Landau levels. Assuming that the interaction strength α in (1.4)-(1.5) is positive (negative) on Σ, we are going to use classic perturbation results in the proof of Theorem 6.7 to show that an accumulation of the discrete eigenvalues of A_α towards the Landau levels Λ_q from above (below, respectively) can be observed. Relying on our analysis of the Toeplitz-type operators of the form $P_q \delta_\Gamma P_q$, we are then going to establish sharp spectral asymptotics on the rate of accumulation of the discrete eigenvalues of A_α towards the Landau levels Λ_q. In Theorem 6.8 we are going to provide an upper bound for the rate of accumulation towards the Landau levels, which even remains true for sign-changing α. In order to obtain lower bounds we require $\alpha \not\equiv 0$ to be sign-definite, as it is still an open problem to show that an eigenvalue accumulation remains present for sign-changing α. In Theorem 6.9 we provide exact spectral asymptotics for the case that α is uniformly positive (uniformly negative) on a closed subarc $\Gamma \subset \Sigma$ of positive measure. More precisely, we arrive at the following result:

Theorem 4. *Let $\alpha \in L^\infty(\Sigma; \mathbb{R})$ be uniformly positive on $\Gamma = \mathrm{supp}\,\alpha$. Then for each $q \in \mathbb{N}_0$ the eigenvalues $\{\lambda_k(q)\}_{k\in\mathbb{N}}$ of A_α lying in the interval $(\Lambda_q, \Lambda_q + B]$ satisy*

$$\lim_{k\to\infty} (k! |\lambda_k(q) - \Lambda_q|)^{1/k} = \frac{B}{2} (\mathrm{Cap}\,(\Gamma))^2, \tag{1.9}$$

where $\mathrm{Cap}\,(\Gamma)$ *denotes the logarithmic capacity of* Γ (*see Def. 2.15*).

Organization of the master's thesis. Section 2 contains some preliminary materials that are needed in the proofs of the main results of this master's thesis. In Section 2.1 and 2.2 we are going to collect some basic definitions from operator theory and provide some of the well-known results from the theory of sesquilinear forms. Section 2.3 contains $\mathcal{C}^{k,\mu}$ domains and tubular coordinates around curves. In Section 2.4 we cover complex curve integrals and provide elementary estimates for analytic functions using Cauchy's integral formula. In Section 2.5 we are going to investigate the n-th root asymptotics of the leading coefficients of orthonormal polynomials with respect to a given Hausdorff measure. Section 2.6 and 2.7 contain elementary results from spectral and perturbation theory of self-adjoint operators under compact perturbations. In Section 2.8 we will give an overview of classical Sobolev spaces and recall some of their well-known properties. Section 2.9 contains properties of Schatten-von Neumann ideals.

Section 3 will be devoted to magnetic Sobolev spaces, which form the magnetic counterpart to classical Sobolev spaces. Basic definitions and elementary results for magnetic Sobolev spaces are collected in Section 3.1. We will see in Section 3.2 that classical and magnetic Sobolev spaces are locally equivalent and use this knowledge in Section 3.3 to construct bounded Dirichlet and Neumann trace operators on Lipschitz domains. Section 3.4 will then be devoted to deriving a version of Green's first identity for the magnetic gradient given in (1.2).

In Section 4 we are going to cover different classes of Landau Hamiltonians. Section 4.1 contains basic material regarding the properties of the unperturbed Landau Hamiltonian. In Section 4.2 we will study the Landau Hamiltonian with Dirichlet boundary conditions on a bounded $\mathcal{C}^{1,1}$ domain. In Section 4.3 we define the Landau Hamiltonian A_α with a δ-potential from (1.6) and prove its self-adjointness. In Section 4.4 we are going to use the quadratic forms of the unperturbed Landau Hamiltonian A_0 and the Landau Hamiltonian A_α with a singular potential to derive the compact factorization (1.7) of their resolvent difference W_λ. Section 4.5 will then be about the thorough analysis of the resolvent difference.

In Section 5 of this master's thesis we introduce Toeplitz operators on compact Lipschitz domains and $\mathcal{C}^{1,1}$ curves Γ and provide sharp spectral asymptotics for the compressed operators $P_q \delta_\Gamma P_q$ onto the eigenspaces to the q-th Landau Level. In particular, Section 5.2 contains the aforementioned spectral analysis of the Toeplitz-type operators on curves for the slightly more general case, where Γ is a closed subarc of a compact $\mathcal{C}^{1,1}$ curve Σ.

Finally, in Section 6 we are going to derive exact spectral asymptotics for the discrete eigenvalues of the Landau Hamiltonian with a δ-potential. For this we are first going to show in Section 6.1 that the singular values of the compressed resolvent difference $P_q W_\lambda P_q$ from (1.7) can be estimated in terms of the singular values of $P_q \delta_\Gamma P_q$. After that we will use the spectral asymptotics for the Toeplitz-type operators from Section 5.2 to obtain sharp spectral estimates for the eigenvalue clustering at the Landau levels.

2 Preliminaries

This section contains some preliminary material that will be needed for the analysis of the perturbed Landau Hamiltonian. In Section 2.1 we collect some basic definitions that will be needed during the master's thesis and then recall some of the well-known facts about sesquilinear forms in Section 2.2. In Section 2.3 and 2.4 we will introduce $\mathcal{C}^{k,\mu}$ domains and discuss curve integrals in $\mathbb{R}^2$ and $\mathbb{C}$, utilizing Cauchy's integral formula to provide useful estimates for analytic functions on curves. Section 2.5 is devoted to the analysis of the n-th root asymptotics of orthogonal polynomials with respect to the Hausdorff measure of a given Lipschitz curve Γ. In Section 2.6 and 2.7 basic results from spectral and perturbation theory for self-adjoint operators under compact perturbations are provided. Section 2.8 and 2.9 will then be used to recall some of the well-known properties of Sobolev spaces and Schatten-von Neumann ideals.

2.1 Elementary Results and Definitions

This subsection contains some very basic definitions and results from elementary calculus and linear algebra. In the following, let X and Y be two normed spaces and T an operator from X to Y. As usual, we are going to denote the domain of definition of T by $\operatorname{dom}(T) \subset X$ and say that T is *densely defined* if $\operatorname{dom}(T)$ is dense in X. Furthermore, we introduce the following spaces:

$$\begin{aligned} \ker(T) &= \{x \in \operatorname{dom}(T) : Tx = 0\} \subset X, && \text{(Kernel of } T\text{)} \\ \operatorname{ran}(T) &= \{Tx : x \in \operatorname{dom}(T)\} \subset Y, && \text{(Range of } T\text{)} \\ \mathcal{G}(T) &= \{(x, Tx) : x \in \operatorname{dom}(T) \subset X \times Y. && \text{(Graph of } T\text{)} \end{aligned}$$

In the case where $T : \operatorname{dom}(T) \to Y$ is a bounded operator, we are going to denote its operator norm by $\|T\|_{X \to Y}$. If there is no danger of confusion we will omit the declaration of the spaces in the operator norm and just write $\|T\|$ instead of $\|T\|_{X \to Y}$. We will denote the space of all bounded and everywhere defined operators from X to Y by $\mathfrak{B}(X, Y)$ and write $\mathfrak{B}(X)$ instead of $\mathfrak{B}(X, X)$. Now suppose, that $T : \operatorname{dom}(T) \to \mathcal{H}$ is an operator acting in a complex Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$. If T is densely defined, we can define the adjoint operator T^* of T via the relation

$$\begin{aligned} \operatorname{dom}(T^*) &= \{y \in \mathcal{H} : \exists y' \in \mathcal{H} : (Tx, y) = (x, y') \quad \forall x \in \operatorname{dom}(T)\}, \\ T^* y &= y'. \end{aligned}$$

We call T *symmetric*, if $T \subset T^*$ and say that T is *self-adjoint* if $T = T^*$. Recall that an operator T is symmetric if and only if the corresponding form $\operatorname{dom}(T) \ni x \mapsto (Tx, x)$ is real-valued. Assume now that T is a symmetric operator. We use the notation $T > 0$ and say that T is *positive*, if $(Tx, x) > 0$ for all $x \in \operatorname{dom}(T)$. We say that T is *uniformly positive* and write $T \geq c$, if there exists a constant $c > 0$ such that $(Tx, x) \geq c\|x\|^2$ for all $x \in \operatorname{dom}(T)$. In the case where c is allowed to be zero we use the notation $T \geq 0$ and call T a *non-negative* operator.

Recall that for a given vector space V and a subspace $U \subset V$ the quotient space of V by U is defined as the set of equivalence classes

$$V/U = \{v + U : v \in V\}.$$

The *codimension* of U in V is then defined as $\operatorname{codim} U = \dim V/U$. For the convenience of the reader we are going to state the fundamental theorem on homomorphisms here, which shows that any linear map can be made injective by factoring out the kernel from its domain of definition.

Theorem 2.1 ([9, Homomorphiesatz für Ringe]). *Let V and W be two vector spaces and $f : V \to W$ a linear mapping. Then $V/\ker(f) \simeq ran(f)$.*

The next proposition can be seen as an immediate consequence of the fundamental theorem on homomorphisms.

Proposition 2.2. *Let V be a vector space and let $U_1, U_2 \subset V$ be subspaces of finite codimension in V. Then there holds $\operatorname{codim}(U_1 \cap U_2) \leq \operatorname{codim}(U_1) + \operatorname{codim}(U_2)$.*

Proof. Suppose that the subspaces U_1 and U_2 have a finite codimension in V and consider the linear mapping

$$f : \begin{cases} V \to V/U_1 \times V/U_2 \\ f(v) = (v + U_1, v + U_2) \end{cases}.$$

By construction we have $\ker(f) = U_1 \cap U_2$, so by the fundamental theorem on homomorphisms there exists an isomorphism between the spaces $V/(U_1 \cap U_2)$ and $\operatorname{ran}(f)$. In particular, it follows that

$$\dim(V/(U_1 \cap U_2)) \leq \dim(V/U_1 \times V/U_2) = \dim(V/U_1) + \dim(V/U_2),$$

which is the stated inequality. $\square$

2.2 Sesquilinear forms

When studying the self-adjointness and spectral properties of an unbounded operator, it is often useful to consider the corresponding form and resort to the well-established theory of sesquilinear forms. Hence, this subsection is devoted to collecting some of the well-known results for sesquilinear forms, like the first and second representation theorem.

In the following let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and let $\operatorname{dom}(\mathfrak{t}) \subset \mathcal{H}$ be a linear subspace. Recall that a linear mapping $\mathfrak{t} : \operatorname{dom}(\mathfrak{t}) \times \operatorname{dom}(\mathfrak{t}) \to \mathbb{C}$ is called *sesquilinear form*, if for all $u, v, w \in \operatorname{dom}(\mathfrak{t})$ and $\lambda \in \mathbb{C}$ there holds

(i) $\mathfrak{t}[u + \lambda v, w] = \mathfrak{t}[u, w] + \lambda \mathfrak{t}[u, w]$,

(ii) $\mathfrak{t}[u, v + \lambda w] = \mathfrak{t}[u, v] + \overline{\lambda} \mathfrak{t}[u, w]$.

We say that $\mathfrak{t}$ is *densely defined*, if $\operatorname{dom}(\mathfrak{t})$ is dense in $\mathcal{H}$. If $\mathfrak{t}$ additionally satisfies

(iii) $\mathfrak{t}[u, v] = \overline{\mathfrak{t}[v, u]}$,

we call $\mathfrak{t}$ a *symmetric* form. Instead of $\mathfrak{t}[u, u]$ we will write $\mathfrak{t}[u]$ and call $\mathfrak{t}[\cdot]$ the quadratic form associated to $\mathfrak{t}$. Recall that a form $\mathfrak{t}$ is called *semibounded from below*, if there exists a constant $c \in \mathbb{R}$ such that for all $u \in \operatorname{dom}(\mathfrak{t})$ there holds

$$\mathfrak{t}[u] \geq c\|u\|_{\mathcal{H}}^2.$$

For such forms we can introduce the inner product

$$(u, v)_{\mathfrak{t}} := \mathfrak{t}[u, v] + (1 - c)(u, v)$$

for $u, v \in \operatorname{dom}(\mathfrak{t})$ and induced norm $\|\cdot\|_{\mathfrak{t}} = (\cdot, \cdot)_{\mathfrak{t}}^{1/2}$, which makes $(\operatorname{dom}(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}})$ a pre-Hilbert space. The form $\mathfrak{t}$ is called *closed*, if $(\operatorname{dom}(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}})$ is a complete space. The next statement, which is also sometimes known as the KLMN theorem, is a criterion that allows us to tell when the perturbation of a closed and semibounded from below form retains that property.

Theorem 2.3 ([21, Chapter VI, Theorem 1.33]). *Let* $\mathfrak{t} : \operatorname{dom}(\mathfrak{t}) \times \operatorname{dom}(\mathfrak{t}) \to \mathbb{C}$ *be a closed and semibounded from below form. Suppose that* $\mathfrak{t}' : \operatorname{dom}(\mathfrak{t}') \times \operatorname{dom}(\mathfrak{t}') \to \mathbb{C}$ *is a form with* $\operatorname{dom}(\mathfrak{t}) \subset \operatorname{dom}(\mathfrak{t}')$ *that satisfies*

$$|t'[u]| \leq a\|u\|^2 + b\,\mathfrak{t}[u]$$

for all $u \in \operatorname{dom}(\mathfrak{t})$*, where* a, b *are non-negative constants with* $b < 1$*. Then* $\mathfrak{t} + \mathfrak{t}'$ *is closed and semibounded from below as well.*

The next result, which is also commonly known as the first representation theorem for sesquilinear forms, consists of the fact that each densely defined, closed and semibounded from below form in $\mathcal{H}$ induces a self-adjoint operator in $\mathcal{H}$.

Theorem 2.4 ([21, Chapter VI, Theorem 2.1]). *Let* $\mathfrak{t}$ *be a densely defined, closed and semibounded from below sesquillinear form in* $\mathcal{H}$*. Then there exists a unique self-adjoint operator* T *in* $\mathcal{H}$ *with* $\operatorname{dom}(T) \subset \operatorname{dom}(\mathfrak{t})$ *such that*

$$\mathfrak{t}[u, v] = (Tu, v)$$

for all $u \in \operatorname{dom}(T)$ *and* $v \in \operatorname{dom}(\mathfrak{t})$*. Moreover, if for* $u \in \operatorname{dom}(t)$ *there exists a* $w \in \mathcal{H}$ *such that* $\mathfrak{t}[u, v] = (w, v)$ *for all* $v \in \operatorname{dom}(t)$*, then* $u \in \operatorname{dom}(T)$ *and* $Tu = w$*.*

We will conclude this subsection with the second representation theorem, which allows us to characterize a densely defined, non-negative and closed form via the square root of its associated non-negative and self-adjoint operator.

Theorem 2.5 ([21, Chapter VI, Theorem 2.23]). *Let* $\mathfrak{t} : \operatorname{dom}(\mathfrak{t}) \times \operatorname{dom}(\mathfrak{t}) \to \mathbb{C}$ *be a densely defined, closed and symmetric form with* $\mathfrak{t} \geq 0$*, and let* T *be the associated non-negative self-adjoint operator. Then* $\operatorname{dom}(T^{1/2}) = \operatorname{dom}(\mathfrak{t})$ *and there holds*

$$\mathfrak{t}[u, v] = (T^{1/2}u, T^{1/2}v)$$

for all $u, v \in \operatorname{dom}(\mathfrak{t})$*.*

2.3 Lipschitz domains and curves in $\mathbb{R}^2$

In the following subsection we will introduce the notion of a $\mathcal{C}^{k,\mu}$ domain Ω following the lines of the chapter about Lipschitz domains in [27]. For the sake of completeness, we provide a comprehensive definition of $\mathcal{C}^{k,\mu}$ domains where k can be an arbitrary non-negative integer and $\mu \in (0,1]$, even though we will generally only work with $\mathcal{C}^{1,1}$ domains. In this case the boundary of Ω can be locally described as the graph of a function with a Lipschitz continuous derivative. After that we will define curve integrals and tubular coordinates in $\mathbb{R}^2$, which we are also going to need. With this we will be able to construct Sobolev spaces on boundaries in Section 2.8.

Definition 2.6. Let $k \in \mathbb{N}_0$ and assume $\zeta \in \mathcal{C}^k(\mathbb{R};\mathbb{R})$. We then call the set

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 < \zeta(x_1)\}$$

a $\mathcal{C}^k$ hypograph. By adding the condition that the k-th order derivative of ζ is bounded and Hölder continuous with exponent $\mu \in (0,1]$, i.e.

$$|\zeta^k(t) - \zeta^k(s)| \leq M\, |t-s|^{\mu}$$

for all $t, s \in \mathbb{R}$ and some $M > 0$, we define a $\mathcal{C}^{k,\mu}$ hypograph. Note that the boundary of the Lipschitz hypograph Ω is then given by

$$\Sigma = \{x_\Sigma = (x_1, x_2) \in \mathbb{R}^2 : x_2 = \zeta(x_1)\}.$$

In the case where $k = 0$ and $\mu = 1$ we simply call Ω a Lipschitz hypograph. Assuming that ζ is Lipschitz continuous it follows by Rademacher's theorem, that ζ is differentiable almost everywhere with $\|\zeta'\|_\infty \leq M$. In particular, the Hausdorff measure σ and the outward unit normal vector ν are given by

$$\mathrm{d}\sigma(x_\Sigma) = \sqrt{1 + |\zeta'(x_1)|^2}\,\mathrm{d}t, \quad \nu(x_\Sigma) = \frac{(-\zeta'(x_1), 1)^\top}{\sqrt{1 + |\zeta'(x_1)|^2}} \tag{2.1}$$

for $x_\Sigma = (x_1, x_2) \in \Sigma$.

We can now extend the above definition to general domains, using a covering of Ω with appropriate $\mathcal{C}^{k,\mu}$ hypographs.

Definition 2.7. An open set $\Omega \subset \mathbb{R}^2$ with boundary Σ is called a $\mathcal{C}^{k,\mu}$ domain if its boundary Σ is compact and if there exist finite families $\{W_j\}$ and $\{\Omega_j\}$ of sets in $\mathbb{R}^2$ that satisfy the following properties:

(i) The family $\{W_j\}$ forms an open covering of Σ, i.e. $\Sigma \subset \bigcup_j W_j$.

(ii) For each j there exists a map $\kappa_j : \mathbb{R}^2 \to \mathbb{R}^2$ consisting of a rotation plus a translation, such that $\kappa_j(\Omega_j)$ is a $\mathcal{C}^{k,\mu}$ hypograph.

(iii) The set Ω satisfies $W_j \cap \Omega = W_j \cap \Omega_j$ for each j.

Visually speaking, the boundary of a $\mathcal{C}^{k,\mu}$ domain can be locally described as the graph of a $\mathcal{C}^{k,\mu}$ function, after possibly applying a rigid motion to it. In this thesis we are mainly going to focus on $\mathcal{C}^{1,1}$ domains, for which many important smoothness and regularity results regarding Sobolev spaces hold. It is also important to note that in the two-dimensional case each connected part of the boundary of a $\mathcal{C}^{k,\mu}$ can be parametrized by a curve $\gamma \in \mathcal{C}^{k,\mu}(I;\mathbb{R}^2)$. With this in mind we will now introduce curve integrals in $\mathbb{R}^2$.

Definition 2.8. Let $\Gamma \subset \mathbb{R}^2$ be a curve that is given by a continuous parametrization $\gamma : I \to \mathbb{R}^2$. We call Γ a *simple curve*, if γ is an injective function. We say that Γ is a $\mathcal{C}^{k,\mu}$ curve if $\gamma \in \mathcal{C}^{k,\mu}(I;\mathbb{R}^2)$. In the case where $k = 0$ and $\mu = 1$ we simply call Γ a Lipschitz curve. Assuming that γ is at least Lipschitz continuous we can define the curve integral of a function $f \in L^1(\Gamma)$ over Γ as

$$\int_\Gamma f(x_\Gamma)\,\mathrm{d}\sigma(x_\Gamma) := \int_I f(\gamma(t))\,|\dot\gamma(t)|\,\mathrm{d}t. \tag{2.2}$$

If $|\dot\gamma(t)| = 1$ for all $t \in I$ we say that γ is a *natural parametrization* of Γ, which we are going to assume in most cases.

Also note that the measure σ in (2.2) coincides with the canonical Hausdorff measure of Γ, which moreover is independent of the specific choice of the parametrization of Γ. In this thesis we will also make use of tubular coordinates in a small neighborhood around a given curve, which are defined in the following lemma.

Lemma 2.9 ([3, Equation (B.4)]). *Let Σ be a simple and closed $\mathcal{C}^{1,1}$ curve of finite length given in natural parametrization $\gamma = (\gamma_1,\gamma_2) : I \to \mathbb{R}^2$, i.e. $|\dot\gamma(t)| = 1$ for all $t \in I$. Denote by $\nu = (\dot\gamma_2, -\dot\gamma_1)$ the normal vector of Σ and let $\kappa = \dot\gamma_2\ddot\gamma_1 - \dot\gamma_1\ddot\gamma_2$ be the signed curvature of Σ. For $\delta > 0$ sufficiently small consider the open tubular neighborhood $\Sigma_\delta = \{x \in \mathbb{R}^2 : \mathrm{dist}(x,\Sigma) < \delta\}$. Then there holds*

$$\int_{\Sigma_\delta} f(x)\,\mathrm{d}x = \int_\Sigma \int_{-\delta}^{\delta} f(x_\Sigma + t\nu(x_\Gamma))(1 - t\kappa(x_\Sigma))\,\mathrm{d}t\,\mathrm{d}\sigma(x_\Sigma)$$

for all $f \in L^1(\Sigma_\delta)$.

2.4 The Fock space and related estimates

In this subsection we are going to introduce curve integrals in $\mathbb{C}$ and use Cauchy's integral formula to provide useful estimates for analytic functions along curves. We identify $\mathbb{R}^2$ and $\mathbb{C}$ in the standard way by letting $z = x_1 + ix_2$ for $(x_1,x_2) \in \mathbb{R}^2$. The derivatives in x_1 and x_2 will be denoted by $\partial_k = \partial_{x_k}$ and we set $\partial = (\partial_1 - i\partial_2)/2$ as well as $\overline{\partial} = (\partial_1 + i\partial_2)/2$ for the Wirtinger derivatives. We denote by $\mathrm{d}m(z)$ the Lebesgue measure in $\mathbb{C}$.

Definition 2.10. Let $\Gamma \subset \mathbb{C}$ be a curve in the complex plane with Lipschitz continuous parametrization $\zeta : I \to \mathbb{C}$. For $f \in L^2(\mathbb{C})$ we define the curve integral of f over Γ by

$$\int_\Gamma f(\zeta)\,\mathrm{d}\zeta := \int_I f(\zeta(t))\dot\zeta(t)\,\mathrm{d}t.$$

We also introduce the real-valued measure $\mathrm{d}|\zeta| := |\dot{\zeta}(t)|\,\mathrm{d}t$, which corresponds to the Hausdorff measure of the curve Γ from (2.2) in real-valued coordinates.

In this thesis we will often view $\Gamma \subset \mathbb{R}^2 \simeq \mathbb{C}$ as a complex curve, in order to apply Cauchy's integral formula. In the next remark we will establish a connection between the real- and complex-valued representation of a curve and the corresponding curve integral.

Remark 2.11. Let $\Gamma \subset \mathbb{R}^2$ be a Lipschitz curve that is given be the parametrization $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$. By identifying $\mathbb{R}^2$ and $\mathbb{C}$ in the standard way we see that $\zeta(t) = \gamma_1(t) + i\gamma_2(t)$ is a complex parametrization of Γ in $\mathbb{C}$. Since $|\dot{\gamma}(t)| = |\dot{\zeta}(t)|$ for all $t \in I$ it follows that

$$\int_\Gamma f(x_\Gamma)\,\mathrm{d}\sigma(x_\Gamma) = \int_I f(\gamma(t))|\dot{\gamma}(t)|\,\mathrm{d}t = \int_I f(\zeta(t))|\dot{\zeta}(t)|\,\mathrm{d}t = \int_\Gamma f(\zeta)\,\mathrm{d}|\zeta|$$

for all $f \in L^1(\Gamma)$, where $\mathrm{d}|\zeta|$ is given as in Definition 2.10. In particular, $\mathrm{d}|\zeta|$ coincides with the Hausdorff measure $\mathrm{d}\sigma$ of Γ seen in complex coordinates. Hence, we will sometimes write $\mathrm{d}\sigma(z)$ instead of $\mathrm{d}|\zeta|$, when we view Γ as a curve in the complex plane.

In the following lemma we apply the tubular coordinates introduced in Lemma 2.9 in combination with Cauchy's integral formula, in order to establish an upper bound for the supremum of an analytic function on a given $\mathcal{C}^{1,1}$ curve Γ. A similar result can be found in the proof of Proposition 4.1(ii) in [31].

Lemma 2.12. *Let $\Sigma \subset \mathbb{R}^2$ be a simple and closed $\mathcal{C}^{1,1}$ smooth curve of finite length and let $\Gamma \subset \Sigma$ be a closed subarc with $|\Gamma| > 0$. For $\delta > 0$ consider the open tubular neighbourhood $\Gamma_\delta := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < \delta\}$. Then for $\delta > 0$ sufficiently small and any $k \in \mathbb{N}_0$ there exists a constant $c = c(k, \delta, \Gamma) > 0$ such that*

$$\sup_{z\in\Gamma} |\partial^k f(z)|^2 \le c\delta^{-2k-3} \int_{\Gamma_\delta} |f(x)|^2\,\mathrm{d}x$$

for all analytic $f : \mathbb{C} \to \mathbb{C}$.

Proof. Let $\gamma = \gamma_1 + i\gamma_2 : [0, s] \to \mathbb{C}$ be a natural parametrization of Σ in $\mathbb{C}$, that is $|\dot{\gamma}(t)| = 1$ on $[0, s]$. Denote by $\nu = \dot{\gamma}_2 - i\dot{\gamma}_1$ the normal vector of Σ. In the proof we will distinguish between the two cases where Γ is a proper subarc of Σ, i.e. $|\Sigma\backslash\Gamma| > 0$ and the case where Γ and Σ coincide.

Case 1: Γ is a proper subarc of Σ

We are going to construct a closed curve around Γ by cutting off a tubular neighborhood of an appropriate extension $\Gamma(r)$ of Γ. The result will then follow after an application of Cauchy's integral formula. Suppose that $\Gamma = \gamma([a, b])$ for some $0 < a < b < s$ and let $r \in (0, \delta]$. Consider the extended curve $\Gamma(r) = \gamma([a - r, b + r])$ and define the family of curves

$$\begin{aligned}
\gamma_r^1(t) &= \gamma(t) + r\nu(\gamma(t)), && t \in I_1 = [a - r, b + r] \\
\gamma_r^2(t) &= \gamma(b + r) + t\nu(\gamma(b + r)), && t \in I_2 = [r, -r] \\
\gamma_r^3(t) &= \gamma(t) - r\nu(\gamma(t)), && t \in I_3 = [b + r, a - r] \\
\gamma_r^2(t) &= \gamma(a - r) + t\nu(\gamma(a - r)), && t \in I_4 = [-r, r],
\end{aligned} \tag{2.3}$$

which together give a complex parametrization of the boundary of the tubular neighborhood $T_r = \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Gamma(r),\ \ t \in (-r, r)\}$ of $\Gamma(r)$. Since by assumption the derivatives of γ are Lipschitz continuous it follows that we can choose δ sufficiently small such that $|\dot\gamma_r^l(t)| < \frac{3}{2}$ for $t \in I_l$ and $l \in \{1, 2, 3, 4\}$ if $0 < r \leq \delta$. Let now $\gamma_r = \sum_{l=1}^4 \gamma_r^l$ be the formal sum of the above curves. Let $z \in \Gamma$, by Cauchy's integral formula it follows that

$$\partial^k f(z) = \frac{k!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} \,\mathrm{d}\zeta,$$

so we get the estimate

$$\sup_{z\in\Gamma} |\partial^k f(z)|^2 \leq c_1 r^{-2k-2} \int_{\gamma_r} |f(\zeta)|^2 \,\mathrm{d}|\zeta|$$

for some $c_1 = c_1(\Gamma, k) > 0$ and any $r \in (0, \delta]$, provided that $\delta > 0$ is sufficiently small. Multiplying both sides by r^{2k+2} and integrating over r from 0 to δ gives us

$$\begin{aligned} \sup_{z\in\Gamma} |\partial^k f(z)|^2 &\leq c_2 \delta^{-2k-3} \int_0^\delta \int_{\gamma_r} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r \\ &= c_2 \delta^{-2k-3} \sum_{l=1}^4 \int_0^\delta \int_{\gamma_r^l} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r. \end{aligned} \tag{2.4}$$

Let us first consider the integral over the curve γ_r^1. Using the parametrization in (2.3) with $|\dot\gamma_r^1(t)| \leq \frac{3}{2}$ for $t \in I_1$ we obtain

$$\begin{aligned} \int_0^\delta \int_{\gamma_r^1} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r &\leq \frac{3}{2} \int_0^\delta \int_{a-r}^{b+r} |f(\gamma(t) + r\nu(\gamma(t)))|^2 \,\mathrm{d}t \,\mathrm{d}r \\ &\leq \frac{3}{2} \int_{-\delta}^\delta \int_{a-\delta}^{b+\delta} |f(\gamma(t) + r\nu(\gamma(t)))|^2 \,\mathrm{d}t \,\mathrm{d}r. \\ &= \frac{3}{2} \int_{-\delta}^\delta \int_{\Gamma(\delta)} |f(x_\Sigma + r\nu(x_\Sigma))|^2 \,\mathrm{d}\sigma(x_\Sigma) \,\mathrm{d}r. \end{aligned}$$

On the other hand, using the tubular coordinates from Lemma 2.9 on the neighbourhood $T_\delta = \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Gamma(\delta),\ \ t \in (-\delta, \delta)\}$ we get

$$\begin{aligned} \int_{T_\delta} |f(x)|^2 \,\mathrm{d}x &= \int_{-\delta}^\delta \int_{\Gamma(\delta)} |f(x_\Sigma + r\nu(x_\Sigma))|^2 (1 - r\kappa(x_\Sigma)) \,\mathrm{d}\sigma(x_\Sigma) \,\mathrm{d}r \\ &\geq \frac{1}{2} \int_{-\delta}^\delta \int_{\Gamma(\delta)} |f(x_\Sigma + r\nu(x_\Sigma))|^2 \,\mathrm{d}\sigma(x_\Sigma) \,\mathrm{d}r \end{aligned} \tag{2.5}$$

for $\delta > 0$ sufficiently small, since the curvature κ of Γ is bounded. This shows that

$$\int_0^\delta \int_{\gamma_r^1} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r \leq c_3 \int_{T_\delta} |f(x)|^2 \,\mathrm{d}x \tag{2.6}$$

for some constant $c_3 > 0$. In the same way one can show the above inequality for the curve γ_r^3, so let us continue with the inequality for γ_r^2. Using the parametrization from (2.3) we obtain

$$\int_0^\delta \int_{\gamma_r^2} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r \leq \frac{3}{2} \int_0^\delta \int_{-r}^r |f(\gamma(b+r) + t\nu(\gamma(b+r)))|^2 \,\mathrm{d}t \,\mathrm{d}r.$$

Note that we can rewrite

$$\{(r,t) \in \mathbb{R}^2 : 0 < r < \delta, \quad -r < t < r\} = \{(r,t) \in \mathbb{R}^2 : -\delta < t < \delta, \quad |t| < r < \delta\},$$

so Fubini's theorem and the substitution $u = b + r$ give us

$$\begin{aligned}
\int_0^\delta \int_{-r}^r |f(\gamma(b+r) + t\nu(\gamma(b+r)))|^2 \,\mathrm{d}t \,\mathrm{d}r &= \int_{-\delta}^\delta \int_{|t|}^\delta |f(\gamma(b+r) + t\nu(\gamma(b+r)))|^2 \,\mathrm{d}r \,\mathrm{d}t \\
&= \int_{-\delta}^\delta \int_{b+|t|}^{b+\delta} |f(\gamma(u) + t\nu(\gamma(u)))|^2 \,\mathrm{d}u \,\mathrm{d}t \\
&\leq \int_{-\delta}^\delta \int_{a-\delta}^{b+\delta} |f(\gamma(u) + t\nu(\gamma(u)))|^2 \,\mathrm{d}u \,\mathrm{d}t \\
&= \int_{-\delta}^\delta \int_{\Gamma(\delta)} |f(x_\Sigma + r\nu(x_\Sigma))|^2 \,\mathrm{d}\sigma(x_\Sigma) \,\mathrm{d}r.
\end{aligned}$$

Together with (2.5) this now implies that

$$\int_0^\delta \int_{\gamma_r^2} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r \leq c_4 \int_{T_\delta} |f(x)|^2 \,\mathrm{d}x$$

for some constant $c_4 > 0$. In an analogous way the same result can be shown for the curve γ_r^4, so we get

$$\int_0^\delta \int_{\gamma_r^l} |f(\zeta)|^2 \,\mathrm{d}|\zeta| \,\mathrm{d}r \leq c_5 \int_{T_\delta} |f(x)|^2 \,\mathrm{d}x$$

for some constant $c_5 > 0$ and arbitrary $l \in \{1, 2, 3, 4\}$. This in conjunction with (2.4) shows that

$$\sup_{z \in \Gamma} |\partial^k f(z)|^2 \leq c_6 \delta^{-2k-3} \int_{T_\delta} |f(x)|^2 \,\mathrm{d}x$$

for all $\delta > 0$ sufficiently small and $c_6 > 0$, which shows the inequality for the tubular neighborhood. The claimed inequality now follows if we can show $T_{\delta/4} \subset \Gamma_\delta$ for all $\delta > 0$ sufficiently small, since then

$$\sup_{z \in \Gamma} |\partial^k f(z)|^2 \leq c_6 \left(\frac{\delta}{4}\right)^{-2k-3} \int_{T_{\delta/4}} |f(x)|^2 \,\mathrm{d}x \leq c\delta^{-2k-3} \int_{\Gamma_\delta} |f(x)|^2 \,\mathrm{d}x$$

for an appropriate $c > 0$. To see this let $x \in T_{\delta/4}$, then $x = x_\Gamma + r\nu(x_\Gamma)$ for some $x_\Gamma \in \Gamma(\frac{\delta}{4})$ and $r \in (-\frac{\delta}{4}, \frac{\delta}{4})$. Moreover, since $|\dot\gamma(t)| = 1$ for all $t \in [0, s]$ there holds

$$|\gamma(t_1) - \gamma(t_2)| \leq 2|t_1 - t_2|$$

for all $t_1, t_2 \in [0, s]$ and hence $\operatorname{dist}(x_\Gamma, \Gamma) < \frac{\delta}{2}$. In particular, we have

$$\operatorname{dist}(x, \Gamma) \leq \|x - x_\Gamma\| + \operatorname{dist}(x_\Gamma, \Gamma) = |r| + \operatorname{dist}(x_\Gamma, \Gamma) < \delta,$$

which shows that $x \in \Gamma_\delta$, concluding the proof for the case where Γ is a proper subarb of Σ.

Case 2: $\Gamma = \Sigma$
To prove the case where Γ and Σ coincide we set $\gamma_r(t) = \gamma(t) + r\nu(\gamma(t))$ for $t \in [0, s]$, which is a closed curve surrounding every point $x_\Sigma \in \Sigma$. By Cauchy's integral formula we obtain

$$\sup_{z\in\Gamma} |\partial^k f(\zeta)|^2 \leq c_1 r^{-2k-2} \int_{\gamma_r} |f(\zeta)|^2 \, \mathrm{d}|\zeta|$$

for some constant $c_1 > 0$. Integrating both sides of the inequality from 0 to δ with the weight r^{2k+2} we can conclude that

$$\sup_{z\in\Gamma} |\partial^k f(\zeta)|^2 \leq c_2 \delta^{-2k-3} \int_0^\delta \int_{\gamma_r} |f(\zeta)|^2 \, \mathrm{d}|\zeta| \, \mathrm{d}r$$

for an appropriate $c_2 > 0$. Since $|\dot\gamma_r(t)| < \frac{3}{2}$ for $r \in (0, \delta)$ and $\delta > 0$ sufficiently small there also holds

$$\begin{aligned}
\int_0^\delta \int_{\gamma_r} |f(\zeta)|^2 \, \mathrm{d}|\zeta| \, \mathrm{d}r &\leq \frac{3}{2} \int_0^\delta \int_0^s |f(\gamma(t) + r\nu(\gamma(t)))|^2 \, \mathrm{d}|\zeta| \, \mathrm{d}r \\
&= \frac{3}{2} \int_0^\delta \int_\Sigma |f(x_\Sigma + r\nu(x_\Sigma))|^2 \, \mathrm{d}\sigma(x_\Sigma) \, \mathrm{d}r,
\end{aligned}$$

so the statement follows from (2.5) by taking $\Gamma(\delta) = \Sigma$ and the same arguments we used in the first case. $\square$

Next we are going to introduce the Fock or Segal-Bargmann space $\mathcal{F}^2$, which plays an important role in the analysis of the spectrum of the Landau Hamiltonian.

Definition 2.13. Let $B > 0$. The *Fock space* $\mathcal{F}^2$ is defined as the space of all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}} |f(z)|^2 \, e^{-\frac{1}{2}B|z|^2} \, \mathrm{d}m(z) < \infty.$$

If we introduce the inner product

$$(f,g)_{\mathcal{F}^2} = \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\frac{1}{2}B|z|^2}\,\mathrm{d}m(z), \quad f,g \in \mathcal{F}^2,$$

then the Fock-space $\mathcal{F}^2$ becomes a Hilbert space with induced norm $\|\cdot\|_{\mathcal{F}^2} = (\cdot,\cdot)_{\mathcal{F}^2}^{1/2}$.

The analysis of the eigenfunctions of the Landau Hamiltonian shows that the eigenspaces of the Landau Hamiltonian are closely related to the Fock space. More precisely, for each Landau Level $\Lambda_q, q \in \mathbb{N}_0$, there exists an isometric isomorphism from the corresponding eigenspace $\ker(\mathsf{A}_0 - \Lambda_q)$ onto the Fock space, a fact which will be discussed in more detail in Propositon 4.2.

Lemma 2.14. *Let $f \in \mathcal{F}^2$. Then for any $k \in \mathbb{N}_0$ and $R > 0$ there exists a constant $C = C(R,k) > 0$ such that*

$$\sup_{|z|\le R} \left|\partial^k f(z)\right|^2 \le C(R,k) \int_{|z|\ge R} |f(z)|^2 \, e^{-\frac{1}{2}B|z|^2}\,\mathrm{d}m(z). \tag{2.7}$$

In particular,

$$|||f|||_{\mathcal{F}^2}^2 = \int_{|z|\ge R} |f(z)|^2 \, e^{-\frac{1}{2}B|z|^2}\,\mathrm{d}m(z) \tag{2.8}$$

defines a norm on $\mathcal{F}^2$ that is equivalent to $\|\cdot\|_{\mathcal{F}^2}$.

Proof. For ease of notation we will show the proof for $B = 2$, the general result follows by a linear change of coordinates. Let $f \in \mathcal{F}^2$ and $r > 2R$, then by the Cauchy integral formula there holds

$$\partial^k f(z) = \int_{|\zeta|=r} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,\mathrm{d}\zeta$$

for any $k \in \mathbb{N}_0$ if $|z| < r$. Moreover, for $|\zeta| = r$ and $|z| \le R$ we have

$$|\zeta - z|^{k+1} \ge (|\zeta| - |z|)^{k+1} \ge \left(\frac{r}{2}\right)^{k+1},$$

which then implies

$$\begin{aligned}
\sup_{|z|\le R} \left|\partial^k f(z)\right|^2 &\le \left(\int_{|\zeta|=r} \left|\frac{f(\zeta)}{(\zeta - z)^{k+1}}\right| \mathrm{d}|\zeta|\right)^2 \\
&\le \left(\frac{2}{r}\right)^{2k+2} \left(\int_{|\zeta|=r} |f(\zeta)|\,\mathrm{d}|\zeta|\right)^2 \\
&\le 2\pi r \left(\frac{2}{r}\right)^{2k+2} \int_{|\zeta|=r} |f(\zeta)|^2\,\mathrm{d}|\zeta| \\
&= \frac{2^{2k+3}\pi}{r^{2k+1}} \int_{|\zeta|=r} |f(\zeta)|^2\,\mathrm{d}|\zeta|.
\end{aligned}$$

Integrating the last inequality from $2R$ to ∞ with the weight $e^{-r^2} r^{2k+1}$ and using

$$\int_{2R}^{\infty} e^{-r^2} r^{2k+1}\,\mathrm{d}r \geq R^{2k} \int_{2R}^{\infty} e^{-r^2} r dr = \frac{1}{2} R^{2k} e^{-4R^2}$$

we get

$$\begin{aligned}\sup_{|z|\leq R} \left|\partial^k f(z)\right|^2 &\leq \frac{4^{k+2}\pi}{R^{2k}} e^{4R^2} \int_{2R}^{\infty} \int_{|\zeta|=r} |f(\zeta)|^2 e^{-r^2}\,\mathrm{d}|\zeta|\,\mathrm{d}r \\ &\leq C(R,k) \int_{|z|\geq R} |f(z)|^2 e^{-|z|^2}\,\mathrm{d}m(z),\end{aligned}$$

where $C(R,k) = \frac{4^{k+2}\pi}{R^{2k}} e^{4R^2}$, which shows (2.7). To see the equivalence of the norms observe first that

$$\begin{aligned}\|f\|_{\mathcal{F}^2}^2 &= \int_{|z|\leq R} |f(z)|^2 e^{-|z|^2}\,\mathrm{d}m(z) + \int_{|z|\geq R} |f(z)|^2 e^{-|z|^2}\,\mathrm{d}m(z) \\ &\leq c \int_{|z|\geq R} |f(z)|^2 e^{-|z|^2}\,\mathrm{d}m(z) + \int_{|z|\geq R} |f(z)|^2 e^{-|z|^2}\,\mathrm{d}m(z) = (1+c) |||f|||_{\mathcal{F}^2}^2\end{aligned}$$

for some $c > 0$, where we have used (2.7) for $k = 0$. On the other hand it is clear that $\|\cdot\|_{\mathcal{F}^2} \geq |||\cdot|||_{\mathcal{F}^2}$, which finishes the proof. □

2.5 Orthonormal Polynomials and Capacity

In this subsection we will follow the lines of [38] to collect results regarding the logarithmic capacity of a compact set and also discuss the n-th root asymptotics of polynomials, which are orthogonal with respect to some finite Borel measure μ. For this we are going to assume that $\mu \geq 0$ is a compactly supported and finite Borel measure on $\mathbb{C}$. We are going to see, that there exists a close relation between the n-th root asymptotics of the leading coefficients of orthonormal polynomials with respect to the measure μ and the logarithmic capacity of the support of the measure. By identifying $\mathbb{C}$ and $\mathbb{R}^2$ in the standard way by letting $z = x_1 + ix_2$ for $(x_1, x_2) \in \mathbb{R}^2$ we see that each Borel measure μ on $\mathbb{C}$ corresponds to a Borel measure $\mu_{\mathbb{R}}$ on $\mathbb{R}^2$ and vice-verca; in particular, all of the following definitions and results remain true, if the measure μ on $\mathbb{C}$ is replaced by the corresponding measure $\mu_{\mathbb{R}}$ on $\mathbb{R}^2$ and vice-verca. With this in mind we are going to drop the difference in notation and write μ for both measures, as there is no danger of confusion.

To begin, we introduce the logarithmic capacity of a compact set K. For more references on the logarithmic capacity see [38, Appendix A. VIII], [17, §III.1] or [24, Chapter 2].

Definition 2.15. For a measure $\mu \geq 0$ in $\mathbb{R}^2$ we define its *logarithmic energy* as

$$I(\mu) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

The *logarithmic capacity* of a compact set $K \subset \mathbb{R}^2$ is defined as

$$\mathrm{Cap}\,(K) := \sup\{e^{-I(\mu)} : \mu \geq 0 \text{ measure on } \mathbb{R}^2,\ \operatorname{supp}\mu \subset K,\ \mu(K) = 1\}.$$

It is a well-known fact that the above supremum is in fact a maximum, which is attained by the so-called *equilibrium measure*, see e.g. [17, §III.4 Theorem 4.1]. In the following lemma we will state two useful properties of the logarithmic capacity, which are often referred to as *monotonicity* and *continuity* of the capacity, respectively.

Proposition 2.16 ([17, §III]). *Let $K, L \subset \mathbb{R}^2$ be compact sets and consider for $\eta > 0$ the compact neighbourhood $U_\eta(K) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, K) \leq \eta\}$. Then the logarithmic capacity satisfies*

(i) $\mathrm{Cap}\,(K) \leq \mathrm{Cap}\,(L)$ *if* $K \subset L$

(ii) $\mathrm{Cap}\,(U_\eta(K)) \to \mathrm{Cap}\,(K)$ *as* $\eta \to 0^+$.

Assuming that the support of μ consists of infinitely many points, we can apply the Gram–Schmidt orthogonalisation process in $L^2(\mathbb{C}; \mathrm{d}\mu)$ to the sequence $\{z^n\}_{n=0}^\infty$ to form the uniquely existing orthonormal polynomials

$$q_n(\mu; z) = \gamma_n(\mu) z^n + \dots,$$

with $\gamma_n(\mu) > 0$ and $n \in \mathbb{N}_0$ that satisfy the orthogonality relation

$$\int_{\mathbb{C}} q_n(\mu; z)\overline{q_m(\mu; z)}\,\mathrm{d}\mu(z) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}. \tag{2.9}$$

We call $\gamma_n(\mu)$ the leading coefficient of the polynomial $q_n(\mu; z)$ and say that $q_n(\mu; z)$ is a *monic* polynomial if $\gamma_n(\mu) = 1$. In the next lemma we make a statement about the distribution of the roots of the above defined polynomials.

Lemma 2.17 ([38, Corollary 1.1.7]). *For $n \in \mathbb{N}_0$ let $\{q_n(\mu; z)\}$ be the sequence of orthonormal polynomials satisfying* (2.9). *Then for each $n \in \mathbb{N}_0$ all zeros of $q_n(\mu; z)$ are contained in the convex hull of* $\operatorname{supp}\mu$.

There are many results known regarding the n-th root asymptotic behavior of the leading coeffiecients $\gamma_n(\mu)$ of the orthonormal polynomials $q_n(\mu; z)$. We are going to make use of the following known fact.

Theorem 2.18 ([38, Theorem 4.2.1]). *Let $\mu \geq 0$ be a finite Borel measure on $\mathbb{C}$ with compact support. For $z \in \mathbb{C}$ and $r > 0$ denote by $D_r(z) = \{z' \in \mathbb{C} : |z' - z| \leq r\}$ the compact disk centered around z with radius r. Then the condition*

$$\mathrm{Cap}\left(\left\{z \in \mathbb{C} : \limsup_{r \to 0+} \frac{\log \mu(D_r(z))}{\log r} < \infty\right\}\right) = \mathrm{Cap}\,(\operatorname{supp}\mu) \tag{2.10}$$

implies $\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\mathrm{Cap}(\operatorname{supp}\mu)}$.

In the next step we want to apply the above theorem to study the n-th root asymptotics of monic polynomials that are orthogonal with respect to the Hausdorff measure σ of a simple Lipschitz smooth curve Γ. But before we can do so, we need further preparations. In order to apply the above results to polynomials on curves, we will use the identification

$$G \mapsto \mu_\sigma(G) := \int_{\Gamma\cap G} \mathrm{d}\sigma(z), \quad G \subset \mathbb{C}, \tag{2.11}$$

which takes the Hausdorff measure σ of the curve Γ to induce a finite Borel measure on $\mathbb{C}$ with $\operatorname{supp}\mu_\sigma = \Gamma$.

Definition 2.19. For any $n \in \mathbb{N}_0$ let $\mathcal{P}_n$ be the set of all monic polynomials in z of degree n:

$$\mathcal{P}_n := \{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 : a_0, \ldots, a_{n-1} \in \mathbb{C}\}.$$

Assume that Γ is a simple Lipschitz curve of finite length with Hausdorff measure σ and corresponding Borel measure μ_σ. Consider the minimization problem

$$M_n(\Gamma) := \inf_{p\in\mathcal{P}_n} \int_\Gamma |p(z)|^2 \,\mathrm{d}\sigma(z) = \inf_{p\in\mathcal{P}_n} \int_{\mathbb{C}} |p(z)|^2 \,\mathrm{d}\mu_\sigma(z). \tag{2.12}$$

It is possible to construct unique solutions to the above minimization problem by applying the Gram-Schmidt orthogonalisation process in $L^2(\mathbb{C};\mathrm{d}\mu)$ to the sequence $\{z^n\}_{n=0}^\infty$, which in turn yields polynomials $q_n(\mu_\sigma;z)$ that satisfy the orthogonality relation (2.9). Setting $p_n(z) = \gamma_n(\mu_\sigma)^{-1} q_n(\mu_\sigma;z)$, where $\gamma_n(\mu_\sigma)$ is the leading coefficient of $q_n(\mu_\sigma;z)$, yields the minimal polynomial in (2.12).

In the next proposition we cover the n-th root asymptotics of the leading coefficients $\gamma_n(\mu_\sigma)$, by showing that the Borel measure associated to σ satisfies (2.10). The aim of the next result is to establish a connection between the asymptotic behaviour of $M_n(\Gamma)$ and the capacity of the curve Γ. A similar result can be found in [15, Remark 1], where a non-negative Borel measure that is induced by a function $v \in L^1(\mathbb{C})$ is considered.

Proposition 2.20. *Let Γ be a simple and finite Lipschitz curve with Hausdorff measure σ. Let μ_σ be the corresponding Borel measure given by* (2.11)*. For $z \in \mathbb{C}$ and $r > 0$ denote by $D_r(z) = \{z' \in \mathbb{C} : |z' - z| \le r\}$ the compact disk around z with radius r. Then there holds*

$$\left\{ z \in \mathbb{C} : \limsup_{r\to 0+} \frac{\log \mu_\sigma(D_r(z))}{\log r} < \infty \right\} = \operatorname{supp}\mu_\sigma. \tag{2.13}$$

Moreover there holds the n-th root asymptotics

$$\lim_{n\to\infty} M_n(\Gamma)^{1/n} = (\operatorname{Cap}(\Gamma))^2,$$

where $M_n(\Gamma)$ is defined as in (2.12).

Proof. We will first show that (2.13) holds true. Let μ_σ be as in (2.11) and recall that $\operatorname{supp}\mu_\sigma = \Gamma$. Let $r > 0$ and assume first that $z \in \mathbb{C}\backslash\Gamma$. Then $d = \operatorname{dist}(z, \Gamma) > 0$ since Γ is a compact curve. In particular, for any $0 < r < d$ there holds $\Gamma \cap D_r(z) = \emptyset$ so we get

$$\mu_\sigma(D_r(z)) = \int_{\Gamma \cap D_r} \mathrm{d}\sigma(z) = 0,$$

which directly imples that

$$\limsup_{r \to 0+} \frac{\log \mu_\sigma(D_r(z))}{\log r} = \infty.$$

Assume conversely that $z \in \Gamma$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a natural parametrization of Γ and assume that $z = \gamma(t_0)$ for some $t_0 \in (\alpha, \beta)$. Since $|\dot{\gamma}(t)| = 1$ for all $t \in (\alpha, \beta)$ there holds

$$|\gamma(t) - \gamma(t_0)| \leq 2\,|t - t_0|$$

for all $t \in [\alpha, \beta]$, implying $\{t \in I : |t - t_0| \leq r/2\} \subset \{t \in I : |\gamma(t) - \gamma(t_0)| \leq r\}$. For $r > 0$ sufficiently small we then find

$$\mu_\sigma(D_r(z)) = \int_{\zeta \in \Gamma : |z - \zeta| \leq r} \mathrm{d}\sigma(z) = \int_{t \in I : |\gamma(t_0) - \gamma(t)| \leq r} \mathrm{d}t \geq \int_{t \in I : |t - t_0| \leq r/2} \mathrm{d}t = r.$$

From this it follows that

$$\frac{\log \mu_\sigma(D_r(z))}{\log r} \leq 1,$$

provided that $0 < r < 1$ is small enough. This means that

$$\limsup_{r \to 0+} \frac{\log \mu_\sigma(D_r(z))}{\log r} < \infty,$$

showing (2.13). In particular, we can apply Theorem 2.18 to obtain

$$\lim_{n \to \infty} \gamma_n(\mu_\sigma)^{1/n} = (\operatorname{Cap}(\Gamma))^{-1},$$

where $\gamma_n(\mu)$ are the leading coefficients of the polynomials $q_n(\mu_\sigma; z)$ that satisfy the orthogonality relation (2.9). Denote by $p_n(z) = \gamma_n(\mu_\sigma)^{-1} q_n(\mu_\sigma; z)$ the monic polynomials, which minimize (2.12), we then have

$$\begin{aligned} M_n(\Gamma)^{1/n} &= \left(\int_\Gamma |p_n(z)|^2 \, \mathrm{d}\sigma(z) \right)^{1/n} \\ &= \frac{1}{\gamma_n(\mu_\sigma)^{2/n}} \Bigg(\underbrace{\int_\Gamma |q_n(\mu_\sigma; z)|^2 \, \mathrm{d}\sigma(z)}_{1} \Bigg)^{1/n} \to \operatorname{Cap}(\Gamma)^2, \end{aligned}$$

as $n \to \infty$, which is the claimed result. $\square$

To conclude this subsection we prove the following lemma, which we are going to need in the proof of Proposition 5.7.

Lemma 2.21. *Let $\rho > 0$ and suppose that $w \in \mathbb{C}$ with $|w| > \rho$. Then the rational function*

$$r(z) = |w| \frac{z - \rho^2/\overline{w}}{\rho(z-w)}, \quad z \in \mathbb{C} \backslash \{w\},$$

satisfies $|r(z)| \leq 1$ as $|z| \leq r$ and $|r(z)| \geq 1$ as $|z| \geq r$.

Proof. Let $z \in \mathbb{C}\backslash\{w\}$ and rewrite

$$r(z) = |w| \frac{z - \rho^2/\overline{w}}{\rho(z-w)} = \frac{1}{\rho\,|w|} \frac{|w|^2 z - \rho^2 w}{z - w}.$$

A direct calculation now shows

$$\begin{aligned}
|r(z)|^2 = r(z)\overline{r(z)} &= \frac{1}{\rho^2\,|w|^2} \cdot \frac{|w|^2 z - \rho^2 w}{z-w} \cdot \frac{|w|^2\,\overline{z} - \rho^2\overline{w}}{\overline{z} - \overline{w}} \\
&= \frac{1}{\rho^2\,|w|^2} \cdot \frac{|w|^4\,|z|^2 - 2\rho^2\,|w|^2\,\mathfrak{Re}(z\overline{w}) + \rho^4\,|w|^2}{|z|^2 - 2\mathfrak{Re}(z\overline{w}) + |w|^2} \\
&= \frac{|w|^2\,|z|^2/\rho^2 - 2\mathfrak{Re}(z\overline{w}) + \rho^2}{|z|^2 - 2\mathfrak{Re}(z\overline{w}) + |w|^2} =: \frac{p(z)}{q(z)},
\end{aligned}$$

where p and q are the non-negative numerator and denominator of r, respectively. Suppose now that $|z| \leq \rho$, then

$$\begin{aligned}
p(z) - q(z) &= |w|^2 \frac{|z|^2}{\rho^2} + \rho^2 - |z|^2 - |w|^2 \\
&= \left(\frac{|w|^2}{\rho^2} - 1\right) |z|^2 + \rho^2 - |w|^2 \\
&\leq |w|^2 - \rho^2 + \rho^2 - |w|^2 = 0.
\end{aligned}$$

On the other hand, if $|z| \geq \rho$ it follows that

$$\begin{aligned}
p(z) - q(z) &= |w|^2 \frac{|z|^2}{\rho^2} + \rho^2 - |z|^2 - |w|^2 \\
&= \left(\frac{|z|^2}{\rho^2} - 1\right) |w|^2 + \rho^2 - |z|^2 \\
&\geq |z|^2 - \rho^2 + \rho^2 - |z|^2 = 0,
\end{aligned}$$

which shows $r(z) \leq 1$ for $|z| \leq \rho$ and $r(z) \geq 1$ for $|z| \geq \rho$. $\square$

2.6 Spectral theory

In this subsection we will cover basic results from the spectral theory of closed and, in particular, self-adjoint operators. We will start by introducing the resolvent set and spectrum of a closed operator and then state an appropriate version of the spectral theorem for unbounded self-adjoint operators, as it will be used in this master's thesis. After that we will collect some well-known results regarding the spectrum of a compactly perturbed self-adjoint operator.

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let $C : \mathrm{dom}(C) \to \mathcal{H}$ be a closed operator. Recall that a $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(C)$ of C if and only if $(C-\lambda)^{-1}$ is a bounded and everywhere defined operator. The spectrum $\sigma(C)$ of C is then defined as $\sigma(C) = \mathbb{C}\backslash\rho(C)$. Furthermore, we define

$$\begin{aligned}
\sigma_p(C) &= \{\lambda \in \mathbb{C} : \ker(C-\lambda) \neq \{0\}\},\\
\sigma_c(C) &= \{\lambda \in \mathbb{C} : \ker(C-\lambda) = \{0\},\ \overline{\mathrm{ran}(C-\lambda)} = \mathcal{H},\ \mathrm{ran}(C-\lambda) \neq \mathcal{H}\},\\
\sigma_r(C) &= \{\lambda \in \mathbb{C} : \ker(C-\lambda) = \{0\},\ \overline{\mathrm{ran}(C-\lambda)} \neq \mathcal{H}\}.
\end{aligned}$$

Recall that for a self-adjoint operator C we have $\sigma(C) \subset \mathbb{R}$ and $\sigma_r(C) = \emptyset$. If C is self-adjoint, we further define the discrete spectrum $\sigma_d(C)$ of C by

$$\sigma_d(C) = \{\lambda \in \sigma_p(C) : \dim\ker(C-\lambda) < \infty,\ \lambda \text{ is isolated in } \sigma(C)\}$$

and the essential spectrum of C by $\sigma_{ess}(C) = \sigma(C)\backslash\sigma_d(C)$. The first lemma of this subsection is a basic result from elementary calculus, which turns out to be quite useful, when studying the spectral asymptotics of a compact and self-adjoint operator.

Lemma 2.22 ([31, Equation (10)]). *Let $\{b_n\}_n$ be a sequence of positive and non-increasing numbers such that $\limsup_{n\to\infty}[n!b_n]^{1/n} < \infty$. Then there holds*

$$\limsup_{n\to\infty}[n!b_{n+l}]^{1/n} = \limsup_{n\to\infty}[n!b_n]^{1/n}$$

as well as

$$\liminf_{n\to\infty}[n!b_{n+l}]^{1/n} = \liminf_{n\to\infty}[n!b_n]^{1/n}$$

for all $l \in \mathbb{Z}$.

In the following, let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra on $\mathbb{R}$ and let $\mathfrak{B}(\mathcal{H})$ be the space of all bounded and everywhere defined operators in $\mathcal{H}$. Recall that we call a mapping $E : \Sigma \to \mathfrak{B}(\mathcal{H})$, $B \mapsto E_B$ a *spectral measure*, if E_B is an orthogonal projection for each $B \in \Sigma$ and

(i) $E_\emptyset = 0$ (zero operator), $\quad E_\mathbb{R} = 1$ (identity operator)

(ii) For any family of pairwise disjoints sets $\{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ and all $x \in \mathcal{H}$ there holds $\sum_{j=1}^\infty E_{B_j}x = E_{\bigcup_{j=1}^\infty B_j}x$

We now come to the spectral theorem for unbounded self-adjoint operators, which is a deep result from abstract functional analysis, that ensures that any self-adjoint operator in $\mathcal{H}$ has a unique associated spectral measure, that induces the operator via a spectral integral.

Theorem 2.23 ([7, Chapter 6, Theorem 1 and (13)]). *Let $C : \operatorname{dom}(C) \to \mathcal{H}$ be a self-adjoint operator. Then there exists a unique spectral measure E on $\mathcal{H}$ defined on the σ-algebra of Borel subsets of $\mathbb{R}$ such that*

$$C = \int_{\mathbb{R}} t \, \mathrm{d}E(t), \quad \operatorname{dom}(C) = \left\{ f \in \mathcal{H} : \int_{\mathbb{R}} t^2 \, \mathrm{d}(E(t)f, f) < \infty \right\}.$$

Moreover, if $h : \mathbb{R} \to \mathbb{R}$ is a measurable function, then

$$h(C) := \int_{\mathbb{R}} h(t) \, \mathrm{d}E(t), \quad \operatorname{dom}(h(C)) = \left\{ f \in \mathcal{H} : \int_{\mathbb{R}} h(t)^2 \, \mathrm{d}(E(t)f, f) < \infty \right\},$$

defines a self-adjoint operator in $\mathcal{H}$.

For two given self-adjoint operators C and D it is in general not possible to give a clear description of $\sigma(C + D)$, even if their respective spectra are known. However, if we additionally require the two operators to act in closed orthogonal subspaces, one can give an exact characterization of the spectrum of their sum.

Proposition 2.24. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}_1, \mathcal{H}_2$ be closed orthogonal subspaces such that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ as a orthogonal sum. Let C and D be two closed operators that act in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $\sigma(C + D) = \sigma(C) \cup \sigma(D)$.*

Proof. Let us denote by P and Q the orthogonal projections from $\mathcal{H}$ into $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then any $u \in \mathcal{H}$ admits the unique representation $u = Pu + Qu$. So for an arbitrary $u \in \operatorname{dom}(C + D)$ the effect of the operator $C + D$, which is defined on the orthogonal sum $\operatorname{dom}(C) + \operatorname{dom}(D)$, is given by

$$(C + D)u = CPu + DQu.$$

In particular, for any $\lambda \in \mathbb{C}$ we get

$$(C + D - \lambda)u = CPu + DQu - \lambda u = (C - \lambda)Pu + (D - \lambda)Qu.$$

Since $C - \lambda$ and $D - \lambda$ act in orthogonal subspaces the operator $C + D - \lambda$ is boundedly invertible in $\mathcal{H}$ if and only if $C - \lambda$ and $D - \lambda$ are invertible in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. This shows $\rho(C+D) = \rho(C) \cap \rho(D)$, which after taking the complement yields the desired result. □

In the following lemma we will make use of the spectral theorem for self-adjoint compact operators, to decompose a non-negative compact operator $K : \mathcal{H} \to \mathcal{H}$ as the sum of two operators, where one operator has finite rank and the other one an arbitrarily small norm.

Lemma 2.25. *Let K be a self-adjoint, compact and non-negative operator in $\mathcal{H}$. Then for each $\delta > 0$ there exists a decomposition $K = K_1 + K_2$ with operators K_1, K_2 such that $0 \leq K_1 \leq \delta I$ and $\operatorname{rank}(K_2) < \infty$.*

Proof. Without loss of generality let $\operatorname{rank}(K) = \infty$. By the spectral theorem for compact self-adjoint operators there exists a monotonously decreasing sequence of non-negative eigenvalues $\{\lambda_n\}_n$ with $\lim_{n\to\infty} \lambda_n = 0$ and corresponding orthonormal eigenvectors $\{u_n\}_n$ such that

$$Ku = \sum_{n=1}^{\infty} \lambda_n (u, u_n) u_n$$

for $u \in \mathcal{H}$. Since $\lambda_n \to 0$ as $n \to \infty$ there exists an $N \in \mathbb{N}$ such that $\lambda_n \leq \delta$ for all $n \geq N$. Defining

$$K_1 u = \sum_{n=N+1}^{\infty} \lambda_n (u, u_n) u_n, \quad K_2 u = \sum_{n=1}^{N} \lambda_n (u, u_n) u_n,$$

for $u \in \mathcal{H}$ gives us the desired decomposition of K. □

Next we are going to use the properties of the spectral measure of a self-adjoint operator, to define a function which, roughly speaking, counts the eigenvalues of an operator with their respective multiplicities in an interval.

Definition 2.26. Let C be a self-adjoint operator in the Hilbert space $\mathcal{H}$ and let E be the associated spectral measure of C. For an arbitrary set I we define the *eigenvalue counting function* by

$$\pi_I(C) = \dim E_I \mathcal{H}.$$

With Lemma 2.25 in mind it makes sense to study the behaviour of $\pi_I(C + D)$, where C and D are two self-adjoint operators, for the two cases, where D has either finite rank or a bounded spectrum. The following two abstract results further justify this consideration.

Lemma 2.27 ([7], §9.3, Theorem 3). *Let C and D be self-adjoint operators in $\mathcal{H}$. For some $\lambda \in \rho(C) \cap \rho(D)$ let the difference of resolvents*

$$R = (C - \lambda)^{-1} - (D - \lambda)^{-1} \tag{2.14}$$

be of finite rank $r \in \mathbb{N}$. Suppose that the spectrum of C is finite in the bounded interval I. Then the spectrum of D is finite in I and there holds the inequality

$$\pi_I(C) - r \leq \pi_I(D) \leq \pi_I(C) + r. \tag{2.15}$$

If the rank of the difference $C - D$ is finite, it follows by the second resolvent identity that $\operatorname{rank}(R) = \operatorname{rank}(C - D)$. In particular, we can apply Lemma 2.27 and get (2.15) with $r = \operatorname{rank}(C - D)$. On the other hand, if $\operatorname{rank}(C - D) = \infty$, but D still has a bounded spectrum, we can make the following statement.

Lemma 2.28 ([7], §9.4, Theorem 3). *Let C and D be two self-adjoint operators in $\mathcal{H}$. Furthermore, assume that $\sigma(D) \subset [d_1, d_2]$ for some $d_1, d_2 \in \mathbb{R}$. Then for any finite interval $I = (\alpha, \beta)$ there holds*

$$\pi_{(\alpha+d_1,\beta+d_2)}(C + D) \geq \pi_{(\alpha,\beta)}(C).$$

Visually speaking, since $\sigma(D) \subset [-\|D\|, \|D\|]$ for a bounded and self-adjoint operator D, the above lemma consists of the fact that the spectrum of C inside the intervall I cannot disappear under a perturbation by D, but may only be displaced to the left or to the right by at most $\|D\|$.

2.7 Compact perturbations of self-adjoint operators

In this subsection we cover some important results on the behaviour of the spectrum of a self-adjoint operator under compact perturbations. We are interested in the situation, where T is a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\Lambda \in \mathbb{R}$ is an isolated eigenvalue of T of infinite multiplicity with corresponding orthogonal projection P_Λ. Since Λ is isolated in $\sigma(T)$ we can choose constants $\tau_\pm > 0$ such that

$$((\Lambda - 2\tau_-, \Lambda + 2\tau_+)\backslash\{\Lambda\}) \cap \sigma(T) = \emptyset. \tag{2.16}$$

Next, consider a self-adjoint and compact opertator $W : \mathcal{H} \to \mathcal{H}$ with corresponding spectral measure E and set

$$W_+ = \int_0^\infty \lambda \, \mathrm{d}E(\lambda), \quad W_- = -\int_\infty^0 \lambda \, \mathrm{d}E(\lambda), \tag{2.17}$$

for the non-negative and non-positive part of W, respectively. By definition both W_+ and W_- are compact, self-adjoint and non-negative operators in $\mathcal{H}$ and there holds $W = W_+ - W_-$ as well as $|W| = W_+ + W_-$. In particular, the self-adjoint so-called *Toeplitz operators* $P_\Lambda W_\pm P_\Lambda \geq 0$ are compact with eigenvalues

$$\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \ldots \geq 0,$$

which we will order non-increasingly and counted with respective multiplicites. By Weyl's theorem we have $\sigma_{ess}(T+W) = \sigma_{ess}(T)$, so Λ is either an eigenvalue of infinite multiplicity or an accumulation point in the spectrum of $T + W$. We will denote the eigenvalues of $T + W$ in $(\Lambda - \tau_-, \Lambda + \tau_+)$ by

$$\lambda_1^- \leq \lambda_2^- \leq \cdots \leq \Lambda \leq \cdots \leq \lambda_2^+ \leq \lambda_1^+.$$

If there exist only finitely many $\lambda_k^+ > \Lambda$ we set $\lambda_k^+ = \Lambda$ for all larger $k \in \mathbb{N}$ and use the same convention for λ_k^-. In the case where either $W = W_+ \geq 0$ or $W = -W_- \leq 0$, one would expect that the eigenvalues of $T + W$ in $(\Lambda - \tau_-, \Lambda + \tau_+)$ can only accumulate to Λ from above or below, respectively. The next proposition will be a modified version of [7, §9.4 Theorem 7], in which we show this exact observation.

Proposition 2.29. *Let T and $W = W_+ - W_-$ be as above. Then the following holds:*

(i) *The eigenvalues of $T + W_+$ accumulate to Λ only from above.*

(ii) *The eigenvalues of $T - W_-$ accumulate to Λ only from below.*

Proof. We will only prove (i), as (ii) works in the exact same way. We will show that there exists an $\alpha < \Lambda$ such that $\pi_{(\alpha,\Lambda)}(T+W_+) = 0$, which directly implies the statement. Recall that we chose $\tau_- > 0$ such that $(\Lambda - \tau_-, \Lambda) \cap \sigma(T) = \emptyset$, which is possible since Λ is isolated in the spectrum of T. By Lemma 2.25 there exists a decomposition $W_+ = S + R$ where $\sigma(S) \subset [0, \frac{\tau_-}{2}]$ and $\operatorname{rank}(R) < \infty$. Set $D = T + S + R$ and $C = T + S$ and let $\lambda \in \mathbb{C}\backslash\mathbb{R}$. An application of the second resolvent formula shows that

$$\operatorname{rank}((C - \lambda)^{-1} - (D - \lambda)^{-1}) = \operatorname{rank}(R) < \infty,$$

so we can apply Lemma 2.27, which yields

$$\begin{aligned} \pi_{(\Lambda-\tau_-/2,\Lambda)}(T + S + R) &= \pi_{(\Lambda-\tau_-/2,\Lambda)}(D) \\ &\leq \pi_{(\Lambda-\tau_-/2,\Lambda)}(C) + \operatorname{rank}(R) \\ &= \pi_{(\Lambda-\tau_-/2,\Lambda)}(T + S) + \operatorname{rank}(R). \end{aligned} \tag{2.18}$$

Next, we set $C = T$ and $D = S$. Since $\sigma(-S) \subset [-\frac{\tau_-}{2}, 0]$ it follows by Lemma 2.28 that

$$\pi_{(\Lambda-\tau_-/2,\Lambda)}(T + S) \leq \pi_{(\Lambda-\tau_-,\Lambda)}(T) = 0,$$

so in combination with (2.18) we obtain

$$\pi_{(\Lambda-\tau_-/2,\Lambda)}(T + W_+) \leq \operatorname{rank}(R) < \infty.$$

In particular, there holds $\pi_{(\alpha,\Lambda)}(T + W_+) = 0$ for some $\alpha \in (\Lambda - \frac{\tau_-}{2}, \Lambda)$ sufficiently close to Λ, which finishes the proof. □

In the next proposition, which was first elaborated in [31, Proposition 2.2], we give an asymptotic estimate on the rate of accumulation of the eigenvalues of $T + W$ to Λ.

Proposition 2.30. *Let T and $W = W_+ - W_-$ be as above. Then there holds*

(i) *If $W_- = 0$ and $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) = \infty$, then for any $\epsilon > 0$ there exists $l \in \mathbb{N}$ such that for all sufficiently large k there holds*

$$(1 - \epsilon)\mu^+_{k+l} \leq \lambda^+_k - \Lambda \leq (1 + \epsilon)\mu^+_{k-l}.$$

Moreover, the eigenvalues of $T + W_+$ accumulate to Λ from above.

(ii) *If $W_+ = 0$ and $\operatorname{rank}(P_\Lambda W_- P_\Lambda) = \infty$, then for any $\epsilon > 0$ there exists $l \in \mathbb{N}$ such that for all sufficiently large k there holds*

$$(1 - \epsilon)\mu^-_{k+l} \leq \Lambda - \lambda^-_k \leq (1 + \epsilon)\mu^-_{k-l}.$$

Moreover, the eigenvalues of $T - W_-$ accumulate to Λ from below.

Remark 2.31. In the case where $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) < \infty$ or $\operatorname{rank}(P_\Lambda W_- P_\Lambda) < \infty$ in Proposition 2.30 we can still achieve the respective upper estimate

$$\lambda_k^+ - \Lambda \leq (1+\epsilon)\mu_{k-l}^+, \quad \Lambda - \lambda_k^- \leq (1+\epsilon)\mu_{k-l}^-$$

for appropriate $l \in \mathbb{N}$ and all sufficiently large k.

Proof of Proposition 2.30. We will prove (i), the proof for (ii) works analogously. We will first show that the upper bound holds without restricting ourselves to the case where $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) = \infty$, and then show the lower bound under this additional assumption.

Let $S = T + W_+$ and $Q_\Lambda = I - P_\Lambda$. We first define the operators

$$R_\pm = -\epsilon P_\Lambda W_+ P_\Lambda - \epsilon^{-1} Q_\Lambda W_+ Q_\Lambda \mp (P_\Lambda W_+ Q_\Lambda + Q_\Lambda W_+ P_\Lambda)$$

and

$$S_\pm = P_\Lambda(T + (1 \pm \epsilon)W_+)P_\Lambda + Q_\Lambda(T + (1 \pm \epsilon^{-1})W_+)Q_\Lambda.$$

A direct calculation shows $S = S_+ + R_- = S_- - R_+$. Since W_+ is compact and the projections P_Λ and Q_Λ are bounded and self-adjoint, it follows that $R_\pm$ is compact and self-adjoint as well. Another computation shows that $R_\pm$ admits the factorization

$$R_\pm = -(\sqrt{\epsilon}P_\Lambda \pm \epsilon^{-1/2}Q_\Lambda)W_+(\sqrt{\epsilon}P_\Lambda \pm \epsilon^{-1/2}Q_\Lambda).$$

Since the projections P_Λ and Q_Λ are bounded and self-adjoint we get for any $u \in \mathcal{H}$

$$\begin{aligned}(R_\pm u, u) &= -((\sqrt{\epsilon}P_\Lambda \pm \epsilon^{-1/2}Q_\Lambda)W_+(\sqrt{\epsilon}P_\Lambda \pm \epsilon^{-1/2}Q_\Lambda)u, u) \\ &= -(W_+(\sqrt{\epsilon}P_\Lambda \pm \epsilon^{-1/2}Q_\Lambda)u, (\sqrt{\epsilon}P_\Lambda \pm \epsilon^{-1/2}Q_\Lambda)u),\end{aligned}$$

which implies $R_\pm \leq 0$ since $W_+ \geq 0$ by assumption.

In the next step we will determine the spectrum of $S_\pm$ in $(\Lambda, \Lambda + \tau_+)$. We first consider the representation

$$P_\Lambda(T + (1 \pm \epsilon)W_+)P_\Lambda = \Lambda P_\Lambda + (1 \pm \epsilon)P_\Lambda W_+ P_\Lambda.$$

Since the operator $P_\Lambda W_+ P_\Lambda$ is compact it follows by Weyl's theorem that

$$\sigma_{ess}(\Lambda P_\Lambda + (1 \pm \epsilon)P_\Lambda W P_\Lambda) \cap (\Lambda, \Lambda + \tau_+) = \sigma_{ess}(\Lambda P_\Lambda) \cap (\Lambda, \Lambda + \tau_+) = \emptyset.$$

Hence the full spectrum of $S_\pm$ in $(\Lambda, \Lambda + \tau_+)$ is given by the eigenvalues $\Lambda + (1 \pm \epsilon)\mu_n^+$. It remains to discuss the spectrum of $Q_\Lambda(T + (1 \pm \epsilon^{-1})W_+)Q_\Lambda|_{\operatorname{ran}(Q_\Lambda)}$. By assumption

$$\sigma(T|_{\operatorname{ran}(Q_\Lambda)}) \cap (\Lambda - 2\tau_-, \Lambda + 2\tau_+) = \emptyset,$$

so Weyl's theorem implies

$$\sigma_{ess}(Q_\Lambda(T + (1 \pm \epsilon^{-1})W_+)Q_\Lambda|_{\operatorname{ran}(Q_\Lambda)}) = \sigma_{ess}(Q_\Lambda T Q_\Lambda).$$

In particular, the essential spectrum of the operator $Q_\Lambda(T+(1\pm\epsilon^{-1})W_+)Q_\Lambda|_{\mathrm{ran}(Q_\Lambda)}$ in $(\Lambda-2\tau_-,\Lambda+2\tau_+)$ must be empty. Hence $Q_\Lambda(T+(1\pm\epsilon^{-1})W_+)Q_\Lambda|_{\mathrm{ran}(Q_\Lambda)}$ can only have finitely many eigenvalues in $(\Lambda-\tau_-,\Lambda+\tau_+)$. Since $\mathrm{ran}(P_\Lambda)$ and $\mathrm{ran}(Q_\Lambda)$ form orthogonal subspaces it follows by Proposition 2.24 that $\sigma(S_\pm)$ is the union of the spectra of the discussed operators.

Let now denote by $\nu_1^\pm \geq \nu_2^\pm \geq ...$ the eigenvalues of $S_\pm$ in $(\Lambda, \Lambda+\tau_+)$. By the above considerations we can conclude that we can write

$$\nu_k^+ = \Lambda + (1+\epsilon)\mu_{k-i}^+, \quad \nu_k^- = \Lambda + (1-\epsilon)\mu_{k-j}^+, \tag{2.19}$$

for appropriate $i, j \in \mathbb{N}$ and all sufficiently large n. In particular, if $\mathrm{rank}(P_\Lambda W_+ P_\Lambda) < \infty$ we see that the spectrum of $S_\pm$ consists of only finitely many eigenvalues.

In the next step we will show that $\lambda_k^+ \leq \nu_{k-l}^+$ for all sufficiently large k and an appropriate integer $l \in \mathbb{N}$. Let $\delta = (\Lambda - \lambda_1^+ + \tau_+)/2$. By Lemma 2.25 there exist operators $R_-^{(1)}$ and $R_-^{(2)}$ such that $R_- = R_-^{(1)} + R_-^{(2)}$ with $-\delta I \leq R_-^{(1)} \leq 0$ and $\mathrm{rank}(R_-^{(2)}) = r_0 < \infty$. For $\lambda \in (\Lambda, \lambda_1^+)$ we have

$$\pi_{(\lambda,\Lambda+\tau_+)}(S_+) = \pi_{(\lambda,\lambda_1^++2\delta)}(S - R_-^{(1)} - R_-^{(2)}).$$

Since $-\delta I \leq R_-^{(1)} \leq 0$ it follows that $\sigma(-R_-^{(1)}) \subset [0,\delta]$, so we can apply Lemma 2.28 which yields

$$\pi_{(\lambda,\Lambda+\tau_+)}(S_+) = \pi_{(\lambda,\lambda_1^++2\delta)}(S - R_-^{(1)} - R_-^{(2)}) \geq \pi_{(\lambda,\lambda_1^++\delta)}(S - R_-^{(2)}). \tag{2.20}$$

Furthermore, if we set $C = S$ and $D = S - R_-^{(2)}$, we see that both operators are self-adjoint as S and $R_-^{(2)}$ are self-adjoint operators and $R_-^{(2)}$ is additionally bounded. So for any $\mu \in \mathbb{C}\backslash\mathbb{R}$ we have

$$\begin{aligned}(D-\mu)^{-1} - (C-\mu)^{-1} &= (D-\mu)^{-1}(C-D)(C-\mu)^{-1} \\ &= (D-\mu)^{-1}R_-^{(2)}(C-\mu)^{-1}.\end{aligned}$$

Clearly the operator on the right hand side is of rank lesser or equal to $\mathrm{rank}(R_-^{(2)})$. On the other hand, since the resolvents are bijective, the rank of the operator cannot diminish which implies

$$\mathrm{rank}((D-\mu)^{-1} - (C-\mu)^{-1}) = r_0. \tag{2.21}$$

Hence we can apply Lemma 2.27 and get

$$\begin{aligned}\pi_{(\lambda,\lambda_1^++\delta)}(S - R_-^{(2)}) &= \pi_{(\lambda,\lambda_1^++\delta)}(D) \\ &\geq \pi_{(\lambda,\lambda_1^++\delta)}(C) - r_0 \\ &= \pi_{(\lambda,\lambda_1^++\delta)}(S) - r_0 \\ &= \pi_{(\lambda,\Lambda+\tau_+)}(S) - r_0 - r_1,\end{aligned} \tag{2.22}$$

where we have defined

$$r_1 = \pi_{[\lambda_1^+ + \delta, \Lambda + \tau_+)}(S) < \infty.$$

Together the inequalities (2.20) and (2.22) imply that

$$\pi_{(\lambda, \Lambda + \tau_+)}(S_+) \geq \pi_{(\lambda, \Lambda + \tau_+)}(S) - r, \tag{2.23}$$

where $r = r_1 + r_2$. We will now show that the last inequality can be rewritten into the form $\lambda_k^+ \leq \nu_{k-r}^+$ all k sufficiently large. First assume that $N \geq r + 1$ and that $\lambda < \lambda_N$ to ensure that the right hand side in (2.23) is non-negative. Now (2.23) implies that there must be at least $k - r$ eigenvalues $\nu_1^+ \geq \nu_2^+ \geq \ldots \geq \nu_{k-r}^+$ in each interval $[\lambda_k, \infty)$ for $k > N$. In particular, this implies $\lambda_k^+ \leq \nu_{k-r}^+$ for all $k > N$, which combined with the representation of ν_k in (2.19) shows that

$$\lambda_k^+ \leq \Lambda + (1 + \epsilon)\mu_{k-l}^+ \tag{2.24}$$

for some $l \in \mathbb{N}$ and all sufficiently large k, proving the upper bound.

In the next step we want to show that $\lambda_k^+ \geq \nu_{k+l}^-$ for an appropriate integer l and all sufficiently large k under the additional assumption that $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) = \infty$. By Lemma 2.25 there exist operators $R_+^{(1)}$ and $R_+^{(2)}$ such that $R_+ = R_+^{(1)} + R_+^{(2)}$ with $-\delta I \leq R_+^{(1)} \leq 0$ and $\operatorname{rank}(R_+^{(2)}) = r_0 < \infty$. For $\lambda \in (\lambda_1, \Lambda)$ we get

$$\pi_{(\lambda, \Lambda + \tau_+)}(S) = \pi_{(\lambda, \lambda_1^+ + 2\delta)}(S) = \pi_{(\lambda, \lambda_1^+ + 2\delta)}(S_- - R_+^{(1)} - R_+^{(2)}).$$

We want to apply Lemma 2.27. For this let $C = S_- - R_+^{(1)}$ and $D = S_- - R_+^{(1)} - R_+^{(2)}$. Both operators C and D are self-adjoint as a finite sum of self-adjoint operators where $R_+^{(1)}$ and $R_+^{(2)}$ are bounded. For $\mu \in \mathbb{C} \backslash \mathbb{R}$ the second resolvent identity yields

$$\begin{aligned} (D - \mu)^{-1} - (C - \mu)^{-1} &= (D - \mu)^{-1}(C - D)(C - \mu)^{-1} \\ &= (D - \mu)^{-1} R_+^{(2)} (C - \mu)^{-1} \end{aligned}$$

and as in (2.21) we see that

$$\operatorname{rank}((D - \mu)^{-1} - (C - \mu)^{-1}) = r_0. \tag{2.25}$$

Hence we can apply Lemma 2.27 and get

$$\begin{aligned} \pi_{(\lambda, \lambda_1^+ + 2\delta)}(S_- - R_+^{(1)} - R_+^{(2)}) &= \pi_{(\lambda, \lambda_1^+ + 2\delta)}(D) \\ &\geq \pi_{(\lambda, \lambda_1^+ + 2\delta)}(C) - r_0 \\ &= \pi_{(\lambda, \lambda_1^+ + 2\delta)}(S_- - R_+^{(1)}) - r_0. \end{aligned}$$

Furthermore, since $-\delta I \le R_+^{(1)} \le 0$, we have $\sigma(-R_+^{(1)}) \subset [0,\delta]$ and hence by Lemma 2.28 we achieve the inequality

$$\pi_{(\lambda,\lambda_1^+ + 2\delta)}(S_- - R_+^{(1)}) \ge \pi_{(\lambda,\lambda_1^+ + \delta)}(S_-).$$

All together the above inequalities imply

$$\begin{aligned} \pi_{(\lambda,\Lambda+\tau_+)}(S) &\ge \pi_{(\lambda,\lambda_1^+ + \delta)}(S_-) - r_0 \\ &= \pi_{(\lambda,\Lambda+\tau_+)}(S_-) - r_0 - r_1 \end{aligned} \tag{2.26}$$

where we have defined

$$r_1 = \pi_{[\lambda_1^+ + \delta, \Lambda+\tau)}(S_-).$$

As in (2.23) one can inductively show that (2.26) can be equivalently rewritten into the form $\lambda_k^+ \ge \nu_{k+r}^-$ for $r = r_0 + r_1$ and all sufficiently large n. In contrast to (2.23) we require $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) = \infty$ here, as otherwise the right hand side in (2.26) were bounded, and the inductive argument would not work. Together with (2.19) this implies

$$\lambda_k^+ \ge \Lambda + (1-\epsilon)\mu_{k-l}^+$$

for some fixed $l \in \mathbb{N}$ and all sufficiently large k, which together with (2.24) shows that

$$(1-\epsilon)\mu_{k+l}^+ \le \lambda_k^+ - \Lambda \le (1+\epsilon)\mu_{k-l}^+$$

for $l \in \mathbb{N}$ and all sufficiently large k. Moreover it follows by Proposition 2.29 that the eigenvalues of $T + W_+$ inside $(\Lambda - \tau_-, \Lambda + \tau_+)$ can only accumulate towards Λ from above, which finishes the proof. $\square$

We will conclude this subsection with a result that can be seen as a complement of Proposition 2.30, where we can drop the definiteness assumption on W and still obtain one-sided estimates on $\Lambda - \lambda_k^-$ and $\lambda_k^+ - \Lambda$.

Proposition 2.32 ([3, Proposition 2.10]). *Let T and $W = W_+ - W_-$ as above. Then the following holds.*

(i) *For $\epsilon > 0$ there exists $l \in \mathbb{N}$ such that*

$$\lambda_k^+ - \Lambda \le (1+\epsilon)\mu_{k-l}^+$$

for all sufficiently large k.

(ii) *For $\epsilon > 0$ there exists $l \in \mathbb{N}$ such that*

$$\Lambda - \lambda_k^- \le (1+\epsilon)\mu_{k-l}^-$$

for all sufficiently large k.

Proof. We will prove (i), since the proof for (ii) works in a similar fashion. For this proof we will introduce the notation $S_U := T + U$ for a generic compact and self-adjoint perturbation U and denote the eigenvalues of S_U in the interval $[\Lambda, \Lambda + \tau_+)$ by

$$\lambda_1^+(S_U) \geq \lambda_2^+(S_U) \geq \lambda_3^+(S_U) \geq \cdots \geq \Lambda,$$

where the eigenvalues $\lambda_k^+(S_U)$ for $k \in \mathbb{N}$ are repeated according to their multiplicity. The idea of the proof is to use the decomposition

$$T + W = T + W_+ - W_-$$

and apply Proposition 2.30 to the operator $T + W_+$. So let $\epsilon > 0$. By the compactness of W_- we can apply Lemma 2.25 and find a decomposition $W_- = F_- + R_-$ where $\operatorname{rank}(F_-) = r_0 < \infty$ and $0 \leq R_- \leq \tau_+ I$. By Proposition 2.30 and Remark 2.31 there exists $l_0 \in \mathbb{N}$ such that

$$\lambda_k^+(S_{W_+}) = \lambda_k^+(T + W_+) \leq (1 + \epsilon)\mu_{k-l_0}^+ \tag{2.27}$$

for all sufficiently large k. Let now $\lambda \in (\Lambda, \Lambda + \tau)$. Since $\sigma(R_-) \subset [0, \tau_+]$ we can use Lemma 2.28 for $C = S_{W_+ - F_-} - R_-$ and $D = R_-$ to obtain

$$\begin{aligned} \pi_{(\lambda,\Lambda+\tau_+)}(S_W) &= \pi_{(\lambda,\Lambda+\tau_+)}(S_{W_+-F_-} - R_-) \\ &\leq \pi_{(\lambda,\Lambda+2\tau_+)}(S_{W_+-F_-}) \\ &= \pi_{(\lambda,\Lambda+\tau_+)}(S_{W_+-F_-}) + r_1 \end{aligned} \tag{2.28}$$

where we have defined

$$r_1 = \pi_{[\Lambda+\tau_+,\Lambda+2\tau_+)}(S_{W_+-F_-}) < \infty.$$

On the other hand setting $C = S_{W_+}$ and $D = S_{W_+-F_-}$ we see that for any $\mu \in \mathbb{C}\backslash\mathbb{R}$

$$\operatorname{rank}((D - \mu)^{-1} - (C - \mu)^{-1}) = \operatorname{rank}((D - \mu)^{-1} F_- (C - \mu)^{-1}) = r_0,$$

so we can apply Lemma 2.27 and get

$$\pi_{(\lambda,\Lambda+\tau_+)}(S_{W_+-F_-}) \leq \pi_{(\lambda,\Lambda+\tau_+)}(S_{W_+}) + r_0. \tag{2.29}$$

Combining (2.28) and (2.29) we can conclude

$$\pi_{(\lambda,\Lambda+\tau_+)}(S_W) \leq \pi_{(\lambda,\Lambda+\tau_+)}(S_{W_+}) + r_0 + r_1 \tag{2.30}$$

for $\lambda \in (\Lambda, \Lambda + \tau_+)$. As in the proof of Proposition 2.30 one can show that (2.30) can be rewritten into the form $\lambda_k^+(S_W) \leq \lambda_{k-r_0-r_1}^+(S_{W_+})$, which together with (2.27) implies

$$\lambda_k^+(S_W) \leq \lambda_{k-r_0-r_1}^+(S_{W_+}) \leq (1 + \epsilon)\mu_{k-l_0-r_0-r_1}^+,$$

for all k sufficiently large, which proves the statement. $\square$

2.8 Sobolev spaces

In this chapter we are going to collect elementary results about Sobolev spaces, which are a necessary tool in the analysis of the Landau Hamiltonian. To do so we are first going to introduce Sobolev spaces $H^k(\Omega)$ for general open sets $\Omega \subset \mathbb{R}^2$ and non-negative integers k. We will then proceed by extending this definition to general $s \geq 0$ using the Sobolev-Slobodeckij semi-norm. After that we are going to consider Sobolev spaces $H^s(\Sigma)$ on the boundary Σ of a Lipschitz domain Ω and collect results regarding the Dirichlet and Neumann trace of functions in Sobolev spaces. In this subsection we are following the lines of [27, Chapter 3].

To begin, let $\Omega \subseteq \mathbb{R}^2$ be open and $k \in \mathbb{N}_0$. We then introduce the spaces

$$\begin{aligned} \mathcal{C}^k(\Omega) &:= \{f : \Omega \to \mathbb{C} : f \text{ is k-times continuously differentiable}\}, \\ \mathcal{C}^k_0(\Omega) &:= \{f \in \mathcal{C}^k(\Omega) : \operatorname{supp} f \text{ is compact in } \Omega\}, \end{aligned}$$

and set as usual $\mathcal{C}^\infty(\Omega) = \bigcap_{k=0}^\infty \mathcal{C}^k(\Omega)$ and $\mathcal{C}^\infty_0(\Omega) = \bigcap_{k=0}^\infty \mathcal{C}^k_0(\Omega)$ for the space of smooth and the space of test functions, respectively. Additionally we define the space

$$\mathcal{C}^k_0(\overline{\Omega}) := \{f\restriction_\Omega : f \in \mathcal{C}^k_0(\mathbb{R}^2)\},$$

that contains those functions in $\mathcal{C}^k(\Omega)$, which have uniformly continuos partial derivatives up to k-th order on $\overline{\Omega}$ and set $\mathcal{C}^\infty_0(\overline{\Omega}) = \bigcap_{k=0}^\infty \mathcal{C}^k_0(\overline{\Omega})$.

Definition 2.33. For $\Omega \subseteq \mathbb{R}^2$ open we introduce the classical Sobolev-spaces

$$H^k(\Omega) := \{f \in L^2(\Omega) : \partial^{\boldsymbol{\alpha}} f \in L^2(\Omega) \text{ for } |\boldsymbol{\alpha}| \leq k, \ \boldsymbol{\alpha} \in \mathbb{N}_0^2\}$$

with the inner product

$$(f,g)_{H^k(\Omega)} := \sum_{\boldsymbol{\alpha} \leq |k|} (\partial^{\boldsymbol{\alpha}} f, \partial^{\boldsymbol{\alpha}} g)_{L^2(\Omega)}$$

and induced norm $\|\cdot\|_{H^k(\Omega)} = (\cdot,\cdot)^{1/2}_{H^k(\Omega)}$, which makes $(H^k(\Omega), \|\cdot\|_{H^k(\Omega)})$ a Hilbert space. As usual, we define $H^k_0(\Omega)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ with respect to $\|\cdot\|_{H^k(\Omega)}$ and recall that there holds $H^k(\mathbb{R}^2) = H^k_0(\mathbb{R}^2)$ for all $k \in \mathbb{N}_0$. In addition to that we define

$$H^1_\Delta(\Omega) := \{f \in H^1(\Omega) : \Delta f \in L^2(\Omega)\}.$$

with the inner product

$$(f,g)_{H^1_\Delta(\Omega)} := (f,g)_{H^1(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)}$$

and induced norm $\|\cdot\|_{H^1_\Delta(\Omega)} = (\cdot,\cdot)^{1/2}_{H^1_\Delta(\Omega)}$ for all $f, g \in H^1_\Delta(\Omega)$.

In the case where $\Omega = \mathbb{R}^2$ it turns out that the existence of Δf in $L^2(\mathbb{R}^2)$ already implies the existence of all the weak derivatives of second order, which we will formulate in the following lemma.

Lemma 2.34 ([4, Lemma 8.2.3]). *The mapping* $f \mapsto \|\Delta f\|_{L^2(\mathbb{R}^2)}$ *is a norm in* $H_0^2(\mathbb{R}^2)$, *which is equivalent to* $\|\cdot\|_{H^2(\mathbb{R}^2)}$. *In particular, there holds* $H^1_\Delta(\mathbb{R}^2) = H^2(\mathbb{R}^2)$.

Next we are going to define Sobolev spaces of fractional order $s \geq 0$, using the approach via the Sobolev-Slobodeckij semi-norm.

Definition 2.35. Let $\Omega \subset \mathbb{R}^2$ be open and $0 < \mu < 1$. For $f \in L^2(\Omega)$ the Sobolev-Slobodeckij semi-norm of f is defined as

$$|f|_{\mu,\Omega} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{2+2\mu}} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/2}.$$

For $s = k + \mu$, where $k \in \mathbb{N}_0$, we define

$$H^s(\Omega) = \{u \in H^k(\Omega) : |\partial^{\boldsymbol{\alpha}} f|_{\mu,\Omega} < \infty \text{ for all } |\boldsymbol{\alpha}| = k\},$$

and equip this space with the norm

$$\|f\|_{H^s(\Omega)} = \left(\|u\|^2_{H^k(\Omega)} + \sum_{|\boldsymbol{\alpha}|=k} |\partial^{\boldsymbol{\alpha}} f|^2_{\mu,\Omega} \right)^{1/2}.$$

Even though we are using the Sobolev-Slobodeckij norm as a means to introduce Sobolev spaces of fractional order, we are generally not interested in calculating the norm $|f|_{\mu,\mathbb{R}^2}$ for $f \in H^s(\mathbb{R}^2)$. Instead we are using the following result to give an esimate of this norm, which is also sometimes known as Sobolev's Lemma.

Lemma 2.36 ([40, Satz 11.18e]). *Let* $k \geq 1$. *Then for any* $0 \leq s < k$ *and arbitrary* $\epsilon > 0$ *there exists a constant* $c = c(\epsilon) > 0$ *such that*

$$\|f\|^2_{H^s(\mathbb{R}^2)} \leq \epsilon \|f\|^2_{H^k(\mathbb{R}^2)} + c(\epsilon)\|f\|^2_{L^2(\mathbb{R}^2)}$$

for all $f \in H^k(\mathbb{R}^2)$.

Even though we have introduced Sobolev spaces on arbitary open sets $\Omega \subset \mathbb{R}^2$, we are generally more interested in the case where the boundary of Ω is at least $\mathcal{C}^{0,1}$, or even better $\mathcal{C}^{1,1}$ smooth, in the sense of Definition 2.7. Assuming that Ω has a Lipschitz smooth boundary Σ, we are able to construct Sobolev spaces $H^s(\Sigma)$.

Recall that we defined a $\mathcal{C}^{k,1}$ hypograph as the set

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 < \zeta(x_1) \text{ for all } x_1 \in \mathbb{R}\},$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is a k-times differentiable function such that the k-th derivative $\zeta^{(k)}$ is Lipschitz continuous and thus differentiable almost everywhere by Rademacher's theorem. Assuming that Ω is a $\mathcal{C}^{k-1,1}$ hypograph for $k \geq 1$, we can construct Sobolev spaces on its boundary Σ as follows. For $f \in L^2(\Sigma)$ we define the function

$$f_\zeta(t) = f(t, \zeta(t)) \text{ for } t \in \mathbb{R},$$

and introduce the space

$$H^s(\Sigma) = \{f \in L^2(\Gamma) : f_\zeta \in H^s(\mathbb{R})\} \tag{2.31}$$

for $0 < s \le k$, which we are going to endow with the inner product

$$(f, g)_{H^s(\Sigma)} = (f_\zeta, g_\zeta)_{H^s(\mathbb{R})}.$$

In the case where $\kappa(\Omega)$ is a Lipschitz hypograph for a rigid motion $\kappa : \mathbb{R}^2 \to \mathbb{R}^2$, we define $H^s(\Sigma)$ in the same way except that $f_\zeta(t) = f(\kappa^{-1}(t, \zeta(t)))$. We can now generalize this definition for general $\mathcal{C}^{k-1,1}$ domains, via the means of a partition of unity.

Lemma 2.37 ([1, Theorem 3.15]). *Let K be a compact subset of $\mathbb{R}^2$ and let $\{\mathcal{O}_j\}$ be a family of open sets in $\mathbb{R}^2$ that form an open cover of K, i.e. $K \subset \bigcup_j \mathcal{O}_j$. Then there exists a family of functions $\{\psi_j\}$ in $\mathcal{C}_0^\infty(\mathbb{R}^2)$ that have the following properties:*

(i) *For each j and for all $x \in \mathbb{R}^2$ there holds $0 \le \psi_j(x) \le 1$.*

(ii) *For each j we have $\operatorname{supp} \psi_j \subset \mathcal{O}_j$.*

(iii) *For every $x \in K$ there holds $\sum_j \psi_j(x) = 1$.*

We call $\{\psi_j\}$ a partition of unity subordinate to $\{\mathcal{O}_j\}$.

Since the boundary of a $\mathcal{C}^{k-1,1}$ domain can be locally described by a $\mathcal{C}^{k-1,1}$ hypograph, we can now use a partition of unity to globalize the coordinates of the hypograph, which are only given in a neighbourhood of each point. In this way we can extend the definition of $H^s(\Sigma)$ to the case where Σ is the boundary of a general $\mathcal{C}^{k-1,1}$ domain.

Definition 2.38. Let Ω be a $\mathcal{C}^{k-1,1}$ domain with $k \ge 1$ and let $\{W_j\}$ and $\{\Omega_j\}$ be given as in Definition 2.7. Suppose that $\{\psi_j\}$ is a partition of unity subordinate to the open cover $\{W_j\}$ of the boundary Σ of Ω, i.e. $\psi_j \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\operatorname{supp} \Psi_j \subset W_j$ for each j and $\sum_j \psi_j(x) = 1$ for all $x \in \Sigma$. We then define the inner product

$$(f, g)_{H^s(\Sigma)} = \sum_j (\psi_j f, \psi_j g)_{H^s(\Sigma_j)},$$

for $f, g \in L^2(\Sigma)$, where $\Sigma_j = \partial\Omega_j$ and define the space

$$H^s(\Sigma) = \{f \in L^2(\Sigma) : \|f\|_{H^s(\Sigma)} < \infty\}$$

for $0 < s \le k$.

It is important to note that this definition, does not depend on the particular choice of the covering $\{W_j\}$ or the partition of unity. We can now define $H^{-s}(\Sigma)$ via duality $H^{-s}(\Sigma) = [H^s(\Sigma)]'$ and introduce the duality product

$$\langle f, g\rangle_{H^{-s}(\Sigma)\times H^s(\Sigma)} := f(g)$$

for $f \in H^s(\Sigma)$ and $g \in H^{-s}(\Sigma)$ if $|s| \le k$. Further references regarding this construction can be found in [27, Chapter 3].

There exists a wide field of results that are known for Sobolev spaces, which only hold true, if the domain Ω is assumed to be sufficiently smooth. For example one can show the existence of bounded Dirichlet and Neumann trace operators, if the boundary Σ is at least Lipschitz smooth.

Lemma 2.39 ([27, Theorem 3.37]). *Let $\Omega \subseteq \mathbb{R}^2$ be a $\mathcal{C}^{k-1,1}$ domain with boundary Σ. Then for $\frac{1}{2} < s \leq k$ there exists a bounded operator $\gamma_{\mathrm{D}} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Sigma)$ that satisfies $\gamma_{\mathrm{D}} f = f|_\Sigma$ for all $f \in \mathcal{C}^\infty(\overline{\Omega}) \cap H^s(\Omega)$. This operator has a continuous right inverse $\mathcal{E} : H^{s-\frac{1}{2}}(\Sigma) \to H^s(\Omega)$, i.e. $\gamma_{\mathrm{D}}\mathcal{E}f = f$ for all $f \in H^{s-\frac{1}{2}}(\Sigma)$.*

We can also generalize the conormal derivative $\partial_\nu f$ of a function $f \in H^1_\Delta(\Omega)$ via the bounded Neumann trace operator, assuming that the boundary Σ of Ω is at least Lipschitz regular. However, it is important to note that the Neumann trace of a function in $H^1_\Delta(\Omega)$ will generally only be an element of $H^{-1/2}(\Sigma)$.

Lemma 2.40 ([27, Lemma 4.3]). *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with Lipschitz boundary Σ and $f \in H^1(\Omega)$ such that $\Delta f \in L^2(\Omega)$. Then there exists a unique $g \in H^{-\frac{1}{2}}(\Sigma)$ such that*

$$(\nabla f, \nabla v)_{L^2(\Omega)} = (-\Delta f, v)_{L^2(\Omega)} + \langle g, \gamma_{\mathrm{D}} v\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}$$

for all $v \in H^1(\Omega)$. This g is uniquely determined by f and Δf and there holds the estimate

$$\|g\|_{H^{-1/2}(\Sigma)} \leq C\|f\|_{H^1(\Omega)} + C\|\Delta f\|_{L^2(\Omega)}.$$

It is a well-known fact that the space of smooth functions lies dense $L^2(\Omega)$ with respect to the norm $\|\cdot\|_{L^2(\Omega)}$. This is generally not true for the space $(H^k(\Omega), \|\cdot\|_{H^k(\Omega)})$, due to the ill-behaviour of the derivatives close to the boundary. The next result consists of the fact that for a Lipschitz domain Ω we can at least expect $\mathcal{C}_0^\infty(\overline{\Omega})$ to be dense in $H^k(\Omega)$ with respect to $\|\cdot\|_{H^k(\Omega)}$.

Lemma 2.41 ([27, Theorem 3.29]). *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with Lipschitz boundary. Then $\mathcal{C}_0^\infty(\overline{\Omega})$ lies dense in $H^s(\Omega)$ with respect to $\|\cdot\|_{H^s(\Omega)}$ for $s \geq 0$.*

We will conclude this subsection about Sobolev spaces with a suitable version of the well-known Rellich-Kondrachov embedding theorem, which we are going to use to show the compactness of the Toeplitz-type operators in Section 5.1.

Theorem 2.42 ([1, Theorem 6.3]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Then the embedding $H^k(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

2.9 Schatten-von Neumann ideals

In this subsection we are following the lines of [3, Chapter 2.2] to introduce Schatten-von Neumann ideals, which are used to characterize the rate at which the singular values of a compact operator tend to zero. After that we are going to state a useful result on the Schatten-von Neumann property of operators that map into Sobolev spaces $H^s(\Sigma)$ with $s > 0$, which is provided in Proposition 2.47.

In this subsection $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{K}$ are always assumed to be separable Hilbert spaces. Recall that we write $\mathcal{B}(\mathcal{H}, \mathcal{G})$ for the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$ and use the shortened notation $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. We will denote the space of all compact operators from $\mathcal{H}$ to $\mathcal{G}$ as $\mathfrak{S}_\infty(\mathcal{H}, \mathcal{G})$ and write $\mathfrak{S}_\infty(\mathcal{H}) := \mathfrak{S}_\infty(\mathcal{H}, \mathcal{H})$. Recall that for any compact operator $K \in \mathfrak{S}_\infty(\mathcal{H}, \mathcal{G})$ its singular values are defined as the eigenvalues of the self-adjoint non-negative operator $(K^*K)^{1/2} \in \mathfrak{S}_\infty(\mathcal{H})$. We will denote the singular values of K by $s_k(K)$ for $k \in \mathbb{N}$ and order them non-increasingly, with multiplicities taken into account. It is a well-known fact that $s_k(K) = s_k(K^*)$ for any compact operator $K \in \mathfrak{S}_\infty(\mathcal{H}, \mathcal{G})$.

Definition 2.43. For $p > 0$ we define the Schatten-von Neumann ideal of order p as the space

$$\mathfrak{S}_p(\mathcal{H}, \mathcal{G}) := \left\{ K \in \mathfrak{S}_\infty(\mathcal{H}, \mathcal{G}) : \sum_{k=1}^{\infty} s_k(K)^p < \infty \right\}.$$

In a similar way we introduce the weak Schatten-von Neumann ideal of order p, which is defined by

$$\mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G}) := \left\{ K \in \mathfrak{S}_\infty(\mathcal{H}, \mathcal{G}) : s_K(K) = \mathcal{O}(k^{-1/p}) \right\}.$$

We will now collect a few useful properties of the Schatten-von Neumann ideals, which we are going to need. As the name suggests the Schatten-von Neumann ideals form two-sided ideals, which we will formulate in the following lemma.

Lemma 2.44 ([3, Section 2.2]). *Let $C \in \mathcal{B}(\mathcal{H})$ and $D \in \mathcal{B}(\mathcal{G})$. Then the following holds.*

(i) *If $K \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$, then $DKC \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$.*

(ii) *If $K \in \mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G})$, then $DKC \in \mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G})$.*

Since the singular values of a compact operator K always tend to zero, it is easy to see that the spaces $\mathfrak{S}_p(\mathcal{H}, \mathcal{G})$ and $\mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G})$ grow in size, as p increases.

Lemma 2.45 ([3, Section 2.2]). *Let $0 < p < q$. Then the (weak) Schatten-von Neumann ideals are ordered in the following way.*

(i) *$\mathfrak{S}_p(\mathcal{H}, \mathcal{G}) \subset \mathfrak{S}_q(\mathcal{H}, \mathcal{G})$ and $\mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G}) \subset \mathfrak{S}_{q,\infty}(\mathcal{H}, \mathcal{G})$.*

(ii) *$\mathfrak{S}_p(\mathcal{H}, \mathcal{G}) \subset \mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G})$ and $\mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G}) \subset \mathfrak{S}_q(\mathcal{H}, \mathcal{G})$.*

The next lemma will be useful in Section 4.5, where we are going to study the resolvent difference between the unperturbed Landau Hamiltonian and its counterpart with a δ-potential.

Lemma 2.46 ([3, Section 2.2]). *Let $p, q > 0$ and let $r > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then for $K_1 \in \mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G})$ and $K_2 \in \mathfrak{S}_{q,\infty}(\mathcal{G}, \mathcal{K})$ the product of these operators satisfies*

$$K_2 K_1 \in \mathfrak{S}_{r,\infty}(\mathcal{H}, \mathcal{K}).$$

The next proposition consists of an abstract result on the Schatten-von Neumann property of operators that map into Sobolev spaces $H^s(\Sigma)$ with $s > 0$.

Proposition 2.47 ([3, Proposition 2.5]). *Let $k \in \mathbb{N}$ and suppose for $k \in \mathbb{N}$ that Σ is the boundary of a bounded $\mathcal{C}^{k,1}$ domain Ω. Let $C \in \mathcal{B}(\mathcal{H}, L^2(\Sigma))$ be a linear operator such that $ran(C) \subset H^{l/2}(\Sigma)$ for some $l \in \{1, \dots, 2k+1\}$. Then*

$$C \in \mathfrak{S}_{2/l,\infty}(\mathcal{H}, L^2(\Sigma)).$$

We conclude this subsection with a technical result, which allows us to characterize the total variation of the discrete spectrum of an operator under a trace class perturbation.

Proposition 2.48 ([3, Proposition 2.11]). *Let C and D be two bounded and self-adjoint operators in $\mathcal{H}$ such that $D - C \in \mathfrak{S}_1(\mathcal{H})$. Then there holds*

$$\sum_{\lambda \in \sigma_{disc}(C)} \operatorname{dist}(\lambda, \sigma(D)) < \infty.$$

3 Magnetic Sobolev Spaces

In this section we are going to introduce the magnetic Sobolev spaces $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ and $\mathcal{H}_{\mathbf{A}}(\Omega)$, which in a way form the magnetic counterpart to classical Sobolev spaces and thus provide us with the perfect framework to study the Landau Hamiltonian. To start, we provide basic definitions and properties of magnetic Sobolev spaces in Section 3.1. In Section 3.2 we will see that functions in magnetic Sobolev spaces coincide locally with classical Sobolev functions on bounded domains. We are then going to exploit this fact in Section 3.3 in order to construct bounded Dirichlet and Neumann trace operators on Lipschitz domains, which will also allow us in Section 3.4 to provide a generalized Green's formula for functions whose weak magnetic derivatives exist in $L^2(\Omega)$.

3.1 Basic Definitions

In this subsection we will define magnetic Sobolev spaces, which form the magnetic counterpart to Sobolev spaces and provide an appropriate version of the diamagnetic inequality, which is essential in the analysis of the Landau Hamiltonian.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^2$ be an open set. Let $B > 0$ and set $\nabla_{\mathbf{A}} := i\nabla + \mathbf{A}$ where $\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1)^\top$. We introduce the magnetic Sobolev space of first order on Ω as the space

$$\mathcal{H}^1_{\mathbf{A}}(\Omega) := \{f \in L^2(\Omega) : |\nabla_{\mathbf{A}} f| \in L^2(\Omega)\}.$$

By endowing $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ with the inner product

$$(f, g)_{\mathcal{H}^1_{\mathbf{A}}(\Omega)} := (f, g)_{L^2(\Omega)} + (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\Omega)}$$

for $f, g \in \mathcal{H}^1_{\mathbf{A}}(\Omega)$ the magnetic Sobolev space becomes a Hilbert space. We can also give the magnetic counterpart of the Sobolev space $H^1_0(\Omega)$ and define the subspace $\mathcal{H}^1_{\mathbf{A},0}(\Omega)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ with respect to $\|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)}$. In addition to that we introduce the subspace

$$\mathcal{H}^1_{\mathbf{A},C}(\Omega) := \{f \in \mathcal{H}^1_{\mathbf{A}}(\Omega) : \operatorname{supp} f \text{ is compact in } \mathbb{R}^2\}$$

of functions in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ with compact support.

It is important to note here, that we only require the support to be compact in $\mathbb{R}^2$, *not* compact in Ω. In particular, $\operatorname{supp} f$ may contain points lying on the boundary of Ω. Following the above definition, we can now go ahead and introduce magnetic Sobolev spaces of second order, which in addition to $|\nabla_{\mathbf{A}} f| \in L^2(\Omega)$ also require the existence of the distribution $\nabla^2_{\mathbf{A}} f$ in the L^2-sense.

Definition 3.2. For an open set $\Omega \subset \mathbb{R}^2$ the magnetic Sobolev spaces of second order is defined as the space

$$\mathcal{H}_{\mathbf{A}}(\Omega) := \{f \in \mathcal{H}^1_{\mathbf{A}}(\Omega) : \nabla^2_{\mathbf{A}} f \in L^2(\Omega)\}.$$

If we endow $\mathcal{H}_{\mathbf{A}}(\Omega)$ with the inner product

$$(f,g)_{\mathcal{H}_{\mathbf{A}}(\Omega)} := (f,g)_{\mathcal{H}^1_{\mathbf{A}}(\Omega)} + (\nabla^2_{\mathbf{A}} f, \nabla^2_{\mathbf{A}} g)_{L^2(\Omega)}$$

for $f, g \in \mathcal{H}_{\mathbf{A}}(\Omega)$, the magnetic Sobolev space of second order becomes a Hilbert space. Furthermore, we introduce the subspace

$$\mathcal{H}_{\mathbf{A},C}(\Omega) := \{f \in \mathcal{H}_{\mathbf{A}}(\Omega) : \operatorname{supp} f \text{ is compact in } \mathbb{R}^2\}$$

of functions in $\mathcal{H}_{\mathbf{A}}(\Omega)$ with compact support.

Recall again, that we only require $\operatorname{supp} f$ to be compact in $\mathbb{R}^2$, but not necessarily in Ω. In the next proposition, we state two well-known variants of the so-called diamagnetic inequalty, which can also be found in [2, Theorem 2.5] and [25, Theorem 7.21] for further reference.

Proposition 3.3 ([3, Proposition 2.2]). *Let $-\Delta$ be the self-adjoint Laplace operator in $L^2(\mathbb{R}^2)$ defined on $H^2(\mathbb{R}^2)$. Let $\beta > 0$ and $\lambda < 0$. Then for any $f \in L^2(\mathbb{R}^2)$ one has pointswise a.e. in $\mathbb{R}^2$*

$$|(\mathsf{A}_0 - \lambda)^{-\beta} f| \leq (-\Delta - \lambda)^{-\beta} |f| \,. \tag{3.1}$$

Moreover, for $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ there holds $|f| \in H^1(\mathbb{R}^2)$ as well as the pointwise estimate $|\nabla |f(x)|| \leq |\nabla_{\mathbf{A}} f(x)|$ for a.e. $x \in \mathbb{R}^2$.

3.2 Local Equivalence

As an important observation, magnetic Sobolev spaces are generally not contained in classical Sobolev spaces, as the existence of $|(i\nabla + \boldsymbol{A})f|$ in $L^2(\Omega)$ does not immediately imply $|\nabla f| \in L^2(\Omega)$, which is due to the fact that the vector potential $\mathbf{A}$ becomes unbounded, if the domain Ω is unbounded as well. On the other hand, since $\mathbf{A}$ is a smooth vector potential, we find that the restriction $|\mathbf{A}f| \restriction_B$ to a bounded subset $B \subset \Omega$ is again a function in $L^2(B)$. With this in mind it is only natural to propose that classic and magnetic Sobolev spaces coincide on bounded sets, which we are going to show in this subsection.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Then there holds $\mathcal{H}^1_{\mathbf{A}}(\Omega) = H^1(\Omega)$ as well as $\mathcal{H}_{\mathbf{A}}(\Omega) = H^1_{\Delta}(\Omega)$ and the respective norms are equivalent.*

Proof. The statement $H^1(\Omega) = \mathcal{H}^1_{\mathbf{A}}(\Omega)$ follows from the fact, that $\mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is bounded inside Ω, implying that for any $f \in L^2(\Omega)$ we also have $|\mathbf{A}f| \in L^2(\Omega)$. Furthermore, for any $f \in H^1(\Omega)$ there holds

$$\|\nabla f\|_{L^2(\Omega)} \leq \|\nabla_{\mathbf{A}} f\|_{L^2(\Omega)} + \|\mathbf{A}\|_\infty \|f\|_{L^2(\Omega)} \leq (1 + \|\mathbf{A}\|_\infty)\|f\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)},$$

as well as

$$\|\nabla_{\mathbf{A}} f\|_{L^2(\Omega)} \leq \|\mathbf{A}\|_\infty \|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)} \leq (1 + \|\mathbf{A}\|_\infty)\|f\|_{H^1(\Omega)}.$$

This proves that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)}$ are in fact equivalent. Now let $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Since $\operatorname{div}\mathbf{A} = 0$ we get

$$\begin{aligned}(f, \nabla_{\mathbf{A}}^2 \varphi)_{L^2(\Omega)} &= (f, \left(-\Delta + 2i\mathbf{A}\cdot\nabla + \mathbf{A}^2\right)\varphi)_{L^2(\Omega)} \\ &= (f, -\Delta\varphi)_{L^2(\Omega)} + ((2i\mathbf{A}\cdot\nabla + \mathbf{A}^2)f, \varphi)_{L^2(\Omega)},\end{aligned}$$

where we used the already proven fact $\mathcal{H}^1_{\mathbf{A}}(\Omega) = H^1(\Omega)$ and that $\mathbf{A}$ is bounded in Ω. This shows that Δf exists if and only if $\nabla_{\mathbf{A}}^2 f$ exists and there holds

$$\nabla_{\mathbf{A}}^2 f = -\Delta f + (2i\mathbf{A}\cdot\nabla + \mathbf{A}^2)f,$$

which in combination with the first part, after a few analogous estimates, yields the stated norm equivalence of $\|\cdot\|_{H^1_\Delta(\Omega)}$ and $\|\cdot\|_{\mathcal{H}_{\mathbf{A}}(\Omega)}$. ☐

We can now use the above result to show that compactly supported functions in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ and $\mathcal{H}_{\mathbf{A}}(\Omega)$ also lie in the classical Sobolev spaces of the respective order.

Corollary 3.5. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set. Then there holds $\mathcal{H}^1_{\mathbf{A},C}(\Omega) \subseteq H^1(\Omega)$ as well as $\mathcal{H}_{\mathbf{A},C}(\Omega) \subseteq H^1_\Delta(\Omega)$.*

Proof. The statement is clear for bounded sets Ω by Lemma 3.4. So suppose that Ω is unbounded and $f \in \mathcal{H}^1_{\mathbf{A},C}(\Omega)$. Let $D \subset \mathbb{R}^2$ be an open disk such that $\operatorname{supp} f \subseteq D$ and set $B = \Omega \cap D$. Then by the boundedness of B it follows by Lemma 3.4 that

$$f\restriction_B \in \mathcal{H}^1_{\mathbf{A}}(B) = H^1(B). \tag{3.2}$$

Since $\operatorname{supp} f$ is contained inside D, it follows that the Dirichlet trace of $f\restriction_B$ vanishes on $\partial B \cap \Omega$. On the other hand there holds $f = 0$ on $\Omega \backslash B$, so it follows with (3.2) that $f \in H^1(\Omega)$. Since $f \in \mathcal{H}^1_{\mathbf{A},C}(\Omega)$ was arbitrary, this shows that $\mathcal{H}^1_{\mathbf{A},C}(\Omega) \subset H^1(\Omega)$. An analogous argument can be made to show that $\mathcal{H}_{\mathbf{A},C}(\Omega) \subset H^1_\Delta(\Omega)$, which finishes the proof. ☐

As an immediate consequence of Lemma 3.4, we obtain the density of $\mathcal{C}_0^\infty(\overline{\Omega})$ in the magnetic Sobolev spaces, in the case where Ω is a bounded Lipschitz domain.

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in $(\mathcal{H}^1_{\mathbf{A}}(\Omega), \|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)})$ and $(\mathcal{H}_{\mathbf{A}}(\Omega), \|\cdot\|_{\mathcal{H}_{\mathbf{A}}(\Omega)})$.*

Proof. By Lemma 3.4 there holds $\mathcal{H}^1_{\mathbf{A}}(\Omega) = H^1(\Omega)$ as well as $\mathcal{H}_{\mathbf{A}}(\Omega) = H^1_\Delta(\Omega)$ and the respective norms are equivalent. Since $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in the classical Sobolev spaces for Lipschitz domains Ω, the result follows. ☐

In the case where $\Omega = \mathbb{R}^2$ one obtains the following result, which we will only state here without giving a proof.

Lemma 3.7 ([25, Theorem 7.22]). *The space of test functions $\mathcal{C}_0^\infty(\mathbb{R}^2)$ lies dense in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ with respect to $\|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)}$.*

In the next step we will show that functions in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ and $\mathcal{H}_{\mathbf{A}}(\Omega)$ can be approximated by functions in $\mathcal{H}^1_{\mathbf{A},C}(\Omega)$ and $\mathcal{H}_{\mathbf{A},C}(\Omega)$, respectively, by using a sequence of smooth cut-off functions.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^2$ be an open set. Then the following holds*

(i) $\mathcal{H}^1_{\mathbf{A},C}(\Omega)$ *is dense in* $(\mathcal{H}^1_{\mathbf{A}}(\Omega), \|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)})$

(ii) $\mathcal{H}_{\mathbf{A},C}(\Omega)$ *is dense in* $(\mathcal{H}_{\mathbf{A}}(\Omega), \|\cdot\|_{\mathcal{H}_{\mathbf{A}}(\Omega)})$

Proof. We will consider the case where Ω is unbounded, as for the bounded case we already have $\mathcal{H}^1_{\mathbf{A},C}(\Omega) = \mathcal{H}^1_{\mathbf{A}}(\Omega)$ and $\mathcal{H}_{\mathbf{A},C}(\Omega) = \mathcal{H}_{\mathbf{A}}(\Omega)$.

So let Ω be unbounded and $f \in \mathcal{H}^1_{\mathbf{A}}(\Omega)$. Choose $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ such that $\chi = 1$ on the unit disk $\{x = (x_1, x_2) \in \mathbb{R}^2 : |x| \leq 1\}$. We define the sequence $\chi_k(x) := \chi(\frac{x}{k})$ for $x \in \Omega$, which satisfies

$$\sup_{x\in\Omega} \left|\partial^m_{x_1}\partial^n_{x_2}\chi_k(x)\right| \leq \frac{1}{k^{m+n}} \sup_{x\in\mathbb{R}^2} \left|\partial^m_{x_1}\partial^n_{x_2}\chi(x)\right| \leq \frac{C}{k^{m+n}}$$

for all $m, n \in \mathbb{N}_0$ and $C(m,n) > 0$. We will show that $f_k := \chi_k f \in \mathcal{H}^1_{\mathbf{A},C}(\Omega)$ is a sequence that fulfils the desired properties. By the dominated convergence theorem it is clear that $f_k \to f$ in $L^2(\Omega)$. For the magnetic gradient there holds

$$(i\nabla + \boldsymbol{A})f_k = \chi_k(i\nabla + \boldsymbol{A})f + i(\nabla\chi_k)f,$$

which then implies

$$\|(i\nabla + \boldsymbol{A})f_k - (i\nabla + \boldsymbol{A})f\|_{L^2(\Omega)} \leq \|(1-\chi_k)(i\nabla + \boldsymbol{A})f\|_{L^2(\Omega)} + \frac{C}{k}\|f\|_{L^2(\Omega)}.$$

By the dominated convergence theorem the right hand side tends to 0 as $k \to \infty$ which together with $f_k \to f$ in $L^2(\Omega)$ means

$$\lim_{k\to\infty} \|f_k - f\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)} = 0.$$

Assume now that $f \in \mathcal{H}_{\mathbf{A}}(\Omega)$. We must show that $(i\nabla + \boldsymbol{A})^2 f_k \to (i\nabla + \boldsymbol{A})^2 f$ in $L^2(\Omega)$. First observe that for any $\varphi \in \mathcal{C}^\infty_0(\Omega)$ there holds

$$(i\nabla + \boldsymbol{A})^2\chi_k\varphi = \chi_k(i\nabla + \boldsymbol{A})^2\varphi - (\Delta\chi_k)\varphi + 2i\nabla\chi_k \cdot \nabla_{\mathbf{A}}\varphi$$

in the strong sense. Rearranging these terms yields

$$\chi_k(i\nabla + \boldsymbol{A})^2\varphi = (i\nabla + \boldsymbol{A})^2\chi_k\varphi + (\Delta\chi_k)\varphi - 2i\nabla\chi_k \cdot (i\nabla + \boldsymbol{A})\varphi. \tag{3.3}$$

Furthermore, there holds

$$(i\nabla + \boldsymbol{A})(\varphi\nabla\chi_k) = \nabla\chi_k \cdot (i\nabla + \boldsymbol{A})\varphi + i\varphi\Delta\chi_k,$$

or equivalently

$$-2i\nabla\chi_k \cdot (i\nabla + \boldsymbol{A})\varphi = -2i(i\nabla + A)(\varphi\nabla\chi_k) - 2\varphi\Delta\chi_k,$$

which inserted in (3.3) yields

$$\chi_k(i\nabla + \boldsymbol{A})^2\varphi = (i\nabla + \boldsymbol{A})^2\chi_k\varphi - (\Delta\chi_k)\varphi - 2i(i\nabla + \boldsymbol{A})(\varphi\nabla\chi_k).$$

This means that for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ we get

$$\begin{aligned}(f_k, (i\nabla + \boldsymbol{A})^2\varphi)_{L^2(\Omega)} &= (f, \chi_k(i\nabla + \boldsymbol{A})^2\varphi)_{L^2(\Omega)} \\ &= (f, (i\nabla + \boldsymbol{A})^2\chi_k\varphi - (\Delta\chi_k)\varphi - 2i(i\nabla + \boldsymbol{A})(\varphi\nabla\chi_k))_{L^2(\Omega)} \\ &= (\chi_k(i\nabla + \boldsymbol{A})^2 f - (\Delta\chi_k)f + 2i\nabla\chi_k(i\nabla + \boldsymbol{A})f, \varphi)_{L^2(\Omega)}.\end{aligned}$$

This shows that there holds

$$(i\nabla + \boldsymbol{A})^2 f_k = \chi_k(i\nabla + \boldsymbol{A})^2 f - (\Delta\chi_k)f + 2i\nabla\chi_k(i\nabla + \boldsymbol{A})f \in L^2(\Omega).$$

In particular, we get the estimate

$$\begin{aligned}\left\|(i\nabla + \boldsymbol{A})^2 f_k - (i\nabla + \boldsymbol{A})^2 f\right\|_{L^2(\Omega)} \leq \left\|(1-\chi_k)(i\nabla + \boldsymbol{A})^2 f\right\|_{L^2(\Omega)} + \frac{C}{k^2}\|f\|_{L^2(\Omega)} \\ + \frac{C}{k}\|(i\nabla + \boldsymbol{A})f\|_{L^2(\Omega)}.\end{aligned}$$

Clearly the right hand side tends to 0 as $k \to \infty$ by the dominated convergence theorem, which concludes the proof. □

3.3 Dirichlet and Neumann trace

In this subsection, we will prove the existence of bounded Dirichlet and Neumann trace operators for magnetic Sobolev spaces on Lipschitz boundaries. We have seen in the last subsection that the magnetic Sobolev spaces coincide locally with their classical counterparts, that is the weak deriatives of functions in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ and $\mathcal{H}_{\mathbf{A}}(\Omega)$ exist in $L^2(B)$ on any bounded subset $B \subset \Omega$. Given some (possibly unbounded) Lipschitz domain $\Omega \subset \mathbb{R}^2$ with compact boundary Σ, we can choose an open disk $D \subset \mathbb{R}^2$ such that $\Sigma \subset D$ and instead consider the bounded set $B = \Omega \cap D$. This way we can construct Dirichlet and Neumann trace operators on $\mathcal{H}^1_{\mathbf{A}}(B) = H^1(B)$ and $\mathcal{H}_{\mathbf{A}}(B) = H^1_\Delta(B)$, which exist in the classical Sobolev space theory. With this in mind our aim will be to first construct bounded trace operators $\widetilde{\gamma}^{\mathbf{A}}_{\mathrm{D}} : \mathcal{H}^1_{\mathbf{A},C}(\Omega) \to H^{1/2}(\Sigma)$ and $\widetilde{\gamma}^{\mathbf{A}}_{\mathrm{N}} : \mathcal{H}_{\mathbf{A},C}(\Omega) \to H^{-1/2}(\Sigma)$ and then use a density argument to extend them to the entire magnetic Sobolev spaces.

We will start by showing that the extension of the Dirichlet trace $f|_\Sigma$ from functions in $\mathcal{C}_0^\infty(\overline{\Omega})$ to functions in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ gives us a well-defined and bounded operator.

Proposition 3.9. *Let Ω be a Lipschitz domain with boundary Σ. Then there exists a bounded linear and surjective operator $\gamma_{\mathrm{D}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A}}(\Omega) \to H^{1/2}(\Sigma)$ such that $\gamma_{\mathrm{D}}^{\mathbf{A}} f = f|_\Sigma$ for $f \in \mathcal{C}_0^\infty(\overline{\Omega})$. Moreover this operator is compact from $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ to $L^2(\Sigma)$.*

Proof. We start off by defining the map $\widetilde{\gamma}_{\mathrm{D}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A},C}(\Omega) \ni f \mapsto \gamma_{\mathrm{D}} f \in H^{1/2}(\Sigma)$ where $\gamma_{\mathrm{D}} : H^1(\Omega) \to H^{1/2}(\Sigma)$ denotes the trace operator for classic Sobolev spaces, which is well-defined since $\mathcal{H}^1_{\mathbf{A},C}(\Omega) \subset H^1(\Omega)$ by Corollary 3.5. To show the boundedness of $\widetilde{\gamma}_{\mathrm{D}}^{\mathbf{A}}$ let D be an open disk in $\mathbb{R}^2$ such that $\Sigma \subset D$ and set $B = \Omega \cap D$. By Lemma 3.4 the restriction operator $R : \mathcal{H}^1_{\mathbf{A}}(\Omega) \to H^1(B), f \mapsto f\restriction_B$ is bounded and by construction it follows that $\gamma_{\mathrm{D}}(Rf) = \gamma_{\mathrm{D}} f$ for any $f \in \mathcal{H}^1_{\mathbf{A},C}(\Omega)$. In particular, there holds

$$\|\gamma_{\mathrm{D}} f\|_{H^{1/2}(\Sigma)} = \|\gamma_{\mathrm{D}} Rf\|_{H^{1/2}(\Sigma)} \leq c_1 \|Rf\|_{H^1(B)} \leq c_2 \|f\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)}$$

for some constants $c_1, c_2 > 0$, which shows the boundedness of $\widetilde{\gamma}_{\mathrm{D}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A},C}(\Omega) \to H^{1/2}(\Sigma)$. As $\mathcal{H}^1_{\mathbf{A},C}(\Omega)$ is dense in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ by Lemma 3.8, the operator $\widetilde{\gamma}_{\mathrm{D}}^{\mathbf{A}}$ can be continuously extended to a bounded operator $\gamma_{\mathrm{D}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A}}(\Omega) \to H^{1/2}(\Sigma)$. To see that $\gamma_{\mathrm{D}}^{\mathbf{A}}$ is surjective, choose a cut-off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ on the open disk D. Then for any $f \in H^1(\Omega)$ we have $\chi\restriction_\Omega f \in \mathcal{H}^1_{\mathbf{A}}(\Omega)$ with $\gamma_{\mathrm{D}}^{\mathbf{A}}(\chi\restriction_\Omega f) = \gamma_{\mathrm{D}} f$, which shows $\operatorname{ran}(\gamma_{\mathrm{D}}^{\mathbf{A}}) = \operatorname{ran}(\gamma_{\mathrm{D}}) = H^{1/2}(\Sigma)$. The compactness of $\gamma_{\mathrm{D}} : \mathcal{H}^1_{\mathbf{A}}(\Omega) \to L^2(\Sigma)$ follows from the fact that the embedding $H^{1/2}(\Sigma) \hookrightarrow L^2(\Sigma)$ is compact. □

We have now established a bounded trace operator for Lipschitz domains, but we are also going to need the trace $f|_\Sigma$ for globally defined functions $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. In this case we can use the diamagnetic inequality to extend the trace map $\mathcal{C}_0^\infty(\mathbb{R}^2) \ni f \mapsto f|_\Sigma \in H^{1/2}(\Sigma)$ by continuity to a bounded operator on $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$.

Proposition 3.10. *Let Σ be the boundary of a Lipschitz domain Ω in $\mathbb{R}^2$. Then there exists a bounded linear operator $\gamma_{\mathrm{D}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to H^{1/2}(\Sigma)$ such that $\gamma_{\mathrm{D}}^{\mathbf{A}} f = f|_\Sigma$ for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. In particular, $\gamma_{\mathrm{D}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to L^2(\Sigma)$ is a compact operator. Moreover for all $\epsilon > 0$ there exists $c(\epsilon) > 0$ such that*

$$\|f|_\Sigma\|_{L^2(\Sigma)} \leq \epsilon \|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2)} + c(\epsilon)\|f\|^2_{L^2(\mathbb{R}^2)} \tag{3.4}$$

holds for all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$.

Proof. Fix $\epsilon > 0$, $s \in (\frac{1}{2}, 1)$ and let $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. By the boundedness of the Dirichlet trace operator $\gamma_{\mathrm{D}} : H^s(\mathbb{R}^2) \to L^2(\Sigma)$ and Lemma 2.36 it follows that there exists a constant $c(\epsilon) > 0$, which does not depend on f, such that

$$\|f|_\Sigma\|^2_{L^2(\Sigma)} = \||f|\,|_\Sigma\|^2_{L^2(\Sigma)} \leq c\||f|\|_{H^s(\mathbb{R}^2)} \leq \epsilon\|\nabla\,|f|\|^2_{L^2(\mathbb{R}^2)} + c(\epsilon)\|f\|^2_{L^2(\mathbb{R}^2)}.$$

Applying the diamagnetic inequality from Lemma 2.36 gives us

$$\|f|_\Sigma\|^2_{L^2(\Sigma)} \leq \epsilon\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2)} + c(\epsilon)\|f\|^2_{L^2(\mathbb{R}^2)},$$

which by the density of $\mathcal{C}_0^\infty(\mathbb{R}^2)$ in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ shows the inequality stated in (3.4) for all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$.

In order to show that $\gamma_{\mathrm{D}}^{\mathbf{A}}$ maps functions from $\mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2)$ into $H^{1/2}(\Sigma)$ we choose an open disk $D \subset \mathbb{R}^2$ such that $\Sigma \subset D$ and let $\mathcal{R} : \mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2) \to \mathcal{H}_{\mathbf{A}}^1(B), f \mapsto f\restriction_\Omega$ be the restriction operator onto Ω. By Proposition 3.9 the local trace operator $\gamma_{\mathrm{D},B}^{\mathbf{A}} : \mathcal{H}_{\mathbf{A}}^1(B) \to H^{1/2}(\Sigma)$ is bounded, so we have the mapping property $\gamma_{\mathrm{D},B}^{\mathbf{A}}\mathcal{R} : \mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2) \to H^{1/2}(\Sigma), f \mapsto f|_\Sigma$, which finishes the proof. $\square$

We will now use a similar strategy as we did in the proof of Proposition 3.9 to show that the conormal derivative $\partial_\nu f$ for smooth functions f can be extended to a bounded operator $\gamma_{\mathrm{N}}^{\mathbf{A}} : \mathcal{H}_{\mathbf{A}}(\Omega) \to H^{1/2}(\Sigma)$.

Proposition 3.11. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with Lipschitz boundary Σ and unit normal field ν. Then there exists a bounded operator $\gamma_{\mathrm{N}}^{\mathbf{A}} : \mathcal{H}_{\mathbf{A}}(\Omega) \to H^{-1/2}(\Sigma)$ such that*

$$(\nabla_{\mathbf{A}}^2 f, g)_{L^2(\Omega)} = (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\Omega)} - \left\langle \gamma_{\mathrm{N}}^{\mathbf{A}} f - i\nu \cdot \mathbf{A}\gamma_{\mathrm{D}}^{\mathbf{A}} f, \gamma_{\mathrm{D}}^{\mathbf{A}} g \right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \tag{3.5}$$

for all $f \in \mathcal{H}_{\mathbf{A},C}(\Omega)$ and $g \in \mathcal{H}_{\mathbf{A}}^1(\Omega)$.

Proof. Assume first that Ω is bounded, then we have $\mathcal{H}_{\mathbf{A},C}(\Omega) \subset H_\Delta^1(\Omega)$ as well as $\mathcal{H}_{\mathbf{A},C}^1(\Omega) \subset H^1(\Omega)$ by Corollary 3.5 and the statement follows directly from Lemma 2.40 after integration by parts.

Suppose now that Ω is unbounded and let $f \in \mathcal{H}_{\mathbf{A},C}(\Omega)$ and $g \in \mathcal{H}_{\mathbf{A},C}^1(\Omega)$. In particular, by Corollary 3.5 there holds $f \in H_\Delta^1(\Omega)$ as well as $g \in H^1(\Omega)$. By Lemma 2.40 there exists $h \in H^{-1/2}(\Sigma)$ such that

$$(\nabla f, \nabla g)_{L^2(\Omega)} = (-\Delta f, g)_{L^2(\Omega)} + \left\langle h, \gamma_{\mathrm{D}}^{\mathbf{A}} g \right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}. \tag{3.6}$$

We will define the operator $\widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}}$ via $\widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} f := h$ for $f \in \mathcal{H}_{\mathbf{A},C}(\Omega)$. Using (3.6) gives us

$$\begin{aligned}(\nabla_{\mathbf{A}}^2 f, g)_{L^2(\Omega)} &= (-\Delta f + 2i\mathbf{A} \cdot \nabla f + \mathbf{A}^2 f, g)_{L^2(\Omega)} \\ &= (\nabla f, \nabla g)_{L^2(\Omega)} + (2i\mathbf{A} \cdot \nabla f, g)_{L^2(\Omega)} + (\mathbf{A} f, \mathbf{A} g)_{L^2(\Omega)} \\ &\qquad - \left\langle \widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} f, \gamma_{\mathrm{D}}^{\mathbf{A}} g \right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}.\end{aligned} \tag{3.7}$$

Using the classic divergence theorem for functions in $H^1(\Omega)$ in conjunction with $\operatorname{div}\mathbf{A} = 0$ we further get

$$(\mathbf{A} \cdot \nabla f, g)_{L^2(\Omega)} = (\nu \cdot \mathbf{A}\gamma_{\mathrm{D}}^{\mathbf{A}} f, \gamma_{\mathrm{D}}^{\mathbf{A}} g)_{L^2(\Sigma)} - (\mathbf{A} f, \nabla g)_{L^2(\Omega)}.$$

Plugging this into (3.7) yields

$$\begin{aligned}(\nabla_{\mathbf{A}}^2 f, g)_{L^2(\Omega)} &= (\nabla f, \nabla g)_{L^2(\Omega)} + (i\mathbf{A} \cdot \nabla f, g)_{L^2(\Omega)} + (f, i\mathbf{A} \cdot \nabla g)_{L^2(\Omega)} \\ &\qquad + (\mathbf{A} f, \mathbf{A} g)_{L^2(\Omega)} - \left\langle \widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} f - i\nu \cdot \mathbf{A}\gamma_{\mathrm{D}}^{\mathbf{A}} f, \gamma_{\mathrm{D}}^{\mathbf{A}} g \right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \\ &= (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\Omega)} - \left\langle \widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} f - i\nu \cdot \mathbf{A}\gamma_{\mathrm{D}}^{\mathbf{A}} f, \gamma_{\mathrm{D}}^{\mathbf{A}} g \right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}.\end{aligned}$$

This shows the claimed identity for $g \in \mathcal{H}^1_{\mathbf{A},C}(\Omega)$. By density of $\mathcal{H}^1_{\mathbf{A},C}(\Omega)$ in $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ this equality extends to all $g \in \mathcal{H}^1_{\mathbf{A}}(\Omega)$. To show the boundedness of the operator let $D \subset \mathbb{R}^2$ be an open disk such that $\Sigma \subset D$ and choose $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ on D. Define the multiplication operator $\mathcal{J} : L^2(\Omega) \ni f \mapsto \chi\restriction_\Omega \cdot f \in L^2(\Omega)$, which by Lemma 3.4 is bounded from $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ to $H^1(\Omega)$ and vice-versa. Let $\mathcal{E} : H^{1/2}(\Sigma) \to H^1(\Omega)$ be the continuous right inverse of the trace operator. Then by construction there holds

$$\gamma_{\mathrm{D}}^{\mathbf{A}}(\mathcal{J}\mathcal{E}\varphi) = \gamma_{\mathrm{D}}(\chi\restriction_\Omega \mathcal{E}\varphi) = \gamma_{\mathrm{D}}(\mathcal{E}\varphi) = \varphi$$

for any $\varphi \in H^{1/2}(\Sigma)$, implying

$$\begin{aligned}
\left|\langle \widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} f, \varphi\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}\right| &= \left|\langle \widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} f, \gamma_{\mathrm{D}}^{\mathbf{A}}(\mathcal{J}\mathcal{E}\varphi)\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}\right| \\
&\leq \left|(\nabla^2_{\mathbf{A}} f, \mathcal{J}\mathcal{E}\varphi)_{L^2(\Omega)}\right| + \left|(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}}\mathcal{J}\mathcal{E}\varphi)_{L^2(\Omega)}\right| \\
&\qquad + \left|(\nu\cdot \mathbf{A}\gamma_{\mathrm{D}}^{\mathbf{A}} f, \varphi)_{L^2(\Sigma)}\right| \\
&\leq c\left(\left\|\nabla^2_{\mathbf{A}} f\right\|_{L^2(\Omega)} + \|\nabla_{\mathbf{A}} f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)\|\varphi\|_{H^{1/2}(\Sigma)} \\
&\leq c\|f\|_{\mathcal{H}_{\mathbf{A}}(\Omega)}\|\varphi\|_{H^{1/2}(\Sigma)}.
\end{aligned}$$

for some $c > 0$. This shows that $\widetilde{\gamma}_{\mathrm{N}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A},C}(\Omega) \to H^{-1/2}(\Sigma)$ can be extended by continuity to a bounded operator $\gamma_{\mathrm{N}}^{\mathbf{A}} : \mathcal{H}^1_{\mathbf{A}}(\Omega) \to H^{-1/2}(\Sigma)$ that satisfies (3.5). □

Remark 3.12. Assume that $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain with unit normal field ν. We have seen in Corollary 3.6 that functions in $\mathcal{H}_{\mathbf{A}}(\Omega)$ can be locally approximated by sequences of smooth functions, that is $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in $(\mathcal{H}_{\mathbf{A}}(\Omega), \|\cdot\|_{\mathcal{H}_{\mathbf{A}}(\Omega)})$, if Ω is bounded. Hence the Neumann trace operator $\gamma_{\mathrm{N}}^{\mathbf{A}}$ from Proposition 3.11 can be seen as a continuous extension of the normal derivative ∂_ν from $\mathcal{C}_0^\infty(\overline{\Omega})$ to the space $\mathcal{H}_{\mathbf{A}}(\Omega)$. In particular, there holds $\gamma_{\mathrm{N}}^{\mathbf{A}} f = \partial_\nu f$ for $f \in \mathcal{C}_0^\infty(\overline{\Omega})$ - with this in mind we will from now on write ∂_ν instead of $\gamma_{\mathrm{N}}^{\mathbf{A}}$. Recall that the normal derivative satisfies $\partial_{-\nu} = -\partial_\nu$ for smooth functions, which by density translates to functions in $\mathcal{H}_{\mathbf{A}}(\Omega)$, a result which we will use later on in the proof of Lemma 3.14.

3.4 Green's Identity

In this subsection we will use the Neumann trace operator that we acquired in the last subsection, to derive an appropriate version of Green's first identity for functions in magnetic Sobolev spaces. We will derive the formula under the following assumption.

Assumption 3.13. From now on we will assume that $\Omega_i \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\Sigma := \partial\Omega_i$ and set $\Omega_e := \mathbb{R}^2\setminus\overline{\Omega}_i$ for the exterior domain. Denote by ν the unit normal vector field pointing outward from Ω_i (and hence pointing into Ω_e). Using the above notation we introduce the space

$$\mathfrak{D} := \{f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) : \nabla^2_{\mathbf{A}} f\restriction_{\Omega_i} \in L^2(\Omega_i) \text{ and } \nabla^2_{\mathbf{A}} f\restriction_{\Omega_e} \in L^2(\Omega_e)\}. \tag{3.8}$$

The space $\mathfrak{D}$ consists of those functions $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ that can be written as $f = f_i \oplus f_e$, where $f_i \in \mathcal{H}_{\mathbf{A}}(\Omega_i)$ and $f_e \in \mathcal{H}_{\mathbf{A}}(\Omega_e)$. As an important observation, functions in $\mathfrak{D}$ have continuous traces, i.e. $\gamma^{\mathbf{A}}_{\mathrm{D},i} f_i = \gamma^{\mathbf{A}}_{\mathrm{D},e} f_e$.

Recall that by the Propositions 3.9 and 3.10 there exist bounded operators

$$\begin{aligned}\gamma^{\mathbf{A}}_{\mathrm{D},i} &: \mathcal{H}^1_{\mathbf{A}}(\Omega_i) \to H^{1/2}(\Sigma),\\ \gamma^{\mathbf{A}}_{\mathrm{D},e} &: \mathcal{H}^1_{\mathbf{A}}(\Omega_e) \to H^{1/2}(\Sigma),\\ \gamma^{\mathbf{A}}_{\mathrm{D}} &: \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to H^{1/2}(\Sigma),\end{aligned}$$

that act as continuous extensions of the trace map $\varphi \mapsto \varphi|_\Sigma$ for smooth functions φ in their respective subdomain. Since we have

$$\gamma^{\mathbf{A}}_{\mathrm{D},i}(\varphi\restriction_{\Omega_i}) = \gamma^{\mathbf{A}}_{\mathrm{D},e}(\varphi\restriction_{\Omega_e}) = \gamma^{\mathbf{A}}_{\mathrm{D}}\varphi = \varphi|_\Sigma$$

for any $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ we will from now on omit the declaration of the subdomain in the notation, and write γ_{D} for all three operators, as there is no danger of confusion, since $H^1(\Omega)$ and $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ coincide locally. The same follows for the operators

$$\begin{aligned}\partial_{\nu,i} &: \mathcal{H}_{\mathbf{A}}(\Omega_i) \to H^{-1/2}(\Sigma)\\ \partial_{-\nu,e} &: \mathcal{H}_{\mathbf{A}}(\Omega_e) \to H^{-1/2}(\Sigma),\end{aligned}$$

which are natural extensions of the conormal derivative to magnetic Sobolev spaces, so we will write ∂_ν for the interior and $\partial_{-\nu} = -\partial_\nu$ for the exterior Neumann trace.

Under the above assumption we are able to derive Green's first identity for functions that lie in the space $\mathfrak{D}$.

Lemma 3.14. *Suppose that Assumption* 3.13 *holds true. Then for any $f \in \mathfrak{D}$, where $\mathfrak{D}$ is defined as in* (3.8)*, and $g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ there holds*

$$(\nabla^2_{\mathbf{A}} f, g)_{L^2(\mathbb{R}^2)} = (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\mathbb{R}^2)} + \langle \partial_\nu f_e - \partial_\nu f_i, \gamma_{\mathrm{D}} g\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}.$$

In particular, a function $f \in \mathfrak{D}$ lies in $\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ if and only if $\partial_\nu f_e = \partial_\nu f_i$.

Proof. Let $f = f_i \oplus f_e \in \mathcal{D}$ and $g = g_i \oplus g_e \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ with $g_i = g\restriction_{\Omega_i}$ and $g_e = g\restriction_{\Omega_e}$, then by Proposition 3.11 there holds

$$(\nabla^2_{\mathbf{A}} f_i, g_i)_{L^2(\Omega_i)} = (\nabla_{\mathbf{A}} f_i, \nabla_{\mathbf{A}} g_i)_{L^2(\Omega_i)} - \langle \partial_\nu f_i - i\nu\cdot\mathbf{A}\gamma_{\mathrm{D}} f_i, \gamma_{\mathrm{D}} g_i\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}$$

for the interior domain, as well as

$$\begin{aligned}(\nabla^2_{\mathbf{A}} f_e, g_e)_{L^2(\Omega_e)} &= (\nabla_{\mathbf{A}} f_e, \nabla_{\mathbf{A}} g_e)_{L^2(\Omega_e)} - \langle \partial_{-\nu} f_e + i\nu\cdot\mathbf{A}\gamma_{\mathrm{D}} f_e, \gamma_{\mathrm{D}} g_e\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}\\ &= (\nabla_{\mathbf{A}} f_e, \nabla_{\mathbf{A}} g_e)_{L^2(\Omega_e)} + \langle \partial_\nu f_e - i\nu\cdot\mathbf{A}\gamma_{\mathrm{D}} f_e, \gamma_{\mathrm{D}} g_e\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)}\end{aligned}$$

for the exterior domain, where we used that the unit normal field of $\partial\Omega_e$ is given by $-\nu$. Since $h_i|_\Sigma = h_e|_\Sigma$ for any function $h \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ we can add both equations for the interior and the exterior domain, which shows the claimed statement. $\square$

4 The Landau Hamiltonian

In this section we are going to study the Landau Hamiltonian with δ-perturbations that are supported on a $\mathcal{C}^{1,1}$ curve Σ in $\mathbb{R}^2$. Section 4.1 contains some preliminary material concerning the unperturbed Landau Hamiltonian. In Section 4.2 we are going to study the Landau Hamiltonian on a domain Ω with Dirichlet boundary conditions, where Ω is assumed to be either a bounded $\mathcal{C}^{1,1}$ domain in $\mathbb{R}^2$ or the complement of a bounded $\mathcal{C}^{1,1}$ domain. In Section 4.3 we introduce the Landau Hamiltonian A_α with a δ-potential by defining the associated sesquilinear form that corresponds to the formal expression $\mathsf{A}_0 + \alpha\delta_\Sigma$ and show that this operator is self-adjoint. Going into Section 4.4 we will derive a compact Krein-type factorization of the resolvent difference of the unperturbed Landau Hamiltonian A_0 and the Landau Hamiltonian A_α with a δ-potential supported on Σ. We will proceed in Section 4.5 with a rigorous analysis of the resolvent difference of A_0 and A_α.

4.1 The unperturbed Landau Hamiltonian

Following the lines of [3, Chapter 2.1] we will now introduce the unperturbed Landau Hamiltonian, that is the unperturbed magnetic Schrödinger operator with a constant and homogenous magnetic field. For this let $B > 0$ be the strength of the magnetic field and let $\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1)^\top$ be the corresponding vector potential in symmetric gauge. Recall that the magnetic Sobolev space of first order

$$\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) := \{f \in L^2(\mathbb{R}^2) : |\nabla_{\mathbf{A}} f| \in L^2(\mathbb{R}^2)\}$$

given in Definition 3.1, endowed with the inner product

$$(f, g)_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)} := (f, g)_{L^2(\mathbb{R}^2)} + (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\mathbb{R}^2)}$$

forms a Hilbert space. We can now introduce the form

$$\mathfrak{a}_0[f] = \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f(x)|^2 \, \mathrm{d}x, \quad \mathrm{dom}(\mathfrak{a}_0) = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2),$$

which is densely defined, non-negative and closed in $L^2(\mathbb{R}^2)$. In particular, this form gives rise to a unique and self-adjoint operator A_0, which is given by

$$\mathsf{A}_0 f = \nabla^2_{\mathbf{A}} f, \quad \mathrm{dom}(\mathsf{A}_0) = \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2) = \{f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) : \nabla^2_{\mathbf{A}} f \in L^2(\mathbb{R}^2)\} \tag{4.1}$$

and referred to as Landau Hamiltonian. The following proposition recalls the well-known spectral properties of the Landau Hamiltonian.

Proposition 4.1 ([3, Proposition 2.1]). *Let A_0 be the Landau Hamiltonian defined in (4.1). Then there holds*

$$\sigma(\mathsf{A}_0) = \sigma_{ess}(\mathsf{A}_0) = \{B(2q+1) : q \in \mathbb{N}_0\},$$

i.e. the spectrum of the Landau Hamiltonian A_0 consists exclusively of the eigenvalues $\Lambda_q = B(2q+1), q \in \mathbb{N}_0$, which are called Landau levels and have infinite multiplicity.

We will conclude this subsection by summarizing some of the well-known properties of the eigenspaces of the Landau Hamiltonian, following the lines of [31, Section 4.2]. For $q \in \mathbb{N}_0$ let $P_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the orthogonal projection onto the eigenspace $\mathcal{L}_q := \ker(\mathsf{A}_0 - \Lambda_q)$ where $\Lambda_q = B(2q+1)$ are the Landau levels. Using the identification $z = x_1 + ix_2$ for $(x_1, x_2) \in \mathbb{R}^2$ with $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$ we set $\Psi(z) = \frac{1}{4}B|z|^2$ and introduce the creation and annihilation operators

$$\begin{aligned} \mathcal{A}^+ &= -2ie^{\Psi}\partial e^{-\Psi} = -2i\partial + \frac{B}{2}i\overline{z}, \\ \mathcal{A}^- &= -2ie^{-\Psi}\overline{\partial} e^{\Psi} = -2i\overline{\partial} - \frac{B}{2}iz, \end{aligned} \tag{4.2}$$

with $\operatorname{dom}(\mathcal{A}^+) = \operatorname{dom}(\mathcal{A}^-) = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$, which are formally adjoint on $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. One readily verifies that the compositions $\mathcal{A}^+\mathcal{A}^-$ and $\mathcal{A}^-\mathcal{A}^+$ are well-defined on $\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ and satisfy the commutation relation

$$\mathsf{A}_0 u = (\mathcal{A}^-\mathcal{A}^+ - B)u = (\mathcal{A}^+\mathcal{A}^- + B)u \tag{4.3}$$

for all $u \in \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$. Using (4.3) along with the formal adjointness of $\mathcal{A}^+$ and $\mathcal{A}^-$ we can conclude that $\ker(\mathsf{A}_0 - B) = \ker(\mathcal{A}^-)$, which shows that the eigenvalue equation $(\mathsf{A}_0 - B)u = 0$ is equivalent to the Cauchy-Riemann differential equation $\overline{\partial}\left(e^{\Psi}u\right) = 0$. In particular, for $u \in \mathcal{L}_0$ the function $f = e^{\Psi}u$ obeys the Cauchy-Riemann differential equations and thus is an entire function such that $e^{-\Psi}f \in L^2(\mathbb{C})$. Recall that the Fock or Segal-Bargmann space $\mathcal{F}^2$ was introduced in Definition 2.13 as the Hilbert space of all entire functions such that

$$\|f\|^2_{\mathcal{F}^2} = \int_{\mathbb{C}} |f(z)|^2 \, e^{-\frac{1}{2}B|z|^2} \, \mathrm{d}m(z) < \infty.$$

This means solutions $u \in \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ to the equation $(\mathsf{A}_0 - B)u = 0$ can be equivalently rewritten into the form $f = e^{\Psi}u \in \mathcal{F}^2$, implying that there holds $\mathcal{L}_0 = e^{-\Psi}\mathcal{F}^2$ as unitary equivalence. Moreover, one can use (4.3) to show that additional eigenspaces $\mathcal{L}_q = \ker(\mathsf{A}_0 - \Lambda_q)$ for $q \geq 1$ can be obtained via $\mathcal{L}_q = (\mathcal{A}^+)^q\mathcal{L}_0$. The next proposition will provide more details on this.

Proposition 4.2 ([31, Equation (17)]). *Let the creation and annihilation operator be defined as in* (4.2)*. Let $q \in \mathbb{N}_0$ and set $\mathcal{L}_q = \ker(\mathsf{A}_0 - \Lambda_q)$. Then $\mathcal{A}^+$ and $\mathcal{A}^-$ act bijectively between the subspaces $\mathcal{L}_q$ as*

$$\mathcal{A}^+ : \mathcal{L}_q \to \mathcal{L}_{q+1}, \quad \mathcal{A}^- : \mathcal{L}_{q+1} \to \mathcal{L}_q, \quad \mathcal{A}^- : \mathcal{L}_0 \to \{0\}. \tag{4.4}$$

Moreover the spaces $\mathcal{F}^2$ and $\mathcal{L}_q$ are unitary equivalent via the mapping

$$\mathcal{U}_q : \begin{cases} \mathcal{F}^2 \to \mathcal{L}_q \\ f \mapsto C_q^{-1} \, (\mathcal{A}^+)^q \, e^{-\Psi} f \end{cases}, \tag{4.5}$$

where $C_q = \sqrt{q!(2B)^q}$.

Proof. Let $q \in \mathbb{N}_0$ be arbitrary but fixed. For this proof let us set $\mathcal{A}_q^+ = \mathcal{A}^+\restriction_{\mathcal{L}_q}$ as well as $\mathcal{A}_q^- = \mathcal{A}^-\restriction_{\mathcal{L}_{q+1}}$. We are first going to show that $\mathcal{A}_q^+ : \mathcal{L}_q \to \mathcal{L}_{q+1}$ is well-defined. To see this let $u_q \in \mathcal{L}_q$ be arbitrary. Using (4.3) we obtain via a direct computation

$$(\mathcal{A}^+ u_q, \nabla_{\mathbf{A}}^2 \varphi)_{L^2(\mathbb{R}^2)} = (\Lambda_{q+1}\mathcal{A}^+ u_q, \varphi)_{L^2(\mathbb{R}^2)}$$

for arbitrary $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, showing that $\nabla_{\mathbf{A}}^2 \mathcal{A}^+ u_q = \Lambda_{q+1} u_q$, that is $\mathcal{A}^+$ maps functions from $\mathcal{L}_q$ to $\mathcal{L}_{q+1}$. Moreover, by (4.3) it follows that

$$\Lambda_q u_q = \mathsf{A}_0 u_q = (\mathcal{A}_q^- \mathcal{A}_q^+ \ - B) u_q$$

or equivalently

$$(2q+2)Bu = \mathcal{A}_q^- \mathcal{A}_q^+ u,$$

implying that $\mathcal{A}_q^+$ is an injective operator with left inverse $[(2q+2)B]^{-1}\mathcal{A}_q^-$. On the other hand, for arbitrary $u_{q+1} \in \mathcal{L}_{q+1}$, using (4.3) again, we see

$$\Lambda_{q+1} u_{q+1} = \mathsf{A}_0 u_{q+1} = (\mathcal{A}_q^+ \mathcal{A}_q^- + B) u_{q+1},$$

which is equivalent to

$$(2q+2)Bu_{q+1} = \mathcal{A}_q^+ \mathcal{A}_q^- u_{q+1}.$$

In particular, $\mathcal{A}_q^+ : \mathcal{L}_q \to \mathcal{L}_{q+1}$ is surjective with right inverse $[(2q+2)B]^{-1}\mathcal{A}_q^-$ and hence bijective. Moreover, we can conclude from the above calculations that $\mathcal{A}_q^- : \mathcal{L}_{q+1} \to \mathcal{L}_q$ must be bijective as well. Next we are going to show that the mapping $\mathcal{U}_q$ from (4.5) is well-defined and unitary. We have seen that $\mathcal{F}^2$ and $\mathcal{L}_0$ are unitary equivalent via the mapping $\mathcal{F}^2 \ni f \mapsto e^{-\Psi} f \in \mathcal{L}_0$, which together with (4.4) shows that $\mathcal{U}_q : \mathcal{F}^2 \to \mathcal{L}_q$ is a well-defined isomorphism. If we are able to show that the mapping

$$\mathcal{L}_0 \in u_0 \mapsto C_q^{-1}(\mathcal{A}^+)^q u_0 \in \mathcal{L}_q, \quad C_q = \sqrt{q!(2B)^q},$$

is isometric, it follows that $\mathcal{U}_q$ is a unitary operator. To see this let $q \in \mathbb{N}$ and $u_0 \in \mathcal{L}_0$. Now (4.3) implies that

$$(\mathcal{A}^-)^q(\mathcal{A}^+)^q u_0 = (\mathcal{A}^-)^{q-1}[\mathsf{A}_0 + B](\mathcal{A}^+)^{q-1} u_0 = 2Bq(\mathcal{A}^-)^{q-1}(\mathcal{A}^+)^{q-1} u_0,$$

so an inductive arguments yields

$$(\mathcal{A}^-)^q(\mathcal{A}^+)^q u_0 = (2B)^q q! u_0 = C_q^2 u_0.$$

Together with the formal adjointness of $\mathcal{A}^+$ and $\mathcal{A}^-$ on $\mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2)$ this gives us

$$\left\|(\mathcal{A}^+)^q u_0\right\|_{L^2(\mathbb{R}^2)}^2 = ((\mathcal{A}^-)^q(\mathcal{A}^+)^q u_0, u_0)_{L^2(\mathbb{R}^2)} = C_q^2 \|u_0\|_{L^2(\mathbb{R}^2)}^2,$$

which finishes the proof. □

4.2 The Landau Hamiltonian with Dirichlet boundary conditions

In this subsection we will provide a definition and some properties of the Landau Hamiltonian on a domain Ω with Dirichlet boundary conditions. For this we will assume that Ω is either a bounded $\mathcal{C}^{1,1}$ domain or the complement of a bounded $\mathcal{C}^{1,1}$ domain, in which case the boundary $\Sigma = \partial\Omega$ is a compact $\mathcal{C}^{1,1}$ curve. Recall that the magnetic Sobolev space of first order is given by

$$\mathcal{H}^1_{\mathbf{A}}(\Omega) = \{f \in L^2(\Omega) : |\nabla_{\mathbf{A}} f| \in L^2(\mathbb{R}^2)\},$$

where $\nabla_{\mathbf{A}} = i\nabla + \mathbf{A}$, which equipped with the inner product

$$(f, g)_{\mathcal{H}^1_{\mathbf{A}}(\Omega)} := (f, g)_{L^2(\Omega)} + (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\Omega)}.$$

is a Hilbert space. Next, we introduce the symmetric sesquilinear form

$$\mathfrak{a}^{\Omega}_{\mathrm{D}} := (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\Omega)}, \quad \mathrm{dom}(\mathfrak{a}^{\Omega}_{\mathrm{D}}) = \mathcal{H}^1_{\mathbf{A},0}(\Omega),$$

where $\mathcal{H}^1_{\mathbf{A},0}(\Omega)$ is the closure of $\mathcal{C}^\infty_0(\Omega)$ with respect to $\|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)}$ from Definition 3.1. It is easy to see that the above form is densely defined, non-negative and closed in $L^2(\Omega)$. In particular, $\mathfrak{a}^{\Omega}_{\mathrm{D}}$ induces a self-adjoint and non-negtative operator $\mathsf{A}^{\Omega}_{\mathrm{D}}$, which is the Landau Hamiltonian on Ω with Dirichlet boundary conditions on Σ. In the case where Ω is a bounded domain we have $\mathcal{H}^1_{\mathbf{A},0}(\Omega) = H^1_0(\Omega)$ by Lemma 3.4, so the space $\mathcal{H}^1_{\mathbf{A},0}(\Omega)$ is compactly embedded in $L^2(\Omega)$ implying that

$$\sigma_{ess}(\mathsf{A}^{\Omega}_{\mathrm{D}}) = \emptyset.$$

We are going to need the following result in the proof of Proposition 4.16.

Proposition 4.3. *Let A_0 be the Landau Hamiltonian from* (4.1) *and let Σ be the boundary of a bounded $\mathcal{C}^{1,1}$ domain $\Omega_i \subset \mathbb{R}^2$. Then the operator*

$$S := \mathsf{A}_0 \upharpoonright \{f \in \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2) : f|_\Sigma = 0\} \tag{4.6}$$

is densely defined, closed and symmetric. Moreover, for any $q \in \mathbb{N}_0$ there holds

$$\dim\ker(S - \Lambda_q) \le \dim\ker(\mathsf{A}^{\Omega_i}_{\mathrm{D}} - \Lambda_q), \tag{4.7}$$

i.e. the space $\ker(S - \Lambda_q)$ *is finite-dimensional.*

Proof. Since $\mathcal{C}^\infty_0(\mathbb{R}^2\backslash\Sigma)$ lies dense in $L^2(\mathbb{R}^2\backslash\Sigma)$ it follows that S is densely defined. To see that S is a closed operator it suffices to show that $\mathrm{dom}(S)$ is a closed subspace in $\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ with respect to $\|\cdot\|_{\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)}$. For this let $\gamma_{\mathrm{D}} : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to H^{1/2}(\Sigma)$ be the trace operator and $J : \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2) \to \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2), u \mapsto u$ the continuous embedding of $\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. In particular, it follows for $\tilde{\gamma}_{\mathrm{D}} = \gamma_{\mathrm{D}} J : \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2) \to H^{1/2}(\Sigma)$ that $\mathrm{dom}(S) = \tilde{\gamma}_{\mathrm{D}}^{-1}(\{0\})$ is closed in $\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ as the pre-image of a closed set under a continuous operator, so S is closed. Moreover, $S \subset \mathsf{A}_0$ implies the symmetry of S, since A_0 is a self-adjoint operator.

Now let us show the inequality we stated in (4.7). Recall that $\Omega_i \subset \mathbb{R}^2$ is a bounded Lipschitz domain with boundary Σ and $\Omega_e = \mathbb{R}^2 \backslash \overline{\Omega}_i$. Assume that $\dim \ker(\mathsf{A}_{\mathrm{D}}^{\Omega_i}) = k \in \mathbb{N}_0$ and suppose that there exist linearly independent $h_1, \ldots, h_{k+1} \in \ker(S - \Lambda_q)$. Denote $h_j^i = h_j \restriction_{\Omega_i}$ and $h_j^e = h_j \restriction_{\Omega_e}$. Since $h_1, \ldots, h_{k+1} \in \mathrm{dom}(S)$ it follows that $h_j|_\Sigma = 0$ for all $j \in \{1, \ldots, k+1\}$ so $h_1^i, \ldots, h_{k+1}^i \in \ker(\mathsf{A}_{\mathrm{D}}^{\Omega_i} - \Lambda_q)$. In particular, $h_1^i, \ldots, h_{k+1}^i$ must be linearly dependent, so it is no restriction to assume that

$$h_{k+1}^i = \sum_{j=1}^{k} \beta_j h_j^i. \tag{4.8}$$

In the same way it follows that $h_1^e, \ldots, h_{k+1}^e \in \ker(\mathsf{A}_{\mathrm{D}}^{\Omega_e} - \Lambda_q)$ and since $h_j \in \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ it follows from Lemma 3.14 that

$$\partial_\nu h_j^e|_\Sigma = \partial_\nu h_j^i|_\Sigma. \tag{4.9}$$

for all $j \in \{1, \ldots, k+1\}$. Now consider the function

$$g^e := h_{k+1}^e - \sum_{j=1}^{k} \beta_j h_j^e \in \ker(\mathsf{A}_{\mathrm{D}}^{\Omega_e} - \Lambda_q),$$

which by (4.8) and (4.9) satisfies

$$\partial_\nu g_e|_\Sigma = \partial_\nu h_{k+1}^e|_\Sigma - \sum_{j=1}^{k} \beta_k \partial_\nu h_j^e|_\Sigma = \partial_\nu h_{k+1}^i|_\Sigma - \sum_{j=1}^{k} \beta_k \partial_\nu h_j^i|_\Sigma = 0,$$

which means that g_e is a function in in $\ker(\mathsf{A}_{\mathrm{D}}^{\Omega_e} - \Lambda_q)$ such that $g_e|_\Sigma = \partial_\nu g_e|_\Sigma = 0$. In particular, we can extend g_e by zero outside of Ω_e to a function $g \in \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$. Moreover, since $g_e \in \ker(\mathsf{A}_{\mathrm{D}}^{\Omega_e} - \Lambda_q)$ it follows that $g \in \ker(\mathsf{A}_0 - \Lambda_q)$. Now let $\mathcal{U}_q$ be the unitary isomorphism form Proposition 4.2 that maps $\mathcal{F}^2$ onto $\mathcal{L}_q$. Since $g(x) = 0$ for all $x \in \Omega_i$ it follows that $(\mathcal{U}_q g)(x) = 0$ for all $x \in \Omega_i$. Since $\mathcal{U}_q g$ is an entire function this means $\mathcal{U}_q g = 0$ and hence $g = 0$; in particular, $g_e = 0$. We conclude that

$$h_{k+1}^e = \sum_{j=1}^{k} \beta_j h_j^e,$$

which together with (4.8) implies

$$h_{k+1} = \sum_{j=1}^{k} \beta_j h_j,$$

showing that $h_1, \ldots, h_{k+1}$ are linearly dependent - a contradiction, which finishes the proof. $\square$

4.3 The Landau Hamiltonian with singular perturbations

In this subsection we will define and study the Landau Hamiltonian with a δ-potential supported on the boundary Σ of a compact $\mathcal{C}^{1,1}$ domain Ω_i. We are first going to introduce the corresponding sesquilinear form and then use the first representation theorem to show the self-adjointness of the Landau Hamiltonian A_α with a singular interaction given in Definition 4.5, which was stated in Theorem 1.

To start off, we are going to provide the sesquilinear form that will be associated to the Landau Hamiltonian with a δ-potential.

Theorem 4.4. *Let the sesquilinear form* $\mathfrak{a}_\alpha : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \times \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to \mathbb{C}$ *be given by*

$$\mathfrak{a}_\alpha [f, g] = (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\mathbb{R}^2)} + \int_\Sigma \alpha f|_\Sigma \overline{g|_\Sigma} \, \mathrm{d}\sigma, \tag{4.10}$$

where $f, g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ *and* $\alpha \in L^\infty(\Sigma; \mathbb{R})$. *Then* $\mathfrak{a}_\alpha$ *is densely defined, closed and bounded from below. In particular, there exists a self-adjoint operator* $\mathsf{A} : \mathrm{dom}(\mathsf{A}) \to L^2(\mathbb{R}^2)$ *such that*

$$\mathfrak{a}_\alpha [f, g] = (\mathsf{A} f, g)_{L^2(\mathbb{R}^2)}$$

for all $f \in \mathrm{dom}(\mathsf{A})$ *and* $g \in \mathrm{dom}(\mathfrak{a}_\alpha)$.

Proof. Since $\mathcal{C}_0^\infty(\mathbb{R}^2) \subseteq \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$ it follows that $\mathfrak{a}_\alpha$ is densely defined. Let now $\epsilon > 0$. By Proposition 3.10 we can choose $c(\epsilon) > 0$ such that

$$\left| \int_\Sigma \alpha \left| f|_\Sigma \right|^2 \mathrm{d}\sigma \right| \leq \|\alpha\|_{L^\infty(\Sigma)} \left(\epsilon \|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2)} + c(\epsilon) \|f\|^2_{L^2(\mathbb{R}^2)} \right).$$

Since $\epsilon > 0$ can be chosen arbitrarily small, this shows that the form

$$\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \ni f \mapsto \int_\Sigma \alpha \left| f|_\Sigma \right|^2 \mathrm{d}\sigma$$

is relatively bounded with respect to the form $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \ni f \mapsto \|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2)}$ with bound zero. By Theorem 2.3 it follows that $\mathfrak{a}_\alpha$ is closed and bounded from below as sum of the aforementioned forms. The existence of the self-adjoint operator $\mathsf{A} : \mathrm{dom}(\mathsf{A}) \to L^2(\mathbb{R}^2)$ follows from the first representation theorem. □

We will now give a proper definition of the Landau Hamiltonian with a δ-potential.

Definition 4.5. Let $\Omega_i \subset \mathbb{R}^2$ be a bounded $\mathcal{C}^{1,1}$ domain with boundary Σ and set $\Omega_e = \mathbb{R}^2 \backslash \overline{\Omega}_i$. We then introduce the Landau Hamiltonian with a δ-interaction

$$\begin{aligned} &\mathsf{A}_\alpha f := \nabla^2_{\mathbf{A}} f_i \oplus \nabla^2_{\mathbf{A}} f_e, \\ &\mathrm{dom}(\mathsf{A}_\alpha) := \{ f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) : \nabla^2_{\mathbf{A}} f \restriction_{\Omega_{i/e}} \in L^2(\Omega_{i/e}),\ \partial_\nu f_e - \partial_\nu f_i = \alpha f|_\Sigma \}. \end{aligned}$$

Our aim is to prove that the operator A_α is self-adjoint. We will do this by showing that the Landau Hamiltonian with a δ-potential and the self-adjoint operator A induced by the form $\mathfrak{a}_\alpha$ from (4.10) coincide.

Theorem 4.6. *The operator* A_α *from Definition* 4.5 *and the self-adjoint operator* A *corresponding to the form* $\mathfrak{a}_\alpha$ *from* (4.10) *coincide. In particular, the Landau Hamiltonian with a singular interaction is self-adjoint.*

Proof. Let $f \in \operatorname{dom}(\mathsf{A}_\alpha)$ and $g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. Then by Lemma 3.14 there holds

$$\begin{aligned}(\nabla^2_{\mathbf{A}} f, g)_{L^2(\mathbb{R}^2)} &= (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\mathbb{R}^2)} + \langle \partial_\nu f_e - \partial_\nu f_i, g|_\Sigma \rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \\ &= (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g)_{L^2(\mathbb{R}^2)} + \int_\Sigma \alpha f|_\Sigma \overline{g|_\Sigma}\, \mathrm{d}\sigma \\ &= \mathfrak{a}_\alpha [f, g].\end{aligned}$$

Since this is true for any $g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ it follows that $f \in \operatorname{dom}(\mathsf{A})$ and $\mathsf{A}_\alpha f = \mathsf{A} f$.

Conversely, let $f \in \operatorname{dom}(\mathsf{A}) \subseteq \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ and $f_{i/e} = f\restriction_{\Omega_{i/e}}$. Then for any $\varphi \in \mathcal{C}^\infty_0(\Omega_{i/e})$ there holds

$$\begin{aligned}(f_i, \nabla^2_{\mathbf{A}} \varphi)_{L^2(\Omega_{i/e})} &= (f, \nabla^2_{\mathbf{A}} \varphi)_{L^2(\mathbb{R}^2)} \\ &= (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} \varphi)_{L^2(\mathbb{R}^2)} \\ &= \mathfrak{a}_\alpha [f, \varphi] = (\mathsf{A} f, \varphi)_{L^2(\mathbb{R}^2)} = (\mathsf{A} f\restriction_{\Omega_{i/e}}, \varphi)_{L^2(\Omega_{i/e})},\end{aligned}$$

which shows that $\nabla^2_{\mathbf{A}} f\restriction_{\Omega_{i/e}} = \mathsf{A} f\restriction_{\Omega_{i/e}} \in L^2(\Omega_{i/e})$. In particular, $f \in \mathcal{D}$, where $\mathcal{D}$ is defined as in (3.8), so for any $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ we get

$$\begin{aligned}&\langle \alpha f|_\Sigma, \varphi|_\Sigma \rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \\ &\quad = \mathfrak{a}_\alpha [f, \varphi] - (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} \varphi)_{L^2(\mathbb{R}^2)} \\ &\quad = (\mathsf{A} f, \varphi)_{L^2(\mathbb{R}^2)} - (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} \varphi)_{L^2(\mathbb{R}^2)} \\ &\quad = (\mathsf{A} f\restriction_{\Omega_i}, \varphi)_{L^2(\Omega_i)} + (\mathsf{A} f\restriction_{\Omega_e}, \varphi)_{L^2(\Omega_e)} - (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} \varphi)_{L^2(\mathbb{R}^2)} \\ &\quad = (\nabla^2_{\mathbf{A}} f_i, \varphi)_{L^2(\Omega_i)} + (\nabla^2_{\mathbf{A}} f_e, \varphi)_{L^2(\Omega_e)} - (\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} \varphi)_{L^2(\mathbb{R}^2)} \\ &\quad = \langle \partial_\nu f_e - \partial_\nu f_i, \varphi|_\Sigma \rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)},\end{aligned}$$

where we used Lemma 3.14 in the last line, which gives us

$$\langle \partial_\nu f_e - \partial_\nu f_i, \varphi|_\Sigma \rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} = \langle \alpha f|_\Sigma, \varphi|_\Sigma \rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \tag{4.11}$$

for all $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^2)$. Since the trace operator $H^1(\mathbb{R}^2) \ni h \mapsto h|_\Sigma \in H^{1/2}(\Sigma)$ is bounded and surjective and $\mathcal{C}^\infty_0(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, it follows that (4.11) holds true for all $\varphi \in H^{1/2}(\Sigma)$. In particular, $\partial_\nu f_e - \partial_\nu f_i = \alpha f|_\Sigma$ and hence $f \in \operatorname{dom}(\mathsf{A}_\alpha)$ with $\mathsf{A}_\alpha f = \mathsf{A} f$ which shows the reverse inclusion. $\square$

4.4 Stability of the essential spectrum

Recall that the spectrum of the unperturbed Landau Hamiltonian is given by

$$\sigma(\mathsf{A}_0) = \sigma_p(\mathsf{A}_0) = \sigma_{ess}(\mathsf{A}_0) = \bigcup_{q=0}^{\infty}\{\Lambda_q\},$$

where the eigenvalues of infinite multiplicity $\Lambda_q = (2q+1)B$ are called Landau levels. In this subsection we will derive a compact factorization of the resolvent difference

$$W_\lambda = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1}, \tag{4.12}$$

which corresponds to the Krein-type representation that was stated in Theorem 2. An appropriate version of Weyl's theorem then shows the stability of the essential spectrum under singular δ-perturbations.

The derivation of the resolvent difference will make use of the second representation theorem for sesquilinear forms as stated in Theorem 2.5. For ease of notation we will introduce the following operator, which will play in important role in the compact factorization of the resolvent difference of A_0 and A_α.

Lemma 4.7. *Let $\lambda < B$ sufficiently small. Then*

$$G_\alpha(\lambda) := |\alpha|^{1/2}\,\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1/2} : L^2(\mathbb{R}^2) \to L^2(\Sigma) \tag{4.13}$$

is a well-defined, compact operator and there holds

$$((1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))f, f)_{L^2(\mathbb{R}^2)} = \mathfrak{a}_\alpha[(\mathsf{A}_0 - \lambda)^{-1/2} f] - \lambda\|(\mathsf{A}_0 - \lambda)^{-1/2} f\|^2_{L^2(\mathbb{R}^2)},$$

where we defined $J_\alpha = \operatorname{sign}(\alpha)$.

Proof. Since $\lambda < B = \min\sigma(\mathsf{A}_0)$, the operator $(\mathsf{A}_0 - \lambda)^{1/2}$ is well-defined and uniformly positive. By the second representation theorem $\operatorname{dom}(\mathsf{A}_0 - \lambda)^{1/2} = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ which shows that $G_\alpha(\lambda)$ is well-defined. Since $\gamma_{\mathrm{D}} : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to L^2(\Sigma)$ is a compact operator, the first part of the statement follows if we can show that $(\mathsf{A}_0 - \lambda)^{-1/2} : L^2(\mathbb{R}^2) \to \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ is a bounded operator.

So let $f \in L^2(\mathbb{R}^2)$, then the second representation theorem implies

$$\begin{aligned}\|(\mathsf{A}_0 - \lambda)^{-1/2} f\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)} &= \mathfrak{a}_0[(\mathsf{A}_0 - \lambda)^{-1/2} f] + \|(\mathsf{A}_0 - \lambda)^{-1/2} f\|^2_{L^2(\mathbb{R}^2)} \\ &= \|f\|^2_{L^2(\mathbb{R}^2)} + (1+\lambda)\|(\mathsf{A}_0 - \lambda)^{-1/2} f\|_{L^2(\mathbb{R}^2)} \\ &\le C(\lambda)\|f\|^2_{L^2(\mathbb{R}^2)}\end{aligned}$$

for some constant $C = C(\lambda) > 0$, where we used the boundedness of the operator $(\mathsf{A}_0 - \lambda)^{-1/2} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ in the last line.

This shows that the operator $G_\alpha(\lambda)$ is compact with $\|G_\alpha(\lambda)\| \leq C(\lambda)\|\alpha\|_{L^\infty(\Sigma)}$ for some $C(\lambda) > 0$. Furthermore, for any $f \in L^2(\mathbb{R}^2)$ there holds

$$\begin{aligned}
&((1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))f, f)_{L^2(\mathbb{R}^2)} \\
&\quad = \|f\|^2_{L^2(\mathbb{R}^2)} + (J_\alpha G_\alpha(\lambda) f, G_\alpha(\lambda) f)_{L^2(\Sigma)} \\
&\quad = \|f\|_{L^2(\mathbb{R}^2)} + (\alpha(\mathsf{A}_0 - \lambda)^{-1/2} f|_\Sigma, (\mathsf{A}_0 - \lambda)^{-1/2} f|_\Sigma)_{L^2(\Sigma)} \\
&\quad = [\mathfrak{a}_0 - \lambda][(\mathsf{A}_0 - \lambda)^{-1/2} f] + (\alpha(\mathsf{A}_0 - \lambda)^{-1/2} f|_\Sigma, (\mathsf{A}_0 - \lambda)^{-1/2} f|_\Sigma)_{L^2(\Sigma)} \\
&\quad = \mathfrak{a}_\alpha[(\mathsf{A}_0 - \lambda)^{-1/2} f] - \lambda \|(\mathsf{A}_0 - \lambda)^{-1/2} f\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}$$

where we used the second representation theorem in the fourth line, which proves the second statement. □

We are now going to provide two technical results based on the spectral theorem for self-adjoint unbounded operators and the diamagnetic inequality, which we will use to derive an estimate on the norm of $G_\alpha(\lambda)$.

Lemma 4.8. *Let $\lambda < 0$. Then there holds*

$$\left\|(\mathsf{A}_0 - \lambda)^{-\beta} f\right\|^2_{L^2(\mathbb{R}^2)} \leq \frac{1}{|B - \lambda|^{2\beta}} \|f\|^2_{L^2(\mathbb{R}^2)}$$

for all $f \in L^2(\mathbb{R}^2)$ and $\beta > 0$.

Proof. Let E be the spectral measure associated to A_0. Suppose that $f \in L^2(\mathbb{R}^2)$ and $\beta > 0$. Since $\sigma(\mathsf{A}_0) \subset [B, \infty)$ the spectral theorem implies

$$\begin{aligned}
\left\|(\mathsf{A}_0 - \lambda)^{-\beta} f\right\|^2_{L^2(\mathbb{R}^2)} &= \int_B^\infty (t - \lambda)^{-2\beta} \, \mathrm{d}(E(t) f, f) \\
&\leq \frac{1}{|B - \lambda|^{2\beta}} \int_{\mathbb{R}} \mathrm{d}(E(t) f, f) \\
&= \frac{1}{|B - \lambda|^{2\beta}} \|f\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}$$

which shows the stated inequality. □

Since $(\mathsf{A}_0 - \lambda)^{-1/2}$ maps $L^2(\mathbb{R}^2)$ boundedly into $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ it is easy to see that the composition $\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\beta} : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is bounded for $\beta > \frac{1}{2}$. The next lemma uses the diamagnetic inequality to show that this fact remains true for all $\beta > \frac{1}{4}$.

Lemma 4.9. *Let $\lambda < 0$ and $\beta > \frac{1}{4}$. Then $\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\beta} : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is a well-defined and bounded operator.*

Proof. Suppose that $\lambda < 0$ and $\beta > \frac{1}{4}$. Let $-\Delta$ be the free Laplacian in $L^2(\mathbb{R}^2)$ defined on the space $H^2(\mathbb{R}^2)$, then $(-\Delta - \lambda)^{-\beta} : L^2(\mathbb{R}^2) \to H^{2\beta}(\mathbb{R}^2)$ is a well-defined and bounded operator. Moreover, it follows by the diamagnetic inequality (3.1) that for any $f \in L^2(\mathbb{R}^2)$ we have

$$\begin{aligned}\big\|\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\beta} f\big\|^2_{L^2(\Sigma)} &= \int_\Sigma \big|(\mathsf{A}_0 - \lambda)^{-\beta} f\big|^2 \,\mathrm{d}\sigma \le \int_\Sigma \big|(-\Delta - \lambda)^{-\beta}\,|f|\big|^2 \,\mathrm{d}\sigma \\ &= \big\|((-\Delta - \lambda)^{-\beta}\,|f|)|_\Sigma\big\|^2_{L^2(\Sigma)}.\end{aligned}$$

Since the trace operator $H^{2\beta}(\mathbb{R}^2) \ni g \mapsto g|_\Sigma \in L^2(\Sigma)$ is bounded it follows that

$$\big\|((-\Delta - \lambda)^{-\beta}\,|f|)|_\Sigma\big\|^2_{L^2(\Sigma)} \le c_1 \big\|(-\Delta - \lambda)^{-\beta}\,|f|\big\|^2_{H^{2\beta}(\mathbb{R}^2)} \le c_2 \|f\|^2_{L^2(\mathbb{R}^2)},$$

for some constants $c_1, c_2 > 0$, which shows that the operator $\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\beta} : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is well-defined and bounded. $\square$

As $\lambda \to -\infty$ one would expect the norm of $G_\alpha(\lambda)$ to become arbitrarily small. We will now utilize the two results we have just shown to prove that $\|G_\alpha(\lambda)\|$ tends to zero as λ tends to $-\infty$.

Lemma 4.10. *Let $G_\alpha(\lambda) : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ be as in* (4.13) *and let $\lambda < 0$. Then for any $\epsilon \in (0, \frac{1}{4})$ there exists a constant $C > 0$ such that*

$$\|G_\alpha(\lambda)\| \le \frac{C}{|B - \lambda|^{1/4-\epsilon}}$$

Proof. Suppose that $\epsilon \in (0, \frac{1}{4})$. We can then rewrite

$$G_\alpha(\lambda) = |\alpha|^{1/2}\,\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1/2} = |\alpha|^{1/2}\,\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1/4-\epsilon}(\mathsf{A}_0 - \lambda)^{-1/4+\epsilon},$$

so with the help of Lemma 4.8 we obtain

$$\big\|G_\alpha(\lambda)\big\| \le \frac{C}{|B - \lambda|^{1/4-\epsilon}} \big\||\alpha|^{1/2}\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1/4-\epsilon}\big\|_{L^2(\mathbb{R}^2) \to L^2(\Sigma)}.$$

Since the operator $|\alpha|^{1/2}\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1/4-\epsilon} : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ is bounded by Lemma 4.9 the assertion follows. $\square$

We are now able to show that the essential spectrum of the Landau Hamiltonian is stable under singular perturbations. But before we do so we are going to introduce some additional notation.

Definition 4.11. For $\lambda < B$ we introduce the operators

$$\begin{aligned}\gamma(\lambda) &= (\mathsf{A}_0 - \lambda)^{-1}\gamma^*_{\mathrm{D}} : L^2(\Sigma) \to L^2(\mathbb{R}^2), \\ M(\lambda) &= \gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1}\gamma^*_{\mathrm{D}} : L^2(\Sigma) \to L^2(\Sigma).\end{aligned}$$

Note that $\gamma(\lambda)$ as well as $M(\lambda)$ are compact, as the trace map $\gamma_{\mathrm{D}} : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to L^2(\Sigma)$ is compact as well. Moreover, we can use the same strategy as in the proof of Lemma 4.10 to show that for any $\epsilon \in (0, \frac{1}{2})$ there exists a constant $C > 0$ such that

$$\|M(\lambda)\| \leq \frac{C}{|B-\lambda|^{1/2-\epsilon}}. \tag{4.14}$$

We will now show that the resolvent difference of A_0 and A_α admits the compact Krein-type factorization that was stated in Theorem 2.

Theorem 4.12. *Let $\mathcal{H}$ be a Hilbert space and $\alpha \in L^\infty(\Sigma; \mathbb{R})$ such that it can be written as the product $\alpha = \alpha_2\alpha_1$ where $\alpha_1 : L^2(\Sigma) \to \mathcal{H}$ and $\alpha_2 : \mathcal{H} \to L^2(\Sigma)$ are bounded operators. Let $M(\lambda)$ be as in Definition 4.11, then for $\lambda < B$ sufficiently small one has $(1 + \alpha_1 M(\lambda)\alpha_2)^{-1} \in \mathcal{B}(\mathcal{H})$ and the resolvent difference $W_\lambda := (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1}$ admits the compact factorization*

$$W_\lambda = -\gamma(\lambda)\alpha_2(1 + \alpha_1 M(\lambda)\alpha_2)^{-1}\alpha_1\gamma(\lambda)^*. \tag{4.15}$$

In particular, there holds

$$\sigma_{ess}(\mathsf{A}_\alpha) = \sigma_{ess}(\mathsf{A}_0) = \bigcup_{q=0}^{\infty}\{\Lambda_q\}.$$

Proof. Let $\lambda < B$ and set $J_\alpha := \operatorname{sign}(\alpha)$, then $(1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))^{-1} \in \mathfrak{B}(L^2(\Sigma))$ if we choose λ sufficiently small such that $\|G_\alpha(\lambda)\| < 1$, which is possible by Lemma 4.10. Our first goal will be to show that the resolvent difference W_λ admits the factorization

$$W_\lambda = -(\mathsf{A}_0 - \lambda)^{-1/2} G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda)(1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))^{-1}(\mathsf{A}_0 - \lambda)^{-1/2} \tag{4.16}$$

and then rewrite it into the form (4.15). To see (4.16) we will first prove that

$$\lambda < \inf \sigma(\mathsf{A}_\alpha) \tag{4.17}$$

is equivalent to

$$1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda) \geq c \tag{4.18}$$

for some $c > 0$. So assume first $\lambda < \inf \sigma(\mathsf{A}_\alpha)$. Then $(\mathsf{A}_\alpha - \lambda)^{1/2}$ is a self-adjoint and uniformly positive operator with $\operatorname{dom}(\mathsf{A}_\alpha - \lambda)^{1/2} = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. In particular, the associated form $\mathfrak{a}_\alpha - \lambda$ is semibounded from below by a positive constant. Now Lemma 4.7 implies that for any $f \in L^2(\mathbb{R}^2)$ we have

$$\begin{aligned}
((1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))f, f)_{L^2(\mathbb{R}^2)} &= \mathfrak{a}_\alpha[(\mathsf{A}_0 - \lambda)^{-1/2} f] - \lambda\|(\mathsf{A}_0 - \lambda)^{-1/2} f\|^2_{L^2(\mathbb{R}^2)} \\
&= [\mathfrak{a}_\alpha - \lambda][(\mathsf{A}_0 - \lambda)^{-1/2} f] \\
&\geq c\|(\mathsf{A}_0 - \lambda)^{-1/2} f\|^2_{L^2(\mathbb{R}^2)} \\
&\geq c\|f\|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

for some constant $c > 0$, where we also used the fact that $(\mathsf{A}_0 - \lambda)^{-1/2}$ is a uniformly positive operator, showing (4.18). Conversely, suppose that $1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda) \geq c$ in $L^2(\mathbb{R}^2)$ for some $c > 0$ and set $g = (\mathsf{A}_0 - \lambda)^{-1/2} f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. Then by Lemma 4.7

$$
\begin{aligned}
[\mathfrak{a}_\alpha - \lambda][g] &= ((1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))(\mathsf{A}_0 - \lambda)^{1/2} g, (\mathsf{A}_0 - \lambda)^{1/2} g)_{L^2(\mathbb{R}^2)} \\
&\geq c \|(\mathsf{A}_0 - \lambda)^{1/2} g\|^2_{L^2(\mathbb{R}^2)} \\
&\geq c' \|g\|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

for some constant $c' > 0$, since $(\mathsf{A}_0 - \lambda)^{1/2}$ is a uniformly positive operator, which shows the stated equivalence.

Our next step will be to show that the resolvent difference W_λ can be written in the form (4.16). For this assume that $\lambda < B$ is sufficiently small such that $\|G_\alpha(\lambda)\| < 1$, then (4.18) holds and the operator

$$1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

is boundedly invertible since $G_\alpha(\lambda)$ is bounded. We can now define the operator

$$M_\alpha(\lambda) := (1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))^{1/2} (\mathsf{A}_0 - \lambda)^{1/2}, \quad \mathrm{dom}(M_\alpha(\lambda)) = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2),$$

which is closed since its the product of a bounded and bijective operator in $L^2(\mathbb{R}^2)$ and the closed bijective operator $(\mathsf{A}_0 - \lambda)^{1/2} : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. Our aim is to prove that

$$\mathsf{A}_\alpha - \lambda = M_\alpha(\lambda)^* M_\alpha(\lambda), \tag{4.19}$$

where $\mathrm{dom}(M_\alpha(\lambda)^* M_\alpha(\lambda)) = \{f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) : M_\alpha(\lambda) f \in \mathrm{dom}(M_\alpha(\lambda)^*)\}$. For this we define the form

$$\mathfrak{t}[f] := \int_{\mathbb{R}^2} |M_\alpha(\lambda) f|^2 \,\mathrm{d}x, \quad \mathrm{dom}(\mathfrak{t}) = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2),$$

which is non-negative and closed since $M_\alpha(\lambda)$ is closed. Moreover, there holds

$$
\begin{aligned}
\mathfrak{t}[f] &= (M_\alpha(\lambda) f, M_\alpha(\lambda) f)_{L^2(\mathbb{R}^2)} \\
&= ((1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))(\mathsf{A}_0 - \lambda)^{1/2} f, (\mathsf{A}_0 - \lambda)^{1/2} f)_{L^2(\mathbb{R}^2)} \\
&= \mathfrak{a}_\alpha[f] - \lambda \|f\|^2_{L^2(\mathbb{R}^2)} \\
&= [\mathfrak{a}_\alpha - \lambda][f],
\end{aligned}
$$

which shows that $\mathfrak{t} = \mathfrak{a}_\alpha - \lambda$. It remains to prove that the induced self-adjoint operator T of the form $\mathfrak{t}$ and $M_\alpha(\lambda)^* M_\alpha(\lambda)$ coincide. So let first $f \in \mathrm{dom}(M_\alpha(\lambda)^* M_\alpha(\lambda))$ and $g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$, then there holds

$$(M_\alpha(\lambda)^* M_\alpha(\lambda) f, g)_{L^2(\mathbb{R}^2)} = (M_\alpha(\lambda) f, M_\alpha(\lambda) g)_{L^2(\mathbb{R}^2)} = \mathfrak{t}[f, g].$$

Since this is true for arbitrary $g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ we can conclude $f \in \operatorname{dom}(T)$ as well as $Tf = M_\alpha(\lambda)^* M_\alpha(\lambda) f$. Conversely, let $f \in \operatorname{dom}(T)$, then it follows that

$$(Tf, g)_{L^2(\mathbb{R}^2)} = \mathfrak{t}[f, g] = (M_\alpha(\lambda) f, M_\alpha(\lambda) g)_{L^2(\mathbb{R}^2)},$$

which implies $M_\alpha(\lambda) f \in \operatorname{dom}(M_\alpha(\lambda)^*)$ as well as $M_\alpha(\lambda)^* M_\alpha(\lambda) f = Tf$, proving (4.19). By inverting both sides in (4.19) and subtracting $(\mathsf{A}_0 - \lambda)^{-1}$ we get

$$\begin{aligned} W_\lambda &= (\mathsf{A}_0 - \lambda)^{-1/2}(1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))^{-1}(\mathsf{A}_0 - \lambda)^{-1/2} - (\mathsf{A}_0 - \lambda)^{-1} \\ &= -(\mathsf{A}_0 - \lambda)^{-1/2} G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda)(1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))^{-1}(\mathsf{A}_0 - \lambda)^{-1/2} \end{aligned} \tag{4.20}$$

as a first factorization in $L^2(\mathbb{R}^2)$. We now want to use the above factorization to show (4.15). For this choose $\lambda_0 \in (-\infty, 0) \cap \rho(\mathsf{A}_\alpha)$ such that

$$\|\alpha_1\| \|\alpha_2\| \|M(\lambda)\| < 1$$

for all $\lambda < \lambda_0$, which is possible due to (4.14). Then $(1 + \alpha_1 M(\lambda)\alpha_2)^{-1} \in \mathcal{B}(\mathcal{H})$ as well as $(1 + \alpha_2\alpha_1 M(\lambda))^{-1} \in \mathcal{B}(L^2(\Sigma))$. By (4.20) we have

$$\begin{aligned} W_\lambda &= -(\mathsf{A}_0 - \lambda)^{-1/2} G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda)(1 + G_\alpha(\lambda)^* J_\alpha G_\alpha(\lambda))^{-1}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}} \\ &= -(\mathsf{A}_0 - \lambda)^{-1} \gamma_{\mathrm{D}}^* \alpha \gamma_{\mathrm{D}} (\mathsf{A}_0 - \lambda)^{-\frac{1}{2}} \\ &\qquad \times (1 + (\mathsf{A}_0 - \lambda)^{-\frac{1}{2}} \gamma_{\mathrm{D}}^* \alpha \gamma_{\mathrm{D}} (\mathsf{A}_0 - \lambda)^{-\frac{1}{2}})^{-1} (\mathsf{A}_0 - \lambda)^{-\frac{1}{2}}. \end{aligned}$$

On the other hand there holds

$$\begin{aligned} \tilde{W}_\lambda &:= -\, \gamma(\lambda)\alpha_2(1 + \alpha_1 M(\lambda)\alpha_2)^{-1}\alpha_1\gamma(\lambda)^* \\ &= -\, (\mathsf{A}_0 - \lambda)^{-1}\gamma_{\mathrm{D}}^*\alpha_2(1 + \alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1}\gamma_{\mathrm{D}}^*\alpha_2)^{-1}\alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1}. \end{aligned}$$

If we set

$$\begin{aligned} P &= (1 + \alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1}\gamma_{\mathrm{D}}^*\alpha_2)^{-1}, \\ Q &= (1 + (\mathsf{A}_0 - \lambda)^{-\frac{1}{2}}\gamma_{\mathrm{D}}^*\alpha\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}})^{-1}, \end{aligned}$$

we see that

$$\begin{aligned} &\alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}} Q - P\alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}} \\ &\quad = P\Big(\big[1 + \alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-1}\gamma_{\mathrm{D}}^*\alpha_2\big]\alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}} \\ &\qquad\qquad - \alpha_1\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}}\big[1 + (\mathsf{A}_0 - \lambda)^{-\frac{1}{2}}\gamma_{\mathrm{D}}^*\alpha\gamma_{\mathrm{D}}(\mathsf{A}_0 - \lambda)^{-\frac{1}{2}}\big]\Big)Q \\ &\quad = 0, \end{aligned}$$

which proves the statement. By applying an appropriate version of Weyl's theorem the stability of the essential spectrum follows. $\square$

4.5 Analysis of the resolvent difference

In this subsection we will investigate the resolvent difference

$$W_\lambda = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1}, \quad \lambda \in \rho(\mathsf{A}_0) \cap \rho(\mathsf{A}_\alpha),$$

in more detail. For this let $\lambda_0 < \min\{0, \min \sigma(\mathsf{A}_\alpha)\}$ be sufficiently small, such that we attain the compact factorization

$$W_\lambda = -\gamma(\lambda)\alpha_2(1 + \alpha_1 M(\lambda)\alpha_2)^{-1}\alpha_1\gamma(\lambda)^* \tag{4.21}$$

from Theorem 4.12 for all $\lambda \in (-\infty, \lambda_0)$. In particular, we are interested in the definiteness and Schatten-von Neumann property of the resolvent difference.

In order to obtain results on the Schatten-von Neumann property of W_λ we will first study the behaviour of the singular values of $\gamma(\lambda)$. In fact, we obtain the following result.

Proposition 4.13. *Let $\lambda \in \rho(\mathsf{A}_0)$. Then the operator $\gamma(\lambda) \in \mathcal{B}(L^2(\Sigma), L^2(\mathbb{R}^2))$ belongs to the weak Schatten-von Neumann ideal $\mathfrak{S}_{2/3,\infty}(L^2(\Sigma), L^2(\mathbb{R}^2))$.*

Proof. We will show that the adjoint operator $\gamma(\lambda)^* = \gamma_{\mathrm{D}}(\mathsf{A}_0 - \overline{\lambda})^{-1} : L^2(\mathbb{R}^2) \to L^2(\Sigma)$ belongs to the weak Schatten-von Neumann ideal $\mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma))$. First observe that $\operatorname{ran}((\mathsf{A}_0 - \overline{\lambda})^{-1}) = \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$. Moreover, the spaces $\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)$ and $H^1_\Delta(\mathbb{R}^2)$ coincide locally by Lemma 3.4 so we have $\operatorname{ran}(\gamma(\lambda)^*) = \gamma_{\mathrm{D}}(H^1_\Delta(\mathbb{R}^2))$. Furthermore, Lemma 2.34 implies $H^1_\Delta(\mathbb{R}^2) = H^2(\mathbb{R}^2)$ and since Σ is bounded we can conclude $\gamma_{\mathrm{D}}(\mathcal{H}_{\mathbf{A}}(\mathbb{R}^2)) = \gamma_{\mathrm{D}}(H^2(\mathbb{R}^2)) = H^{3/2}(\Sigma)$. With the help of Proposition 2.47, where $k = 1$ and $l = 3$, we now obtain $\gamma(\lambda)^* \in \mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma))$ and hence it follows that $\gamma(\lambda) \in \mathfrak{S}_{2/3,\infty}(L^2(\Sigma), L^2(\mathbb{R}^2))$. □

We are now able to derive the Schatten-von Neumann property for the resolvent difference W_λ by applying the above result to the factorization (4.21).

Proposition 4.14. *Let $\lambda < \min\{0, \min \sigma(\mathsf{A}_\alpha)\}$ and let $\{s_k(W_\lambda)\}_k$ be the singular values of the resolvent difference*

$$W_\lambda = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1}$$

Then there holds $s_k(W_\lambda) = \mathcal{O}(k^{-3})$ and, in particular, W_λ belongs to the weak Schatten-von Neumann ideal $\mathfrak{S}_{p,\infty}(L^2(\mathbb{R}^2))$ for all $p \geq \frac{1}{3}$.

Proof. Using Theorem 4.12 with $\alpha_1 = \alpha$ and $\alpha_2 = 1$ we obtain the factorization

$$W_\lambda = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1} = -\gamma(\lambda)(1 + \alpha M(\lambda))^{-1}\alpha\gamma(\lambda)^*,$$

where $(1 + \alpha M(\lambda))^{-1}\alpha$ is bounded in $L^2(\Sigma)$. Since $\gamma(\lambda) \in \mathfrak{S}_{2/3,\infty}(L^2(\Sigma), L^2(\mathbb{R}^2))$ and $\gamma(\lambda)^* \in \mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma))$ it follows by Lemma 2.44 that

$$(1 + \alpha M(\lambda))^{-1}\alpha\gamma(\lambda)^* \in \mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma)).$$

Thus we can apply Lemma 2.46 with $p = q = \frac{2}{3}$ for the operators $(1 + \alpha M(\lambda))^{-1}\alpha\gamma(\lambda)^*$ and $\gamma(\lambda)$, implying that

$$\gamma(\lambda)(1 + \alpha M(\lambda))^{-1}\alpha\gamma(\lambda)^* \in \mathfrak{S}_{s,\infty}(L^2(\mathbb{R}^2))$$

for any $s \geq \frac{1}{3}$, which proves the assertion. $\square$

The next lemma will provide sign properties for the resolvent difference W_λ.

Lemma 4.15. *Let $\lambda_0 < \inf \sigma(\mathsf{A}_\alpha)$ and $\alpha \in L^\infty(\Sigma; \mathbb{R})$. Then for any $\lambda \in (-\infty, \lambda_0)$ the following holds.*

(i) *If $\alpha(x) \geq 0$ for a.e. $x \in \Sigma$, then $W_\lambda \leq 0$. In the case where $\alpha(x) > 0$ for a.e. $x \in \Sigma$, then $W_\lambda < 0$.*

(ii) *If $\alpha(x) \leq 0$ for a.e. $x \in \Sigma$, then $W_\lambda \geq 0$. In the case where $\alpha(x) < 0$ for a.e. $x \in \Sigma$, then $W_\lambda > 0$.*

Proof. We will only show (i) since the proof for (ii) works analogously. So let $\alpha \geq 0$, then $\mathfrak{a}_\alpha \geq \mathfrak{a}_0$ and hence

$$(\mathsf{A}_\alpha - \lambda)^{-1} \leq (\mathsf{A}_0 - \lambda)^{-1}.$$

This then implies

$$W_\lambda = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1} \leq 0$$

which shows (i). In the case where $\alpha(x) > 0$ for a.e. $x \in \Sigma$, we can use the exact same arguments with strict inequalities, which finishes the proof. $\square$

Recall that for $q \in \mathbb{N}_0$ we denote by $P_q : L^2(\mathbb{R}^2) \to \ker(\mathsf{A}_0 - \Lambda_q)$ the orthogonal projection onto the infinite-dimensional eigenspace corresponding to the Landau level Λ_q of the Landau Hamiltonian. The next lemma consists of the fact that the compression $P_q W_{\lambda_0} P_q$ of the resolvent difference onto $\ker(\mathsf{A}_0 - \Lambda_q)$ is a compact operator of infinite rank.

Proposition 4.16. *Let $\alpha \in L^\infty(\Sigma; \mathbb{R})$ with either $\alpha(x) > 0$ or $\alpha(x) < 0$ for a.e. $x \in \Sigma$. Then there exists $\lambda_0 \in (-\infty, 0)$ such that the operator $P_q W_{\lambda_0} P_q$ has infinite rank.*

Proof. We will show the result for $\alpha > 0$, the proof for $\alpha < 0$ works in a similar fashion. So let $\alpha(x) > 0$ a.e. and choose $\lambda_0 < 0$ such that $\|\alpha\|_\infty \|M(\lambda_0)\| < 1$. By Theorem 4.12 we have

$$-P_q W_{\lambda_0} P_q = P_q \gamma(\lambda_0)\sqrt{\alpha}(1 + \sqrt{\alpha}M(\lambda_0)\sqrt{\alpha})^{-1}\sqrt{\alpha}\gamma(\lambda_0)^* P_q,$$

where W_{λ_0} is compact in $L^2(\mathbb{R}^2)$. Now, let us define the operators

$$C = (1 + \sqrt{\alpha}M(\lambda)\sqrt{\alpha})^{-1}, \quad D = \sqrt{\alpha}C\sqrt{\alpha}.$$

It is clear that C is a non-negative, self-adjoint and bounded operator in $L^2(\Sigma)$ with $0 \in \rho(C)$. Moreover, the operators D and $\sqrt{D}$ are both non-negative and self-adjoint in $L^2(\Sigma)$. In the next step we will show that $0 \notin \sigma_p(D)$ and hence also $0 \notin \sigma_p(\sqrt{D})$. So assume that $D\varphi = 0$ for some $\varphi \in L^2(\Sigma)$, then

$$\begin{aligned}\left|(C\sqrt{\alpha}\varphi, \psi)_{L^2(\Sigma)}\right|^2 &\leq (C\sqrt{\alpha}\varphi, \sqrt{\alpha}\varphi)_{L^2(\Sigma)}(C\psi, \psi)_{L^2(\Sigma)} \\ &= (D\varphi, \varphi)_{L^2(\Sigma)}(C\psi, \psi)_{L^2(\Sigma)} = 0\end{aligned}$$

for all $\psi \in L^2(\Sigma)$ which shows $C\sqrt{\alpha}\varphi = 0$ and hence $\varphi = 0$, so $0 \notin \sigma_p(\sqrt{D})$, implying that $\operatorname{ran}(\sqrt{D})$ is dense in $L^2(\Sigma)$. Now let S be the operator defined in (4.6). We will show that $\operatorname{ran}(P_q\gamma(\lambda_0)\sqrt{D})$ is dense in $\ker(\mathsf{A}_0 - \Lambda_q) \ominus \ker(S - \Lambda_q)$, which is infinite-dimensional by Proposition 4.3.

Assume that $h \in \ker(\mathsf{A}_0 - \Lambda_q) \ominus \ker(S - \Lambda_q)$ satisfies

$$(P_q\gamma(\lambda_0)\sqrt{D}\varphi, h)_{L^2(\mathbb{R}^2)} = 0$$

for all $\varphi \in L^2(\Sigma)$. Then there holds

$$\begin{aligned}0 &= (P_q\gamma(\lambda_0)\sqrt{D}\varphi, h)_{L^2(\mathbb{R}^2)} = (P_q(\mathsf{A}_0 - \lambda_0)^{-1}\gamma_{\mathrm{D}}^*\sqrt{D}\varphi, h)_{L^2(\mathbb{R}^2)} \\ &= \frac{1}{\Lambda_q - \lambda_0}(\sqrt{D}\varphi, \gamma_{\mathrm{D}} h)_{L^2(\Sigma)},\end{aligned}$$

which directly implies $\gamma_{\mathrm{D}} h = 0$ in $L^2(\Sigma)$. Since $h \in \operatorname{dom}(\mathsf{A}_0)$ we get $h \in \operatorname{dom}(S)$ and thus $h \in \ker(S - \Lambda_q)$ implying $h = 0$, which shows the claimed density. We can now rewrite

$$-P_q W_{\lambda_0} P_q = P_q\gamma(\lambda_0)D\gamma(\lambda_0)^* P_q = RR^*,$$

where we have introduced the operator $R = P_q\gamma(\lambda_0)\sqrt{D}$. Since $\ker(RR^*) = \ker(R^*)$ it follows that

$$\overline{\operatorname{ran}(RR^*)} = \overline{\operatorname{ran}(R)}$$

and since $\overline{\operatorname{ran}(R)}$ is infinite dimensional, as we just proved above, it follows directly that $\operatorname{ran}(RR^*) = \operatorname{ran}(P_q W_{\lambda_0} P_q)$ is infinite dimensional as well, so $P_q W_{\lambda_0} P_q$ has infinite rank. $\square$

5 Toeplitz-type operators

In this section we will establish some spectral properties of Toeplitz-type operators related to Landau Hamiltonians. In the following Σ will be the boundary of a bounded $\mathcal{C}^{1,1}$ domain $\Omega_i \subset \mathbb{R}^2$ and $\Gamma \subset \Sigma$ a closed subarc. Typically we are interested in the case where $\Gamma = \operatorname{supp} \alpha$ is the essential support of the interaction strength $\alpha \in L^\infty(\Sigma; \mathbb{R})$ of our δ-perturbation. The aim of this section will be to find exact asymptotics for the singular values of the compressed operators $P_q \delta_\Gamma P_q$ onto the eigenspaces of the Landau Hamiltonian. For this we will first discuss Toeplitz operators on Lipschitz domains in Section 5.1 and then extend our analysis to Toeplitz operators defined on compact $\mathcal{C}^{1,1}$ curves in Section 5.2. Before doing so we will reiterate some properties of the eigenspaces of the Landau Hamiltonian, which we are going to need in the proofs of this section.

Recall that for $q \in \mathbb{N}_0$ we denote by $P_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the orthogonal projection onto the eigenspace $\mathcal{L}_q := \ker(\mathsf{A}_0 - \Lambda_q)$, where $\Lambda_q = B(2q+1)$ are the Landau levels. The Fock space $\mathcal{F}^2$ was introduced in Definition 2.13 as the Hilbert space of all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}} |f(z)|^2 \, e^{-\frac{1}{2}B|z|^2} \, \mathrm{d}m(z) < \infty,$$

which we equipped with the inner product

$$(f, g)_{\mathcal{F}^2} = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\frac{1}{2}|z|^2} \, \mathrm{d}m(z), \quad f, g \in \mathcal{F}^2,$$

and the induced norm $\|\cdot\|_{\mathcal{F}^2} = (\cdot, \cdot)_{\mathcal{F}^2}^{1/2}$. In Proposition 4.2 we have shown that for each $q \in \mathbb{N}_0$ the eigenspaces $\mathcal{L}_q$ of the Landau Hamiltionian are unitarily equivalent to the Fock space via the mapping $\mathcal{U}_q : \mathcal{F}^2 \to \mathcal{L}_q$ given by

$$\mathcal{F}^2 \ni f \mapsto (2B)^{-q/2} (q!)^{-1/2} \left(\mathcal{A}^+\right)^q e^{-\Psi} f \in \mathcal{L}_q,$$

where $\psi(z) = \frac{1}{4}|z|^2$ and $\mathcal{A}^+ : \mathcal{L}_q \to \mathcal{L}_{q+1}$ is the creation operator from (4.2). This unitary equivalence is going to play an important role in the proofs of this section, where we are deriving the exact spectral asymptotics of the singular values of the Toeplitz operators.

5.1 Toeplitz-type operators on Lipschitz domains

In this subsection we will consider Toeplitz-type operators of the form $P_q \chi_K P_q$, where χ_K denotes the characteristic function of a compact set K with Lipschitz boundary. Following the lines of [15] we will see that in this case exact asymptotics on the singular values of $P_q \chi_K P_q$ can be derived.

In the next proposition we introduce the form corresponding to the Toeplitz-type operator $P_q \chi_K P_q$.

Proposition 5.1. *Let $q \in \mathbb{N}_0$ and denote by $\mathrm{d}x$ the Lebesgue measure in $\mathbb{R}^2$ restricted to a bounded Lipschitz domain K. Then the form*

$$\mathfrak{t}_q^K[f] = \int_K |(P_q f)(x)|^2 \,\mathrm{d}x, \quad \mathrm{dom}(\mathfrak{t}_q^K) = L^2(\mathbb{R}^2),$$

is well-defined, symmetric and bounded.

Proof. For $f \in L^2(\mathbb{R}^2)$ there holds the estimate

$$\mathfrak{t}_q^K[f] = \|P_q f\|_{L^2(K)}^2 \le \|P_q f\|_{L^2(\mathbb{R}^2)}^2 \le \|f\|_{L^2(\mathbb{R}^2)}^2$$

which shows that the form $\mathfrak{t}$ is well-defined and bounded. Since $\mathfrak{t}$ is real-valued the symmetry follows straightaway. □

We can now define the Toeplitz operator by applying the first representation theorem to the above form.

Definition 5.2. The bounded and self-adjoint operator in $L^2(\mathbb{R}^2)$ induced by the form $\mathfrak{t}_q^K$ in Proposition 5.1 will be denoted by T_q^K.

Since we are interested in studying the singular values of these Toeplitz-type operators it remains to show that T_q^K is compact. To see this let us write $\mathcal{R} : \mathcal{H}_{\mathbf{A}}(\mathbb{R}^2) \to L^2(K)$ for the bounded restriction operator $\mathcal{R}f = f\restriction_K$. For $f, g \in \mathcal{H}_{\mathbf{A}}^1(\mathbb{R}^2)$ we then get

$$\mathfrak{t}_q^K[f,g] = (P_q f, P_q g)_{L^2(K)} = (\mathcal{R}P_q f, \mathcal{R}P_q g)_{L^2(K)} = ((\mathcal{R}P_q)^*(\mathcal{R}P_q f), g)_{L^2(\mathbb{R}^2)},$$

which yields the representation $T_q^K = (\mathcal{R}P_q)^*(\mathcal{R}P_q)$. Since $\mathcal{R}f \in \mathcal{H}_{\mathbf{A}}(K) \subset H^1(K)$ by Lemma 3.4 and the embedding $H^1(K) \hookrightarrow L^2(K)$ is compact for a Lipschitz domain K it follows that T_q^K is compact as well.

The following proposition from the paper [15] by Filonov and Pushnitski gives us exact asymptotic estimates on the singular values of T_q^K.

Proposition 5.3 ([15, Lemma 1]). *Let $\mathrm{d}x$ be the restriction of the Lebesgue measure onto a bounded Lipschitz domain $K \subset \mathbb{R}^2$. Then for $q \in \mathbb{N}_0$ the eigenvalues of the operator T_q^K satisfy*

$$\lim_{k\to\infty} \left(k! s_k(T_q^K)\right)^{1/k} = \frac{B}{2} \left(\mathrm{Cap}\,(K)\right)^2,$$

where $\mathrm{Cap}\,(K)$ *denotes the logarithmic capacity of K.*

We are not going to prove this result here. Note however, that the proof works very similar as the proof of Proposition 5.8 in the next subsection. In fact, it suffices to replace the Hausdorff measure of the curve Γ in the proof by the restricted Lebesgue measure $\mathrm{d}x$ and one can apply the exact same arguments.

5.2 Toeplitz-type operators on curves

In the following we are going to assume that Γ is a simple curve of finite length with at least Lipschitz regularity. Under this assumptions we can introduce the formal expression $P_q\delta_\Gamma P_q$ via its corresponding quadratic form and show spectral asymptotics for the compression $P_0\delta_\Gamma P_0$ onto the lowest Landau level. After that we will restrict ourselves to the case where Γ is a closed subarc of the boundary Σ of a $\mathcal{C}^{1,1}$ domain. In this particular case we are able to use a reduction to the lowest Landau level to extend the sharp estimates on the singular values of $P_0\delta_\Gamma P_0$ to Toeplitz operators $P_q\delta_\Gamma P_q$ on higher Landau levels as stated in Theorem 3.

In the next proposition we will introduce the form that is associated to the Toeplitz operator $P_q\delta_\Gamma P_q$ and show that it is well-defined.

Proposition 5.4. *Let $q \in \mathbb{N}_0$ and Γ be a simple Lipschitz curve of finite length. Then the form*

$$\mathfrak{t}_q^\Gamma[f] = \int_\Gamma |(P_q f)(x_\Gamma)|^2 \,\mathrm{d}\sigma(x_\Gamma), \quad \mathrm{dom}(\mathfrak{t}_q^\Gamma) = L^2(\mathbb{R}^2),$$

is well-defined, bounded and symmetric.

Proof. By Proposition 3.10 the trace operator $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \ni f \mapsto f|_\Sigma \in L^2(\Sigma)$ is a well-defined and compact map. Moreover, for any $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that

$$\mathfrak{t}_q^\Gamma[f] = \|(P_q f)|_\Gamma\|^2_{L^2(\Gamma)} \le \|(P_q f)|_\Sigma\|^2_{L^2(\Sigma)} \le \epsilon \|\nabla_{\mathbf{A}} P_q f\|^2_{L^2(\mathbb{R}^2)} + c(\epsilon)\|P_q f\|^2_{L^2(\mathbb{R}^2)}$$

for all $f \in L^2(\mathbb{R}^2)$. Furthermore, by the first representation theorem we have

$$\|\nabla_{\mathbf{A}} P_q f\|^2_{L^2(\mathbb{R}^2)} = \mathfrak{a}_0[P_q f, P_q f] = (\mathsf{A}_0 P_q f, P_q f)_{L^2(\mathbb{R}^2)} = \Lambda_q \|P_q f\|^2_{L^2(\mathbb{R}^2)},$$

implying $\mathfrak{t}_q^\Gamma[f] \le c'(\epsilon)\|P_q f\|^2_{L^2(\mathbb{R}^2)}$ for some $c'(\epsilon) > 0$, which shows that t_q^Γ is well-defined and bounded. The symmetry follows since the form is real-valued. $\square$

We can now define the corresponding Toeplitz operator on the curve Γ.

Definition 5.5. The bounded and self-adjoint operator in $L^2(\mathbb{R}^2)$ that is associated to the form $\mathfrak{t}_q^\Gamma$ in Proposition 5.4 will be denoted by T_q^Γ.

In fact, the operator T_q^Γ from the above definition is even compact. Too see this let $\gamma : \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \to L^2(\Gamma)$ be the trace operator, which is compact by Proposition 3.10. Now for any $f, g \in L^2(\mathbb{R}^2)$ we have

$$\mathfrak{t}_q^\Gamma[f, g] = (\gamma P_q f, \gamma P_q g)_{L^2(\Gamma)} = ((\gamma P_q)^*(\gamma P_q f), g)_{L^2(\mathbb{R}^2)},$$

so the operator $T_q^\Gamma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ admits the representation $T_q^\Gamma = (\gamma P_q)^*(\gamma P_q)$, which shows the compactness of the Toeplitz operator.

Recall that the Wirtinger derivatives are given by the pair of complex differential operators $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$.

Proposition 5.6. *Let $q \in \mathbb{N}_0$ and Γ be a simple Lipschitz curve of finite length with Hausdorff measure σ. Let T_q^Γ be the Toeplitz operator from Definition* 5.5*. Suppose that $f \in \mathcal{F}^2$ and $u_q = \mathcal{U}_q f$, where $\mathcal{U}_q : \mathcal{F}^2 \to \mathcal{L}_q$ is the unitary operator from* (4.5)*. Then there holds*

$$(T_q^\Gamma u_q, u_q)_{L^2(\mathbb{R}^2)} = C_q^{-2} \|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\sigma})},$$

where we have defined $\mathrm{d}\tilde{\sigma}(z) = e^{-2\Psi(z)}\,\mathrm{d}\sigma(z)$ *with* $\Psi(z) = \frac{1}{4}B|z|^2$.

Proof. Let $q \in \mathbb{N}_0$, $f \in \mathcal{F}^2$ and set $u_q = \mathcal{U}_q f$. Recall that the spaces $\mathcal{F}^2$ and $\mathcal{L}_q$ are unitarily equivalent via the mapping

$$\mathcal{F}^2 \ni f \mapsto \mathcal{U}_q f = C_q^{-1} \left(\mathcal{A}^+\right)^q e^{-\Psi} f,$$

with $C_q = \sqrt{q!(2B)^q}$, where $\mathcal{A}^+$ is the creation operator from (4.2) given by

$$\mathcal{A}^+ = -2ie^{\Psi}\partial e^{-\Psi} = -2i\partial + \frac{B}{2} i\overline{z}.$$

A direct computation now shows that

$$\begin{aligned}
\mathcal{A}^+ \left(e^{-\Psi} f\right) &= -2ie^{\Psi}\partial \left(e^{-2\Psi} f\right) \\
&= -ie^{\Psi}\left[\partial_1 \left(e^{-2\Psi} f\right) - i\partial_2 \left(e^{-2\Psi} f\right)\right] \\
&= -ie^{-\Psi}\left[-Bx_1 f + \partial_1 f + iBx_2 f - i\partial_2 f\right] \\
&= -ie^{-\Psi}\left(\partial_1 - i\partial_2\right) f + ie^{-\Psi}\left(B(x_1 - ix_2)\right) f \\
&= ie^{-\Psi}\left(-2\partial + B\overline{z}\right) f,
\end{aligned}$$

which by induction implies $(\mathcal{A}^+)^q \left(e^{-\Psi} f\right) = i^q e^{-\Psi}(-2\partial + B\overline{z})^q f$ for all $q \in \mathbb{N}$. Using this identity we get

$$\begin{aligned}
(T_q^\Gamma u_q, u_q)_{L^2(\mathbb{R}^2)} &= \int_\Gamma |(P_q u_q)(x_\Gamma)|^2 \,\mathrm{d}\sigma(x_\Gamma) \\
&= \int_\Gamma |u_q(x_\Gamma)|^2 \,\mathrm{d}\sigma(x_\Gamma) \\
&= \int_\Gamma \left|C_q^{-1} \left(\mathcal{A}^+\right)^q e^{-\Psi(z)} f(z)\right|^2 \mathrm{d}\sigma(z) \\
&= C_q^{-2} \int_\Gamma \left|e^{-\Psi(z)}(-2\partial + B\overline{z})^q f(z)\right|^2 \mathrm{d}\sigma(z) \\
&= C_q^{-2} \|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\sigma})},
\end{aligned}$$

where $\mathrm{d}\sigma(z) = e^{-2\Psi(z)}\,\mathrm{d}\sigma(z)$, which finishes the proof. □

We are now going to provide an exact estimate on the asymptotics of the singular values of T_0^Γ for the case, where Γ is a Lipschitz curve of finite length.

Proposition 5.7. *Let Γ be a simple Lipschitz curve of finite length with Hausdorff measure σ. Then the eigenvalues of the operator T_0^Γ satisfy*

$$\lim_{k\to\infty} \left(k! s_k(T_0^\Gamma)\right)^{1/k} = \frac{B}{2} \left(\mathrm{Cap}\,(\Gamma)\right)^2.$$

Proof. The strategy of the proof will be to show that

$$\left(k! s_k(T_0^\Gamma)\right)^{1/k} = \frac{B}{2} M_k(\Gamma)^{1/k} \left(1 + o(1)\right), \tag{5.1}$$

as $k \to \infty$, where $M_k(\Gamma)$ is given as in (2.12). By Proposition 2.20 there holds

$$\lim_{k\to\infty} M_k(\Gamma)^{1/k} = (\mathrm{Cap}\,(\Gamma))^2,$$

so the statement will then follow by (5.1). For ease of notation we will show the asymptotic esimate under the assumption $B = 2$, the general case will then follow by a linear change of coordinates. So let $f \in \mathcal{F}^2$ and set $u = \mathcal{U}_0 f$, where $\mathcal{U}_0$ is the unitary operator between $\mathcal{F}^2$ and $\mathcal{L}_0$. Then Proposition 5.6 implies

$$(T_0^\Gamma u, u)_{L^2(\mathbb{R}^2)} = \|f\|^2_{L^2(\tilde{\sigma})} = \int_\Gamma |f(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z). \tag{5.2}$$

This means that the quadratic form of the operator $T_0^K \restriction_{\mathcal{L}_0}$ is unitarily equivalent to the form

$$\mathcal{F}^2 \ni f \mapsto \int_\Gamma |f(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z),$$

implying that the non-zero eigenvalues $s_k(T_0^\Gamma)$, $k \in \mathbb{N}$, of T_0^Γ coincide with the singular values of the embedding operator

$$\mathcal{F}^2 \hookrightarrow L^2(\Gamma, e^{-|z|^2} \, \mathrm{d}\sigma(z)).$$

In particular, by applying the min-max principle, we get the following representations for the singular values:

$$s_{k+1}(T_0^\Gamma) = \inf_{L_k^+ \subset \mathcal{F}^2} \sup_{f \in L_k^+ \setminus \{0\}} \frac{\int_\Gamma |f(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z)}{\|f\|^2_{\mathcal{F}^2}}, \quad \mathrm{codim}\ L_k^+ = k, \tag{5.3}$$

$$s_{k+1}(T_0^\Gamma) = \sup_{L_k^- \subset \mathcal{F}^2} \inf_{f \in L_k^- \setminus \{0\}} \frac{\int_\Gamma |f(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z)}{\|f\|^2_{\mathcal{F}^2}}, \quad \dim\ L_k^- = k+1. \tag{5.4}$$

Upper bound: Let $\{p_k\}$ be the sequence of monic polynomials (i.e. the leading coefficient of each p_k is equal to 1) with $\deg(p_k) = k$, that are orthogonal with respect to $\mathrm{d}\sigma(z)$. For $k \in \mathbb{N}$ we fix the polynomial p_k and introduce the space

$$L_k^+ = \{f \in \mathcal{F}^2 : f(z) = p_k(z) g(z), \text{ g is entire function}\}.$$

That is, L_k^+ consists of those functions in $\mathcal{F}^2$ that admit a representation $f(z) = p_k(z)g(z)$ for some entire function g. In particular, we have $\operatorname{codim} L_k^+ = k$. To see this we define the polynomials

$$q_l(z) = \begin{cases} z^l, & l \in \{0, \cdots, k-1\} \\ z^{k-l} p_k(z), & l \geq k \end{cases},$$

which form a basis in $\mathcal{F}^2$. Then for any $f \in \mathcal{F}^2$ it follows that

$$f(z) = \sum_{l=0}^{\infty} a_l q_l(z) = \sum_{l=0}^{k-1} a_l q_l(z) + \sum_{l=k}^{\infty} a_l q_l(z) = \sum_{l=0}^{k-1} a_l z^l + p_k(z) \sum_{l=0}^{\infty} a_{l+k} z^l,$$

where $\{a_l\} \subset \mathbb{C}$. This now implies that

$$\mathcal{F}^2 / L_k^+ = \{f + L_k^+ : f \in \mathcal{F}^2\} = \left\{ \sum_{l=0}^{k-1} a_l z^l + L_k^+ : a_0, \cdots, a_{k-1} \in \mathbb{C} \right\},$$

which shows that $\operatorname{codim} L_k^+ = k$. So let now $\epsilon \in (0, \frac{1}{4})$ and define $R_0 = \max_{z \in \Gamma} |z|$, we will now show that there exists a $K \in \mathbb{N}$ such that

$$\sup_{|z| \leq R_0} |g(z)|^2 \leq (1-\epsilon)^{-2k} \frac{1}{k!} \|p_k g\|_{\mathcal{F}^2}^2 \tag{5.5}$$

for all $k \geq K$ and arbitrary $f = p_k g \in L_k^+$. By Cauchy's integral formula we have

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{p_k(\zeta)(\zeta - z)} \,\mathrm{d}\zeta$$

for all $r > R_0$, where the divisor can't become 0, since all zeros $z_1, \ldots, z_k$ of p_k are contained within a disk of radius R_0 by Lemma 2.17. Setting $R = R_0/\epsilon$ we get the estimate

$$|p_k(\zeta)| = \prod_{l=1}^{k} |\zeta - z_l| \geq \prod_{l=1}^{k} (|\zeta| - |z_l|) \geq ((1-\epsilon)r)^k,$$

if $|\zeta| = r \geq R$. From this it follows that

$$\begin{aligned} \sup_{|z| \leq R_0} |g(z)|^2 &\leq \frac{1}{4\pi^2} \left| \int_{|\zeta|=r} \frac{f(\zeta)}{p_k(\zeta)(\zeta - z)} \,\mathrm{d}\zeta \right|^2 \\ &\leq \frac{r}{2\pi} \int_{|\zeta|=r} \left| \frac{f(\zeta)}{p_k(\zeta)(\zeta - z)} \right|^2 \mathrm{d}|\zeta| \\ &\leq \frac{r^{-2k-1}}{2\pi(1-\epsilon)^{2k+2}} \int_{|\zeta|=r} |f(\zeta)|^2 \,\mathrm{d}|\zeta|. \end{aligned}$$

Multiplying both sides by $e^{-r^2}r^{2k+1}$ and integrating over r from R to ∞ yields

$$\begin{aligned}\sup_{|z|\le R_0}|g(z)|^2\int_R^\infty e^{-r^2}r^{2k+1}\,\mathrm{d}r &\le \frac{1}{2\pi(1-\epsilon)^{2k+2}}\int_R^\infty\int_{|\zeta|=r}|f(\zeta)|^2\,e^{-r^2}\,\mathrm{d}|\zeta| \\ &\le \frac{1}{2\pi(1-\epsilon)^{2k+2}}\|f\|^2_{\mathcal{F}^2}.\end{aligned}\tag{5.6}$$

On the other hand we have

$$\begin{aligned}\int_R^\infty e^{-r^2}r^{2k+1}dr &= \frac{1}{2}k! - \int_0^R e^{-r^2}r^{2k+1}\,\mathrm{d}r \\ &= \frac{1}{2}k! - \frac{1}{2}\int_0^{R^2} e^{-r}r^k\,\mathrm{d}r \\ &\ge \frac{1}{2}k! - \frac{1}{2}R^2\int_0^{R^2} e^{-r}r^{k-1}\,\mathrm{d}r \\ &\ge \frac{1}{2}k! - \frac{1}{2}R^2(k-1)! \\ &= \frac{1}{2}k!\left(1-\frac{R^2}{k}\right) \ge \frac{1}{2\pi}(1-\epsilon)^{-2}k!,\end{aligned}$$

for all sufficiently large k since $\frac{1}{2\pi}(1-\epsilon)^{-2}\le\frac{1}{3}$ for $\epsilon\in(0,\frac{1}{4})$. This in conjunction with (5.6) yields

$$\frac{1}{2\pi}(1-\epsilon)^{-2}k!\sup_{|z|\le R_0}|g(z)|^2 \le \frac{1}{2\pi(1-\epsilon)^{2k+2}}\|f\|^2_{\mathcal{F}^2},$$

which after some rearrangements shows (5.5). Using this inequality on $f=p_kg\in L_k^+$ we obtain

$$\begin{aligned}\int_\Gamma |f(z)|^2\,e^{-|z|^2}\,\mathrm{d}\sigma(z) &\le \|g\|^2_{\mathcal{C}(\Gamma)}\int_\Gamma |p_k(z)|^2\,e^{-|z|^2}\,\mathrm{d}\sigma(z) \\ &\le \|g\|^2_{\mathcal{C}(\Gamma)}M_k(\Gamma) \\ &\le M_k(\Gamma)(1-\epsilon)^{-2k}\frac{\|f\|^2_{\mathcal{F}^2}}{k!},\end{aligned}$$

where the second inequality follows from

$$M_k(\Gamma)=\int_\Gamma |p_k(z)|^2\,\mathrm{d}\sigma(z)\ge\int_\Gamma |p_k(z)|^2\,e^{-|z|^2}\,\mathrm{d}\sigma(z).$$

Using the last inequality together with the representation (5.3) of $s_k(T_0^\Gamma)$ yields

$$(k!s_{k+1}(T_0^\Gamma))^{1/k}\le(1-\epsilon)^{-2}M_k(\Gamma)^{1/k}\tag{5.7}$$

for all sufficiently large k, which shows the upper bound.

Lower Bound: Let L_k^- be the set of all polynomials in z of degree $\leq k$. As above we set $R_0 = \max_{z\in\Gamma} |z|$ and $R = R_0/\epsilon$ for $\epsilon > 0$. To show the lower bound, we will make use of the norm

$$|||f|||_{\mathcal{F}^2}^2 = \int_{|z|\geq R} |f(z)|^2 \, e^{-|z|^2} \, \mathrm{d}m(z),$$

which by Lemma 2.14 is equivalent to $\|\cdot\|_{\mathcal{F}^2}$. Let $q_k \in L_k^- \backslash \{0\}$ be the monic polynomial, which minimizes the ratio

$$\frac{\int_\Gamma |q_k(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z)}{|||q_k|||_{\mathcal{F}^2}^2}.$$

Set $l = \deg q_k \leq k$, we will show that all zeroes $z_1, \dots, z_l$ of q_k are contained in the disk $\{z \in \mathbb{C} : |z| \leq R_0\}$. Suppose that one of the zeros z_1 lies outside of the disk. Consider the modified polynomial

$$\tilde{q}_k(z) = q_k(z) \, |z_1| \, \frac{z - R_0^2/\overline{z}_1}{R_0(z - z_1)},$$

which clearly lies in $L_k^- \backslash \{0\}$. By Lemma 2.21 there holds

$$\frac{|z_1| \left| z - R_0^2/\overline{z}_1 \right|}{R_0 \, |z - z_1|} \leq 1$$

for $|z| \leq R_0$ and

$$\frac{|z_1| \left| z - R_0^2/\overline{z}_1 \right|}{R_0 \, |z - z_1|} \geq 1$$

for $|z| \geq R_0$. In particular, it follows that

$$\frac{\int_\Gamma |\tilde{q}_k(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z)}{|||\tilde{q}_k|||_{\mathcal{F}^2}^2} < \frac{\int_\Gamma |q_k(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z)}{|||q_k|||_{\mathcal{F}^2}^2},$$

since $\Gamma \subset \{z \in \mathbb{C} : |z| \leq R_0\}$ - a contradiction. Thus the estimate

$$|q_k(z)| = \prod_{j=1}^{l} |z - z_j| \leq \prod_{j=1}^{l} |z| \left(1 + \frac{|z_j|}{|z|}\right) \leq |z|^l \, (1+\epsilon)^l$$

holds true for $|z| \geq R$. From this it follows that

$$\begin{aligned}
|||q_k|||_{\mathcal{F}^2}^2 &= \int_{|z|\geq R} |q_k(z)|^2 \, e^{-|z|^2} \, \mathrm{d}m(z) \\
&\leq \int_{|z|\geq R} |z|^{2l} \, (1+\epsilon)^{2l} e^{-|z|^2} \, \mathrm{d}m(z) \\
&\leq (1+\epsilon)^{2l} \int_{\mathbb{C}} |z|^{2l} \, e^{-|z|^2} \, \mathrm{d}m(z) = (1+\epsilon)^{2l} \pi l!.
\end{aligned}$$

On the other hand there holds

$$\int_\Gamma |q_k(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z) \geq e^{-R_0^2} \int_\Gamma |q_k(z)|^2 \, \mathrm{d}\sigma(z) \geq e^{-R_0^2} M_l(\Gamma),$$

where $M_l(\Gamma)$ is defined as in (2.12). Using the representation in (5.4) we obtain the lower bound

$$\begin{aligned} s_{k+1}(T_0^\Gamma) &\geq \inf_{q \in L_k^- \setminus \{0\}} \frac{\int_\Gamma |q(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\sigma(z)}{C(R) \|\!|q|\!\|_{\mathcal{F}^2}^2} \\ &= \min_{0 \leq l \leq k} \frac{\int_\Gamma |q_l(z)|^2 e^{-|z|^2} \, \mathrm{d}\sigma(z)}{C(R) \|\!|q_l|\!\|_{\mathcal{F}^2}^2} \\ &\geq \frac{e^{-R_0^2}}{\pi C(R)} \min_{0 \leq l \leq k} \frac{M_l(\Gamma)}{(1+\epsilon)^{2l} l!}, \end{aligned} \tag{5.8}$$

where $C(R) > 0$ is chosen such that $\|\cdot\|_{\mathcal{F}^2} \leq C(R) \|\!|\cdot|\!\|_{\mathcal{F}^2}$. In the next step we will show that for k sufficiently large, the minimum in (5.8) is attained at $l = k$. Since for any $l \in \mathbb{N}_0$ we have $zp_l \in \mathcal{P}_{l+1}$ we see

$$M_{l+1}(\Gamma) = \inf_{p \in \mathcal{P}_{l+1}} \int_\Gamma |p(z)|^2 \, \mathrm{d}\mu(z) \leq \int_\Gamma |zp_l(z)|^2 \, \mathrm{d}\mu(z) \leq R_0^2 M_l(\Gamma).$$

So for any $l \in \{1, \dots, k\}$ we obtain

$$\begin{aligned} \frac{M_k(\Gamma)}{(1+\epsilon)^{2k} k!} &\leq \frac{R_0^{2l} M_{k-l}(\Gamma)}{(1+\epsilon)^{2k} k!} \\ &= \frac{R_0^{2l} (k-l)!}{(1+\epsilon)^{2l} k!} \frac{M_{k-l}(\Gamma)}{(1+\epsilon)^{2k-2l} (k-l)!}. \end{aligned}$$

Let us set $a_l^{(k)} = \frac{R_0^{2l}(k-l)!}{(1+\epsilon)^{2l} k!}$. We will show that $a_l^{(k)} < 1$ for any $1 \leq l \leq k$ if k is sufficiently large, which then implies that the minimum in (5.8) is attained at $l = k$.

If $R_0 \leq 1 + \epsilon$ the statement is clear, so assume that $R_0 > 1 + \epsilon$. Choose $K \geq \left(\frac{R_0}{1+\epsilon}\right)^4$ and let $k > 2K$, we will show

$$k(k-1) \dots (k-l+1) \geq \left(\frac{R_0}{1+\epsilon}\right)^{2l}$$

for all $l \in \{1, \dots, k\}$. Indeed for $l \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ we have

$$k(k-1) \dots (k-l+1) \geq K^l \geq \left(\frac{R_0}{1+\epsilon}\right)^{4l} > \left(\frac{R_0}{1+\epsilon}\right)^{2l}.$$

For $l \in \{\lceil \frac{k}{2} \rceil, \dots, k\}$ we see that

$$\begin{aligned} k(k-1)\dots(k-l+1) &\geq k(k-1)\dots(k-\lceil \frac{k}{2} \rceil+1) \\ &\geq K^{k/2} \\ &> \left(\frac{R_0}{1+\epsilon}\right)^{2k} \geq \left(\frac{R_0}{1+\epsilon}\right)^{2l}, \end{aligned}$$

which shows that $a_l^{(k)} < 1$ for all $1 \leq l \leq k$ if k is sufficiently large. It follows with (5.8) that

$$\begin{aligned} s_{k+1}(T_0^\Gamma) &\geq C \min_{0 \leq l \leq k} \frac{M_l(\Gamma)}{(1+\epsilon)^{2l} l!} \\ &= C \frac{M_k(\Gamma)}{(1+\epsilon)^{2k} k!} \end{aligned}$$

for some constant $C > 0$ and all sufficiently large k. In particular,

$$(k! s_{k+1}(T_0^\Gamma))^{1/k} \geq C^{1/k} \frac{M_k(\Gamma)^{1/k}}{(1+\epsilon)^2} \geq (1+\epsilon)^{-3} M_k(\Gamma)^{1/k}$$

for k large enough since $\lim_{k\to\infty} C^{1/k} = 1$ for any $C > 0$. Together with the upper bound (5.7) we have

$$(1+\epsilon)^{-3} M_k(\Gamma)^{1/k} \leq (k! s_{k+1}(T_0^\Gamma))^{1/k} \leq (1-\epsilon)^{-2} M_k(\Gamma)^{1/k}$$

for arbitrary $\epsilon \in \left(0, \frac{1}{4}\right)$. Since

$$\lim_{k\to\infty} M_k(\Gamma)^{1/k} = (\mathrm{Cap}\,(\Gamma))^2$$

by Proposition 2.20 the statement follows. $\square$

We will now show the asymptotic estimate for the case $q \geq 1$ by a reduction to the lowest Landau level, using Proposition 5.6. The next result, which corresponds to Theorem 3 of the Introduction, can be shown under the additional assumption that Γ is a closed subarc of the boundary Σ of a $\mathcal{C}^{1,1}$ domain Ω.

Proposition 5.8. *Let Σ be the boundary of a $\mathcal{C}^{1,1}$ domain Ω with Hausdorff measure σ. Suppose that $\Gamma \subset \Sigma$ is a closed subarc with positive measure. Then for any $q \in \mathbb{N}_0$ the eigenvalues of the operator T_q^Γ from Definition* 5.5 *satisfy*

$$\lim_{k\to\infty} \left(k! s_k(T_q^\Gamma)\right)^{1/k} = \frac{B}{2} (\mathrm{Cap}\,(\Gamma))^2 .$$

Proof. For $q = 0$ the result has already been shown in Proposition 5.7. In the case where $q \geq 1$ we apply the reduction to the lowest Landau level.

So let $q \in \mathbb{N}$ and set $u_q = \mathcal{U}_q f$ for $f \in \mathcal{F}^2$, where $U_q : \mathcal{F}^2 \to \mathcal{L}_q$ is the unitary mapping from (4.5). By Proposition 5.6 there holds

$$(T_q^\Gamma u_q, u_q)_{L^2(\mathbb{R}^2)} = C_q^{-2} \|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\sigma})}, \tag{5.9}$$

where $\mathrm{d}\tilde{\sigma}(z) = e^{-2\Psi(z)}\,\mathrm{d}\sigma(z)$ with $\Psi(z) = \frac{1}{4}B|z|^2$. We are going to separately prove the estimate for the upper and lower bound of the quadratic form in (5.9).

Upper bound: Consider the open δ-neighborhood $\Gamma_\delta = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Gamma) < \delta\}$ for $\delta > 0$ sufficiently small. By Lemma 2.12 for each $q \in \mathbb{N}$ there exists a constant $c_1 > 0$ such that such that

$$\sup_{z\in\Gamma} |\partial^q f(z)|^2 \le c_1 \delta^{-2q-3} \int_{\Gamma_\delta} |f(z)|^2\,\mathrm{d}m(z).$$

In particular, it follows that

$$\|\partial^q f\|^2_{L^2(\tilde{\sigma})} = \int_\Gamma |\partial^q f(z)|^2 e^{-2\Psi(z)}\,\mathrm{d}\sigma(z) \le c_2 \delta^{-2q-3} \int_{\Gamma_\delta} |f(z)|^2\,\mathrm{d}\sigma(z),$$

which together with Leibniz' formula yields

$$\|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\sigma})} \le c_3 \delta^{-2q-3} \int_{\Gamma_\delta} |f(z)|^2\,\mathrm{d}m(z) \tag{5.10}$$

for some constants $c_2, c_3 > 0$. On the other hand there holds $u_0 = e^{-\Psi} f$ so we obtain

$$(T_0^{\Gamma_\delta} u_0, u_0)_{L^2(\mathbb{R}^2)} = \int_{\Gamma_\delta} |f(z)|^2\, e^{-2\Psi(z)}\,\mathrm{d}m(z) \ge e^{-2\Psi(r_0)} \int_{\Gamma_\delta} |f(z)|^2\,\mathrm{d}m(z),$$

where $r_0 = \max_{z\in\Gamma_\delta}|z|$ and $T_0^{\Gamma_\delta}$ is the Toeplitz operator from Definition 5.2. The last inequality together with (5.9) and (5.10) yields

$$(T_q^\Gamma u_q, u_q)_{L^2(\mathbb{R}^2)} \le c_4 \delta^{-2q-3} (T_0^{\Gamma_\delta} u_0, u_0)_{L^2(\mathbb{R}^2)},$$

where $c_4 > 0$ does not depend on δ. By the min-max principle it follows that the singular values of the operators satisfy

$$s_k(T_q^\Gamma) \le c\delta^{-2q-3} s_k(T_0^{\Gamma_\delta})$$

for $k \in \mathbb{N}$ and some constant $c > 0$. In particular, we can apply Proposition 5.3 and get

$$\limsup_{k\to\infty} (k! s_k(T_q^\Gamma))^{1/k} \le \limsup_{k\to\infty} (c\delta^{-2q-3} k! s_k(T_0^{\Gamma_\delta}))^{1/k} = \frac{B}{2} (\mathrm{Cap}\,(\Gamma_\delta))^2,$$

where we have used that $\limsup_{k\to\infty} a^{1/k} = 1$ for any $a > 0$. By the continuity of the capacity we have $\mathrm{Cap}\,(\Gamma_\delta) \to \mathrm{Cap}\,(\Gamma)$ as $\delta \to 0^+$, which shows the upper bound.

Lower bound: Let $\gamma : [0,s] \to \mathbb{C}$ be a natural parametrization of Γ. For $f \in \mathcal{F}^2$ and $q \in \mathbb{N}$ we set $u_q = \mathcal{U}_q f$. Since f is analytic there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = (\partial f)(\gamma(t))\dot{\gamma}(t),$$

where $|\dot{\gamma}(t)| = 1$. Given an arbitrary $\beta > 0$ and integer $q \geq 1$, our first aim is to construct subspaces $N(\beta, q) \subset \mathcal{F}^2$ of finite codimension, such that for any f in the subspace there holds

$$\int_\Gamma \left|\partial^k f(z)\right|^2 \mathrm{d}\tilde{\sigma}(z) \leq \beta^2 \int_\Gamma \left|\partial^q f(z)\right|^2 \mathrm{d}\tilde{\sigma}(z)$$

for all $k \in \{0, 1, \dots, q-1\}$. To do so we will first construct appropriate subspaces in $H^1(0,s)$ as follows. First we introduce the densely defined, closed and non-negative form

$$\mathfrak{m}[u,v] := \int_0^s u'(t)\overline{v'(t)}\,\mathrm{d}t, \quad \mathrm{dom}(\mathfrak{m}) = H^1(0,s),$$

which corresponds to the one-dimensional Laplacian $\mathfrak{M}u = -u''$ with Neumann boundary conditions. Since the embedding $H^1(0,s) \hookrightarrow L^2(0,s)$ is compact, it follows that $\mathfrak{M}$ has a purely discrete spectrum of eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ that accumulate at infinity. Let $\{u_l\}$ be the sequence of eigenvalues of $\mathfrak{M}$ corresponding to $\{\lambda_l\}$. For $v_1, \dots, v_l \in L^2(0,s)$ we define the set

$$\mathcal{U}(v_1, \dots, v_l) = \{v \in H^1(0,s) : v \in \mathrm{span}\{v_1, \dots, v_l\}^\perp\}.$$

By the min-max principle there holds

$$\lambda_l = \inf_{v \in \mathcal{U}(u_1, \dots, u_{l-1})} \frac{\mathfrak{m}(v,v)}{\|v\|^2_{L^2(0,s)}} \leq \frac{\mathfrak{m}(u,u)}{\|u\|^2_{L^2(0,s)}}$$

for all $u \in \mathcal{U}(u_1, \dots, u_{l-1})$. This means that for $l \geq 2$ we get

$$\|u\|^2_{L^2(0,s)} \leq \frac{1}{\lambda_l}\|u'\|^2_{L^2(0,s)}$$

if $u \in \mathcal{U}(u_1, \dots, u_{l-1})$. Since $f \circ \gamma \in H^1(0,s)$ for $f \in \mathcal{F}^2$ we can conclude that any $f \in \mathcal{F}^2$ such that $h := f \circ \gamma \in \mathrm{span}\{v_1, \dots, v_l\}^\perp$ satisfies

$$\begin{aligned}
\int_\Gamma |f(z)|^2 \mathrm{d}\tilde{\sigma}(z) &\leq \int_\Gamma |f(z)|^2 \mathrm{d}\sigma(z) \\
&= \int_0^s |h(t)|^2 \mathrm{d}t \\
&\leq \lambda_l^{-1} \int_0^s \left|h'(t)\right|^2 \mathrm{d}t \\
&\leq \lambda_l^{-1} e^{\frac{B}{2}R_0^2} \int_0^s \left|h'(t)\right|^2 e^{-2\Psi(\gamma(t))}\,\mathrm{d}t \\
&= \lambda_l^{-1} e^{\frac{B}{2}R_0^2} \int_\Gamma |\partial f(z)|^2 \mathrm{d}\tilde{\sigma}(z),
\end{aligned}$$

where $R_0 = \max_{z\in\Gamma} |z|$. Moreover, $\lambda_l^{-1} \to 0$ as $l \to \infty$, hence we can choose $l = l(\beta) \in \mathbb{N}$ sufficiently large such that

$$\int_\Gamma |f(z)|^2 \,\mathrm{d}\tilde{\sigma}(z) \le \beta^2 \int_\Gamma |\partial f(z)|^2 \,\mathrm{d}\tilde{\sigma}(z)$$

if $f \circ \gamma \in \operatorname{span}\{v_1, \dots, v_l\}^\perp$. Using an inductive argument it follows that

$$\int_\Gamma |\partial^k f(z)|^2 \,\mathrm{d}\tilde{\sigma}(z) \le \beta^2 \int_\Gamma |\partial^q f(z)|^2 \,\mathrm{d}\tilde{\sigma}(z) \quad k \in \{0, 1, \dots, q-1\} \tag{5.11}$$

for all $f \in \mathcal{F}^2$ such that $\partial^k f \circ \gamma \in \mathcal{U}(u_1, \dots, u_{l-1})$ for every $k \in \{0, 1, \dots, q-1\}$. With this observation in mind we introduce the subspace

$$N(\beta, q) := \{f \in \mathcal{F}^2 : \partial^k f \circ \gamma \in \operatorname{span}\{v_1, \dots, v_{l(\beta)}\}^\perp \quad \forall k \in \{0, \dots, q-1\}\},$$

where $l = l(\beta) \in \mathbb{N}$ is chosen such that (5.11) holds. We will now show that the space $N(\beta, q)$ has a finite codimension in $\mathcal{F}^2$. To do so fix $l \in \mathbb{N}$ and define the spaces

$$N_k = \{f \in \mathcal{F}^2 : \partial^k f \circ \gamma \in \operatorname{span}\{v_1, \dots, v_l\}^\perp\}$$

for $k \in \{0, \dots, q-1\}$. Note that there holds

$$N(\beta, q) = \bigcap_{k=0}^{q-1} N_k,$$

so it suffices to show that each N_k has a finite codimension in $\mathcal{F}^2$. To see this let $\Pi_l : H^1(0, s) \to \operatorname{span}\{v_1, \dots, v_{l(\beta)}\}$ be the orthogonal projection of $H^1(0, s)$ onto the subspace $\operatorname{span}\{v_1, \dots, v_{l(\beta)}\}$ and define the mappings

$$B_k : \begin{cases} \mathcal{F}^2 \to \operatorname{span}\{v_1, \dots, v_{l(\beta)}\} \\ f \mapsto \Pi_l(\partial^k f \circ \gamma) \end{cases} \quad \text{for } k \in \{0, 1, \dots, q-1\}.$$

By construction $B_k f = 0$ implies $f \in N_k$, so $\ker(B_k) = N_k$. It follows by the fundamental theorem on homomorphisms that the spaces $\mathcal{F}^2/\ker(B_k)$ and $\operatorname{ran}(B_k)$ are isomorphic. Since the dimension of $\operatorname{ran}(B_k)$ cannot exceed l it follows that that the codimension of N_k in $\mathcal{F}^2$ must be finite. By Proposition 2.2

$$\operatorname{codim}(N(\beta, q)) = \operatorname{codim}\left(\bigcap_{k=0}^{q-1} N_k\right) \le \sum_{k=0}^{q-1} \operatorname{codim}(N_k) < \infty,$$

showing that $N(\beta, q)$ is space of finite codimension in $\mathcal{F}^2$ with the property

$$\int_\Gamma |\partial^k f(z)|^2 \,\mathrm{d}\tilde{\sigma}(z) \le \beta^2 \int_\Gamma |\partial^q f(z)|^2 \,\mathrm{d}\tilde{\sigma}(z), \quad k \in \{0, 1, \dots, q-1\} \tag{5.12}$$

for all $f \in N(\beta, q)$. We will now use the above inequality to show that

$$(T_q^\Gamma, u_q)_{L^2(\mathbb{R}^2)} \geq C(T_0^\Gamma u_0, u_0)_{L^2(\mathbb{R}^2)} + (Fu_0, u_0)_{L^2(\mathbb{R}^2)},$$

for some constant $C > 0$, where F will be an operator of finite rank in $L^2(\mathbb{R}^2)$. First note that $N(\beta, q)$ is a closed space, so we can consider the orthogonal decomposition $\mathcal{F}^2 = N(\beta, q) \oplus N(\beta, q)^\perp$. Let us introduce the corresponding orthogonal projections $\Pi_q^+ : \mathcal{F}^2 \to N(\beta, q)$ and $\Pi_q^- = I - \Pi_q^+ : \mathcal{F}^2 \to N(\beta, q)^\perp$ and write

$$\mathcal{F}^2 \ni f = f_+ + f_- \in N(\beta, q) \oplus N(\beta, q)^\perp,$$

where $f_+ = \Pi_q^+ f$ and $f_- = \Pi_q^- f$. Recall that for $q \in \mathbb{N}_0$ we set $u_q = \mathcal{U}_q f \in \mathcal{L}_q$, where $\mathcal{U}_q$ is the unitary map between $\mathcal{F}^2$ and $\mathcal{L}_q$ given in (4.5). Using the above orthogonal decomposition of $\mathcal{F}^2$ we can rewrite

$$u_q = \mathcal{U}_q f = \mathcal{U}_q \Pi_q^+ f + \mathcal{U}_q \Pi_q^- f = \mathcal{U}_q \Pi_q^+ \mathcal{U}_q^{-1} u_q + \mathcal{U}_q \Pi_q^- \mathcal{U}_q^{-1} u_q,$$

which gives rise to the orthogonal projections $\mathcal{Q}_q^+ = \mathcal{U}_q \Pi_q^+ \mathcal{U}_q^{-1}$ and $\mathcal{Q}_q^- = \mathcal{U}_q \Pi_q^- \mathcal{U}_q^{-1}$ in $\mathcal{L}_q$ and the decomposition $u_q = u_q^+ + u_q^-$ with $u_q^+ = \mathcal{Q}_q^+ u$ and $u_q^- = \mathcal{Q}_q^- u$. This means

$$\begin{aligned}(T_q^\Gamma u_q, u_q)_{L^2(\mathbb{R}^2)} &= (T_q^\Gamma u_q^+, u_q^+)_{L^2(\mathbb{R}^2)} + (T_q^\Gamma u_q^+, u_q^-)_{L^2(\mathbb{R}^2)} \\ &\quad + (T_q^\Gamma u_q^-, u_q^+)_{L^2(\mathbb{R}^2)} + (T_q^\Gamma u_q^-, u_q^-)_{L^2(\mathbb{R}^2)} \\ &= \|(2\partial - B\overline{z})^q f_+\|^2_{L^2(\tilde{\sigma})} + (\mathcal{Q}_q^- T_q^\Gamma \mathcal{Q}_q^+ u_q, u_q)_{L^2(\mathbb{R}^2)} \\ &\quad + (\mathcal{Q}_q^+ T_q^\Gamma \mathcal{Q}_q^- u_q, u_q)_{L^2(\mathbb{R}^2)} + (\mathcal{Q}_q^- T_q^\Gamma \mathcal{Q}_q^- u_q, u_q)_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{5.13}$$

We can now apply inequality (5.12) to the first term on the right hand side for a sufficiently small chosen $\beta > 0$, which gives us

$$\begin{aligned}\|(2\partial - B\overline{z})^q f_+\|^2_{L^2(\tilde{\sigma})} &\geq \big(\|(2\partial)^q f_+\|_{L^2(\tilde{\sigma})} - \sum_{k=0}^{q-1} C_{q,k} \|\partial^q f_+\|_{L^2(\tilde{\sigma})}\big)^2 \\ &\geq \|(2\partial)^q f_+\|^2_{L^2(\tilde{\sigma})} \big(1 - \sum_{k=0}^{q-1} C_{q,k}\beta\big)^2 \\ &= c_1 \|\partial^q f_+\|^2_{L^2(\tilde{\sigma})}\end{aligned}$$

for some constant $c_1 \geq 0$. Applying (5.12) again for $k = 0$ gives us

$$\begin{aligned}\|(2\partial - B\overline{z})^q f_+\|^2_{L^2(\tilde{\sigma})} &\geq c_1 \|\partial^q f_+\|^2_{L^2(\tilde{\sigma})} \\ &\geq c_2 \|f_+\|^2_{L^2(\tilde{\sigma})} \\ &= c_3 (T_0^\Gamma u_0^+, u_0^+)_{L^2(\mathbb{R}^2)} \\ &= c_3 (T_0^\Gamma (u_0 - u_0^-), u_0 - u_0^-)_{L^2(\mathbb{R}^2)} \\ &= c_3 \big[(T_0^\Gamma u_0, u_0)_{L^2(\mathbb{R}^2)} - (T_0^\Gamma u_0, u_0^-)_{L^2(\mathbb{R}^2)} \\ &\qquad - (T_0^\Gamma u_0^-, u_0)_{L^2(\mathbb{R}^2)} + (T_0^\Gamma u_0^-, u_0^-)_{L^2(\mathbb{R}^2)}\big] \\ &= c_3 \big[(T_0^\Gamma u_0, u_0)_{L^2(\mathbb{R}^2)} - (\mathcal{Q}_q^- T_0^\Gamma u_0, u_0)_{L^2(\mathbb{R}^2)} \\ &\qquad - (T_0^\Gamma \mathcal{Q}_q^- u_0, u_0)_{L^2(\mathbb{R}^2)} + (\mathcal{Q}_q^- T_0^\Gamma \mathcal{Q}_q^- u_0, u_0)_{L^2(\mathbb{R}^2)}\big]\end{aligned}$$

for some constants $c_2, c_3 > 0$. Together with (5.13) we now obtain

$$(T_q^\Gamma u_q, u_q)_{L^2(\mathbb{R}^2)} \geq c(T_0^\Gamma u_0, u_0)_{L^2(\mathbb{R}^2)} + (F u_0, u_0)_{L^2(\mathbb{R}^2)}, \tag{5.14}$$

for a constant $c > 0$, where the operator $F : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is given by

$$F = \mathcal{Q}_q^- T_q^\Gamma \mathcal{Q}_q^+ + \mathcal{Q}_q^+ T_q^\Gamma \mathcal{Q}_q^- + 2\mathcal{Q}_q^- T_q^\Gamma \mathcal{Q}_q^- - \mathcal{Q}_q^- T_0^\Gamma - T_0^\Gamma \mathcal{Q}_q^-.$$

Since the orthogonal projection $\mathcal{Q}_q^-$ has a finite rank, it follows that F is an operator of finite rank as well. After an application of the min-max principle on (5.13) we can conclude that the singular values of T_q^Γ satisfy

$$s_k(T_q^\Gamma) \geq c s_{k-r}(T_0^\Gamma), \quad k \in \mathbb{N}$$

for k sufficiently large, where $r = \operatorname{rank}(F)$. Thus we can apply Proposition 5.7 and get

$$\begin{aligned}
\liminf_{k\to\infty} (k! s_k(T_q^\Gamma))^{1/k} &\geq \liminf_{k\to\infty} (ck! s_k(T_0^\Gamma))^{1/k} \\
&= \liminf_{k\to\infty} (k! s_k(T_0^\Gamma))^{1/k} \\
&= \frac{B}{2}(\operatorname{Cap}(\Gamma))^2,
\end{aligned}$$

where we have used the fact that $\liminf_{k\to\infty} a^{1/k} = 1$ for any $a > 0$, which shows the statement. □

6 Asymptotics Estimates and Eigenvalue Clustering

In the final section of this master's thesis we are going to derive local spectral properties of the perturbed Landau Hamiltonian, following the lines of [3]. We start in Section 6.1 with a thorough spectral analysis of the compression $P_q W_\lambda P_q$ of the resolvent difference from (4.21) onto the eigenspaces of the Landau Hamiltonian. Section 6.2 then contains the main results regarding the local spectral clustering of the eigenvalues of the Landau Hamiltonian A_α with a singular potential towards the Landau levels. In particular, Theorem 4 can be found in the final section of this thesis.

6.1 Asymptotic estimates for the compressed resolvent difference

In this subsection we are analyzing the spectral properties of the compressed resolvent difference $P_q W P_q$ onto the eigenspaces $\ker(\mathsf{A}_0 - \Lambda_q)$ of the Landau Hamiltonian. To proceed, we fix some $\lambda_0 < \min\{0, \min \sigma(\mathsf{A}_\alpha)\}$ sufficiently small such that $\|\alpha\|_\infty \|M(\lambda_0)\| < 1$, which is possible due to (4.14), and consider again the resolvent difference

$$W_{\lambda_0} = (\mathsf{A}_\alpha - \lambda_0)^{-1} - (\mathsf{A}_0 - \lambda_0)^{-1} = -\gamma(\lambda_0)(1 + \alpha M(\lambda_0))^{-1} \alpha \gamma(\lambda_0)^*. \tag{6.1}$$

For convenience we will use the notation $W := W_{\lambda_0}$ and write $W = W_+ - W_-$ as well as $|W| = W_+ + W_-$, where $W_+ \geq 0$ denotes the non-negative part of W and $W_- \geq 0$ the non-positive part of W. Our aim will be to establish sharp spectral asymptotics on the singular values of the compressed operators $P_q W_\pm P_q$ and $P_q |W| P_q$ under different sign conditions on the interaction strength $\alpha \in L^\infty(\Sigma; \mathbb{R})$.

In the first proposition of this section we will establish an estimate of the operator $P_q |W| P_q$ in terms of the Toeplitz operator T_q^Γ introduced in Definition 5.5.

Proposition 6.1. *Let $\alpha \in L^\infty(\Sigma; \mathbb{R})$ with $\Gamma = \operatorname{supp} \alpha$ and $|\Gamma| > 0$. Then there holds*

(i) $P_q |W| P_q \leq c T_q^\Gamma$ *and* $P_q W_\pm P_q \leq c_\pm T_q^\Gamma$ *for some* $c_\pm, c > 0$.

(ii) *If α is non-negative (non-positive) on Γ and uniformly positive (uniformly negative) on a closed subset $\Gamma' \subset \Gamma$ with $|\Gamma'| > 0$ then $P_q |W| P_q \geq c T_q^{\Gamma'}$.*

Proof. To begin, let $\Gamma_* \subset \Sigma$ with $|\Gamma_*| > 0$ and consider the bounded operator

$$D_{\Gamma_*} := P_q \gamma(\lambda_0) \chi_{\Gamma_*} \gamma(\lambda_0)^* P_q,$$

where χ_{Γ_*} denotes the characteristic function of Γ_*. Then for any $f \in L^2(\mathbb{R}^2)$ we find

$$\begin{aligned}
(D_{\Gamma_*} f, f)_{L^2(\mathbb{R}^2)} &= (P_q \gamma(\lambda_0) \chi_{\Gamma_*} \gamma(\lambda_0)^* P_q f, f)_{L^2(\mathbb{R}^2)} \\
&= (\chi_{\Gamma_*} \gamma(\lambda_0)^* P_q f, \gamma(\lambda_0)^* P_q f)_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{(\Lambda_q - \lambda_0)^2} (\chi_{\Gamma_*} (P_q f)|_\Sigma, (P_q f)|_\Sigma)_{L^2(\Sigma)} \\
&= \frac{1}{(\Lambda_q - \lambda_0)^2} \|(P_q f)|_{\Gamma_*}\|^2_{L^2(\Gamma_*)} \\
&= \frac{t_q^{\Gamma_*}[f]}{(\Lambda_q - \lambda_0)^2}.
\end{aligned}$$

Hence we obtain the operator relation

$$D_{\Gamma_*} = \frac{T_q^{\Gamma_*}}{(\Lambda_q - \lambda_0)^2}. \tag{6.2}$$

(i) We will prove the claim for W_+, the proof for W_- follows analogously. Consider the mappings

$$\begin{aligned} \alpha_1 : \quad & L^2(\Sigma) \to L^2(\Gamma), \quad \alpha_1\Phi = (\alpha\Phi)|_\Gamma \\ \alpha_2 : \quad & L^2(\Gamma) \to L^2(\Sigma), \quad \alpha_2\Phi = \begin{cases} \Phi & \text{on } \Gamma \\ 0 & \text{on } \Sigma\backslash\Gamma \end{cases}. \end{aligned}$$

It is clear that $\alpha = \alpha_2\alpha_1$, thus by Theorem 4.12 we get

$$W = \gamma(\lambda_0)C\gamma(\lambda_0)^*,$$

where $C := -\alpha_2(1 + \alpha_1 M(\lambda_0)\alpha_2)^{-1}\alpha_1 \in \mathcal{B}(L^2(\Sigma))$. Since the resolvent difference W is self-adjoint, it follows that C is a self-adjoint operator. Let C_+ be the non-negative part of C, then we get the estimate $C_+ \leq \|C\|$ in the operator sense. Furthermore, we see that

$$W_+ = \gamma(\lambda_0)C_+\gamma(\lambda_0)^* = \gamma(\lambda_0)\chi_\Gamma C_+\chi_\Gamma\gamma(\lambda_0)^*,$$

so we can calculate

$$\begin{aligned} (P_qW_+P_qf, f)_{L^2(\mathbb{R}^2)} &= (C_+\chi_\Gamma\gamma(\lambda_0)^*P_qf, \chi_\Gamma\gamma(\lambda_0)^*P_qf)_{L^2(\Sigma)} \\ &\leq \|C\|(P_q\gamma(\lambda_0)\chi_\Gamma\gamma(\lambda_0)^*P_qf, f)_{L^2(\mathbb{R}^2)} \\ &= \|C\|(D_\Gamma f, f)_{L^2(\mathbb{R}^2)}, \end{aligned}$$

which together with (6.2) implies $P_qW_+P_q \leq c_+T_q^\Gamma$, where $c_+ = \frac{\|C\|}{(\Lambda_q - \lambda_0)^2}$.

(ii) Assume now that $\alpha \geq 0$ on Γ and that α is uniformly positive on a subset $\Gamma' \subset \Gamma$ such that $|\Gamma'| > 0$. We define the mappings

$$\begin{aligned} \alpha_1 : \quad & L^2(\Sigma) \to L^2(\Gamma), \quad \alpha_1\Phi = (\sqrt{\alpha}\Phi)|_\Gamma \\ \alpha_2 : \quad & L^2(\Gamma) \to L^2(\Sigma), \quad \alpha_2\Phi = \begin{cases} \sqrt{\alpha}\Phi & \text{on } \Gamma \\ 0 & \text{on } \Sigma\backslash\Gamma \end{cases}, \end{aligned}$$

which are adjoints to each other and satisfy $\alpha = \alpha_2\alpha_1$. Using (4.15) again shows that

$$W = -\gamma(\lambda_0)\alpha_2\hat{C}\alpha_1\gamma(\lambda_0)^*,$$

where $\hat{C} = (1 + \alpha_1 M(\lambda_0)\alpha_2)^{-1} \in \mathcal{B}(L^2(\Gamma))$ is a self-adjoint and uniformly positive operator in $L^2(\Gamma)$. In particular, W is non-positive. This means that for $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned}
(P_q|W|P_q f, f)_{L^2(\mathbb{R}^2)} &= (P_q W_- P_q f, f)_{L^2(\mathbb{R}^2)} \\
&= (P_q \gamma(\lambda_0)\alpha_2 \hat{C} \alpha_1 \gamma(\lambda_0)^* P_q f, f)_{L^2(\mathbb{R}^2)} \\
&= (\hat{C}\alpha_1\gamma(\lambda_0)^* P_q f, \alpha_1\gamma(\lambda_0)^* P_q f)_{L^2(\Gamma)} \\
&\geq (\inf \sigma(\hat{C})) \cdot (\inf_{x \in \Gamma'} \alpha(x))(\chi_{\Gamma'}\gamma(\lambda_0)^* P_q f, \chi_{\Gamma'}\gamma(\lambda_0)^* P_q f)_{L^2(\Gamma)} \\
&\geq (\inf \sigma(\hat{C})) \cdot (\inf_{x \in \Gamma'} \alpha(x))(D_{\Gamma'} f, f)_{L^2(\Gamma')} \\
&= c'(T_q^{\Gamma'} f, f)_{L^2(\Gamma')},
\end{aligned}$$

where we choose

$$c' = \frac{\inf \sigma(\hat{C})}{(\Lambda_q - \lambda_0)^2} \cdot \inf_{x \in \Gamma'} \alpha(x) > 0,$$

which proves the inequality in (ii). □

Using the above proposition we can immediately conclude the following results. The next corollary is an immediate consequence of Proposition 6.1(i) and the spectral asymptotics for T_q^Γ established in Proposition 5.8.

Corollary 6.2. *Let $\alpha \in L^\infty(\Sigma; \mathbb{R})$ with $\Gamma = \operatorname{supp} \alpha$ and $|\Gamma| > 0$. Then for $q \in \mathbb{N}_0$ the following estimate holds for the singular values of $P_q|W|P_q$:*

$$\limsup_{k \to \infty} (k! s_k(P_q|W|P_q))^{1/k} \leq \frac{B}{2}(\operatorname{Cap}(\Gamma))^2.$$

In particular, we have

$$\limsup_{k \to \infty} (k! s_k(P_q W_\pm P_q))^{1/k} \leq \frac{B}{2}(\operatorname{Cap}(\Gamma))^2.$$

for the singular values of $P_q W_\pm P_q$.

Proof. By Proposition 6.1(i) there holds $P_q|W|P_q \leq c T_q^\Gamma$ as well as $P_q W_\pm P_q \leq c_\pm T_q^\Gamma$ for some constants $c, c_\pm > 0$. This, in particular, implies

$$\limsup_{k \to \infty} (k! s_k(P_q|W|P_q))^{1/k} \leq \limsup_{k \to \infty} (c \cdot k! s_k(T_q^\Gamma))^{1/k}.$$

By Proposition 5.8 there holds

$$\limsup_{k \to \infty} (c \cdot k! s_k(T_q^\Gamma))^{1/k} = \lim_{k \to \infty} (k! s_k(T_q^\Gamma))^{1/k} = \frac{B}{2}(\operatorname{Cap}(\Gamma))^2,$$

since $\lim_{k \to \infty} c^{1/k} = 1$, which shows the statement for $P_q|W|P_q$. The same argument holds for the operator $P_q W_\pm P_q$ which finishes the proof. □

The next corollary can be seen as a consequence of Proposition 6.1(ii) and Proposition 5.8, assuming that α is a sign-definite function.

Corollary 6.3. *Let $\alpha \in L^\infty(\Sigma;\mathbb{R})$ with $\Gamma = \operatorname{supp}\alpha$ and $|\Gamma| > 0$. Furthermore, suppose that α is non-negatve (non-positive) on Γ and uniformly positive (uniformly-negative) on a closed subarc $\Gamma' \subset \Gamma$ with $|\Gamma'| > 0$. Then for $q \in \mathbb{N}_0$ the singular values of $P_q|W|P_q$ satisfy*

$$\liminf_{k\to\infty}(k!s_k(P_q|W|P_q))^{1/k} \geq \frac{B}{2}(\mathrm{Cap}(\Gamma'))^2.$$

In particular, the operator $P_q|W|P_q$ has infinite rank.

Proof. By Proposition 6.1(ii) we have $P_q|W|P_q \geq c'T_q^{\Gamma'}$ for some constant $c' > 0$. In composition with Proposition 5.8 we now have

$$\liminf_{k\to\infty}(k!s_k(P_q|W|P_q))^{1/k} \geq \liminf_{k\to\infty}(k!s_k(T_q^{\Gamma'}))^{1/k} = \frac{B}{2}(\mathrm{Cap}(\Gamma'))^2,$$

since $\lim_{k\to\infty} c^{1/k} = 1$, which shows the statement. □

Using the two corollaries from above we can obtain exact spectral asymptotics for $P_q|W|P_q$ assuming that α is uniformly positive or uniformly negative on Γ. In the next theorem we will see that we can achieve the same results under the slightly weaker assumption that α is uniformly positive or uniformly negative on the interior of Γ. For the following theorem $D_\epsilon(x)$ will denote the disk of radius $\epsilon > 0$ centered at x

Theorem 6.4. *Let $\alpha \in L^\infty(\Sigma, \mathbb{R})$ and $\Gamma = \operatorname{supp}\alpha$. Suppose that α is non-negative (non-positive) on Γ and uniformly positive (uniformly negative) on the truncated arc $\Gamma_\epsilon = \{x \in \Gamma : D_\epsilon(x) \cap \Sigma \subset \Gamma\}$ for all $\epsilon > 0$ sufficiently small. Then for $q \in \mathbb{N}_0$ there holds the estimate*

$$\lim_{k\to\infty}(k!s_k(P_q|W|P_q))^{1/k} = \frac{B}{2}(\mathrm{Cap}\,(\Gamma))^2$$

for the singular values of the operator $P_q|W|P_q$.

Proof. By Corollary 6.2 we already have

$$\limsup_{k\to\infty}(k!s_k(P_q|W|P_q))^{1/k} \leq \frac{B}{2}(\mathrm{Cap}(\Gamma))^2$$

and from Corollary 6.3 we get

$$\liminf_{k\to\infty}(k!s_k(P_q|W|P_q))^{1/k} \geq \frac{B}{2}(\mathrm{Cap}(\Gamma_\epsilon))^2. \tag{6.3}$$

We will now show that

$$\liminf_{\epsilon\to 0^+} \mathrm{Cap}(\Gamma_\epsilon) = \mathrm{Cap}(\Gamma),$$

from which the claim will follow by (6.3). Since $\Gamma_\epsilon \subset \Gamma$ it follows from Proposition 2.16(i) that $\mathrm{Cap}\,(\Gamma_\epsilon) \leq \mathrm{Cap}\,(\Gamma)$.

To show the inverse inequality consider the equilibrium measure μ for Γ, which was introduced in Definition 2.15. Without loss of generality we can assume that μ has no point mass, which would imply $I(\mu) = \infty$ and hence $\operatorname{Cap}(\Gamma) = 0$, which is the trivial case. By the dominated convergence theorem we see that

$$\mu(\Gamma_\epsilon) = \int_{\mathbb{R}^2} \chi_{\Gamma_\epsilon} \,\mathrm{d}\mu(x) \to \int_{\mathbb{R}^2} \chi_\Gamma \,\mathrm{d}\mu(x) = \mu(\Gamma)$$

as $\epsilon \to 0^+$. This means that for $\epsilon > 0$ sufficiently small we can introduce the measure

$$\mu_\epsilon(M) := \frac{1}{\mu(\Gamma_\epsilon)} \mu(M \cap \Gamma_\epsilon),$$

which is well-defined on Borel sets $M \subset \mathbb{R}^2$. In fact $\mu_\epsilon \geq 0$ with $\operatorname{supp} \mu_\epsilon = \Gamma_\epsilon$ and $\mu_\epsilon(\Gamma_\epsilon) = 1$. Applying the dominated convergence theorem again gives us

$$\begin{aligned} I(\mu_\epsilon) &= \frac{1}{\mu(\Gamma_\epsilon)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \chi_{\Gamma_\epsilon}(x) \chi_{\Gamma_\epsilon}(y) \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\ &\to \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) = I(\mu) \end{aligned}$$

as $\epsilon \to 0^+$, which proves that $\liminf_{\epsilon \to 0^+} \operatorname{Cap}(\Gamma_\epsilon) \geq \operatorname{Cap}(\Gamma)$, showing the statement. □

Remark 6.5. The above theorem allows us to drop the requirement that α is uniformly positive on Γ and still obtain exact spectral asymptotics for the singular values of $P_q |W| P_q$. In particular, a continuous interaction strength α only has to be positive on the interior of Γ and may be allowed to vanish at the endpoints.

6.2 Eigenvalue Clustering at Landau Levels

In the final section of this thesis we are going to use the results on the spectral asymptotics of the compressed resolvent $P_q W P_q$ from the last subsection to derive local spectral properties of the Landau Hamiltonian A_α with a singular interaction. Our strategy will be to interpret the resolvent difference

$$W = (\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1} \tag{6.4}$$

as a compact perturbation of the resolvent $(\mathsf{A}_0 - \lambda)^{-1}$. We then apply Proposition 2.32, in order to give an estimate on the rate of accumulaton of the eigenvalues of the Landau Hamiltonian A_α with a δ-potential at the Landau levels in terms of the singular values of $P_q W P_q$. Throughout this section we fix $\lambda_0 < \min\{0, \min \sigma(\mathsf{A}_\alpha)\}$, which ensures that the resolvent difference W admits the compact factorization as in (4.15).

The first proposition shows that for a sign-definite interaction strength α we can exclude an accumulation of the eigenvalues from one side to each Landau level.

Proposition 6.6. *Let $\alpha \in L^\infty(\Sigma, \mathbb{R})$ and $q \in \mathbb{N}_0$. Then there holds:*

(i) *If α is non-negative, then the eigenvalues of A_α do not accumulate to the Landau levels Λ_q from below.*

(ii) *If α is non-positive, then the eigenvalues of A_α do not accumulate to the Landau levels Λ_q from above.*

Proof. We will show (i), since the proof for (ii) works in the same way. By Lemma 4.15 we know that

$$(\mathsf{A}_\alpha - \lambda_0)^{-1} - (\mathsf{A}_0 - \lambda_0)^{-1} = -\gamma(\lambda_0)(1 + \alpha M(\lambda_0))^{-1}\alpha\gamma(\lambda_0)^* \leq 0.$$

Applying Proposition 2.29(ii) with

$$T = (\mathsf{A}_0 - \lambda_0)^{-1}, \quad W_- = (\mathsf{A}_0 - \lambda_0)^{-1} - (\mathsf{A}_\alpha - \lambda_0)^{-1},$$

shows that there is no accumulation of eigenvalues of $(\mathsf{A}_\alpha - \lambda_0)^{-1}$ from above to the eigenvalues $(\Lambda_q - \lambda_0)^{-1}$. Hence the eigenvalues of A_α do not accumulate to the Landau levels Λ_q from below. □

Under the additional assumption that α is either strictly positive or negative we can always observe an accumulation of eigenvalues to the Landau levels.

Theorem 6.7. *Let $\alpha \in L^\infty(\Sigma, \mathbb{R})$ and $q \in \mathbb{N}_0$. Then there holds*

(i) *Suppose that $\alpha(x) > 0$ for a.e. $x \in \Sigma$. Then the eigenvalues of A_α accumulate to Λ_q from above.*

(ii) *Suppose that $\alpha(x) < 0$ for a.e. $x \in \Sigma$. Then the eigenvalues of A_α accumulate to Λ_q from below.*

Proof. We will prove (i), as (ii) can be shown analogously. Recall that

$$(\mathsf{A}_\alpha - \lambda_0)^{-1} - (\mathsf{A}_0 - \lambda_0)^{-1} = -\gamma(\lambda_0)(1 + \alpha M(\lambda_0))^{-1}\alpha\gamma(\lambda_0)^* < 0$$

by Lemma 4.15(i) and since $\alpha > 0$ it folows from Proposition 4.16 that $P_q W P_q$ has an infinite rank. Setting

$$T = (\mathsf{A}_0 - \lambda_0)^{-1}, \quad W_- = (\mathsf{A}_0 - \lambda_0)^{-1} - (\mathsf{A}_\alpha - \lambda_0)^{-1},$$

we can apply Proposition 2.30, which shows that the eigenvalues of $(\mathsf{A}_\alpha - \lambda_0)^{-1}$ accumulate to the eigenvalues $(\Lambda_q - \lambda_0)^{-1}$ of $(\mathsf{A}_0 - \lambda_0)^{-1}$ from below. In particular, the eigenvalues of A_α must accumulate to the Landau levels Λ_q from above. □

In the following we will give some results on the rate of accumulation of the eigenvalues of A_α to the Landau levels. Before doing so we will introduce the following notation for the sake of convenience:

$$I_q^+ = (\Lambda_q, \Lambda_q + B],\ q \in \mathbb{N}_0,$$
$$I_q^- = \begin{cases} (-\infty, \Lambda_0), & q = 0 \\ (\Lambda_q - B, \Lambda_q), & q \geq 1 \end{cases}.$$

The so defined intervals are disjoint and satisfy

$$\mathbb{R} = \bigcup_{q=0}^{\infty} I_q^- \cup \bigcup_{q=0}^{\infty} I_q^+ \cup \bigcup_{q=0}^{\infty} \{\Lambda_q\}.$$

In the following theorem we provide an upper bound on the accumulation of the eigenvalues of the Landau Hamiltonian A_α with a δ-potential to each Landau level and show regularized summability of the discrete spectrum over all clusters.

Theorem 6.8. *Let $q \in \mathbb{N}_0$ and let $\{\lambda_k^\pm(q)\}_{k\in\mathbb{N}_0}$ be the sequence of eigenvalues of A_α contained in the interval $I_q^\pm$ counted with multiplicites. Suppose that $\lambda_k^\pm(q)$ are ordered in such a way that $|\lambda_k^\pm(q) - \Lambda_q|$ is a non-increasing sequence. Then the eigenvalues $\lambda_k^\pm(q)$ of A_α satisfy*

(i) $\sum_{q=0}^{\infty} \frac{1}{(2q+1)^2} \left(\sum_k |\lambda_k^+(q) - \Lambda_q| + \sum_k |\lambda_k^-(q) - \Lambda_q|\right) < \infty.$

(ii) $\limsup_{k\to\infty} (k!|\lambda_k^\pm(q) - \Lambda_q|)^{1/k} \leq \frac{B}{2}(\mathrm{Cap}\,(\Gamma))^2.$

Proof. (i) For this proof we set $C = (\mathsf{A}_\alpha - \lambda_0)^{-1}$ and $D = (\mathsf{A}_0 - \lambda_0)^{-1}$. By applying Proposition 4.14, we see that $C - D = W \in \mathfrak{S}_{p,\infty}(L^2(\mathbb{R}^2))$ for $p \geq 1/3$. In particular, $C - D \in \mathfrak{S}_{1,\infty}(L^2(\mathbb{R}^2))$. Next we are going to use that the spectrum of $D := (\mathsf{A}_0 - \lambda_0)^{-1}$ is given by the eigenvalues $\{(\Lambda_q - \lambda_0)^{-1}\}_{q\in\mathbb{N}_0}$, which have infinite multiplicity. Recall that $\lambda_0 < \min\{0, \min\sigma(\mathsf{A}_\alpha)\}$. So for $q \in \mathbb{N}_0$ we have

$$\begin{aligned}
\mathfrak{d}_k^+(q) &:= \mathrm{dist}\left(\frac{1}{\lambda_k^+(q) - \lambda_0}, \sigma(D)\right) \\
&= \min\left\{\frac{1}{\lambda_k^+(q) - \lambda_0} - \frac{1}{\Lambda_{q+1} - \lambda_0}, \frac{1}{\Lambda_q - \lambda_0} - \frac{1}{\lambda_k^+(q) - \lambda_0}\right\} \\
&= \min\left\{\frac{\Lambda_{q+1} - \lambda_k^+(q)}{(\lambda_k^+(q) - \lambda_0)(\Lambda_{q+1} - \lambda_0)}, \frac{\lambda_k^+(q) - \Lambda_q}{(\Lambda_q - \lambda_0)(\lambda_k^+(q) - \lambda_0)}\right\}.
\end{aligned}$$

Since $\lambda_k^+(q) \in I_q^+ = (\Lambda_q, \Lambda_q + B]$ and $\Lambda_{q+1} = \Lambda_q + 2B$ there holds

$$\lambda_k^+(q) - \Lambda_q \leq \Lambda_{q+1} - \lambda_k^+(q),$$

which then implies

$$\begin{aligned}
\mathfrak{d}_k^+(q) &\geq (\lambda_k^+(q) - \Lambda_q) \min\left\{\frac{1}{(\lambda_k^+(q) - \lambda_0)(\Lambda_{q+1} - \lambda_0)}, \frac{1}{(\Lambda_q - \lambda_0)(\lambda_k^+(q) - \lambda_0)}\right\} \\
&= \frac{\lambda_k^+(q) - \Lambda_q}{(\lambda_k^+(q) - \lambda_0)(\Lambda_{q+1} - \lambda_0)}.
\end{aligned}$$

Next we choose $c > 0$ such that

$$\Lambda_q - \lambda_0 \leq c\Lambda_q \Longleftrightarrow c \geq 1 - \frac{\lambda_0}{\Lambda_q} = 1 + \frac{|\lambda_0|}{\Lambda_q}, \tag{6.5}$$

where the constant $c > 0$ can be chosen independent of $q \in \mathbb{N}_0$ since $\frac{\lambda_0}{\Lambda_q}$ is a decreasing sequence. Furthermore, there holds

$$\lambda_k^+(q) - \lambda_0 \leq \Lambda_{q+1} - \lambda_0 \leq c\Lambda_{q+1} \leq c^2 \Lambda_q \tag{6.6}$$

if we additionally choose $c > 0$ such that $\Lambda_{q+1} \leq c\Lambda_q$, which is possible since $\frac{\Lambda_{q+1}}{\Lambda_q}$ is decreasing. Choosing $c > 0$ sufficiently large such that (6.5) and (6.6) are satisfied we arrive at the inequality

$$\mathfrak{d}_k^+(q) \geq \frac{c_+(\lambda_k^+(q) - \Lambda_q)}{\Lambda_q^2},$$

which holds for all $q \in \mathbb{N}_0$, where we have chosen $c_+ = c^{-3}$. In a similar fashion one can show for $q \in \mathbb{N}_0$ that

$$\begin{aligned}
\mathfrak{d}_k^-(q) &:= \operatorname{dist}\left(\frac{1}{\lambda_k^-(q) - \lambda_0}, \sigma(D)\right) \\
&= \min\left\{\frac{1}{\lambda_k^-(q) - \lambda_0} - \frac{1}{\Lambda_q - \lambda_0}, \frac{1}{\Lambda_{q-1} - \lambda_0} - \frac{1}{\lambda_k^-(q) - \lambda_0}\right\} \\
&\geq \frac{c_-(\Lambda_q - \lambda_k^-(q))}{\Lambda_q^2},
\end{aligned}$$

for some constant $c_- > 0$. Set $C = (\mathsf{A}_\alpha - \lambda_0)^{-1}$, then $D - C = W \in \mathfrak{S}_{1,\infty}(L^2(\mathbb{R}^2))$, so

$$\begin{aligned}
\sum_{\lambda \in \sigma_{disc}(C)} \operatorname{dist}(\lambda, \sigma(D)) &= \sum_{q=0}^{\infty} \sum_k (\mathfrak{d}_k^+(q) + \mathfrak{d}_k^-(q)) \\
&\geq \sum_{q=0} \frac{c}{B^2(2q+1)^2} \sum_k (|\lambda_k^+(q) - \Lambda_q| + |\lambda_k^-(q) - \Lambda_q|),
\end{aligned}$$

and since by the series on the left hand side is finite by Proposition 2.48 the assertion follows.

(ii) Let us set

$$\begin{aligned}
&W = W_{\lambda_0}, \quad T = (\mathsf{A}_0 - \lambda_0)^{-1}, \quad \Lambda = \frac{1}{\Lambda_q - \lambda_0}, \\
&P_\Lambda = P_q, \quad \epsilon = \frac{1}{2}, \quad \tau_\pm = \pm\frac{1}{2}\left[\frac{1}{\Lambda_q \mp B - \lambda_0} - \frac{1}{\Lambda_q - \lambda_0}\right],
\end{aligned}$$

where W is defined as in (6.1), and use Proposition 2.32. Since the eigenvalues of $T + W$ in $(\Lambda - 2\tau_-, \Lambda + 2\tau_+)$ are given by

$$\frac{1}{\lambda_1^+(q) - \lambda_0} \leq \frac{1}{\lambda_2^+(q) - \lambda} \leq \cdots \leq \Lambda \leq \cdots \leq \frac{1}{\lambda_2^-(q) - \lambda_0} \leq \frac{1}{\lambda_1^-(q) - \lambda_0},$$

it follows from Proposition 2.32 that there exists $l = l(q) \in \mathbb{N}$ such that

$$\left| \frac{1}{\lambda_k^{\pm}(q) - \lambda_0} - \frac{1}{\Lambda_q - \lambda_0} \right| \leq \frac{3}{2} s_{k-l}(P_q W_{\mp} P_q)$$

for all sufficiently large $k \in \mathbb{N}$. Moreover, by Lemma 2.22 and Corollary 6.2, we find

$$\limsup_{k\to\infty} (k! s_{k-l}(P_q W_{\mp} P_q))^{1/k} = \limsup_{k\to\infty} (k! s_k(P_q W_{\mp} P_q))^{1/k} \leq \frac{B}{2} (\mathrm{Cap}\,(\Gamma))^2.$$

Another calculation now shows

$$\begin{aligned}
&\limsup_{n\to\infty} (k! \left|\lambda_k^{\pm}(q) - \Lambda_q\right|)^{1/k} \\
&\qquad = \limsup_{k\to\infty} (\lambda_k^{\pm}(q) - \lambda_0)^{1/k} (\Lambda_q - \lambda_0)^{1/k} \left(k! \left| \frac{1}{\lambda_k^{\pm}(q) - \lambda_0} - \frac{1}{\Lambda_q - \lambda_0} \right| \right)^{1/k} \\
&\qquad \leq \limsup_{k\to\infty} (k! s_{k-l}(P_q W_{\mp} P_q))^{1/k} \\
&\qquad = \limsup_{k\to\infty} (k! s_k(P_q W_{\mp} P_q))^{1/k} \leq \frac{B}{2} (\mathrm{Cap}\,(\Gamma))^2,
\end{aligned}$$

where we have used the boundedness of $(\lambda_k^{\pm}(q) - \lambda_0)$ and that $\limsup_{k\to\infty} a^{1/k} = 1$ for any positive number $a > 0$, which shows (ii). $\square$

It is important to note here, that the above result does not require α to be sign-definite. In particular, we can always achieve an upper bound on the rate of accumulation of the eigenvalues of A_α to the Landau levels, regardless of the definiteness of α. Under the additional assumption that α is sign-definite on Γ we are able to obtain a respective lower bound on the rate of accumulation. The following theorem can be seen as a generalization of Theorem 4 in the Introduction.

Theorem 6.9. *Let $\alpha \in L^\infty(\Sigma; \mathbb{R})$ and $\Gamma = \operatorname{supp} \alpha$. Suppose that α is non-negative (non-positive) on Γ and uniformly positive (uniformly negative) on the truncated arc $\Gamma_\epsilon = \{x \in \Gamma : D_\epsilon(x) \cap \Sigma \subset \Gamma\}$ for all $\epsilon > 0$ sufficiently small. Then the eigenvalues $\{\lambda_k(q)\}_{k\in\mathbb{N}_0}$ of A_α lying in the interval I_q^+ (I_q^-, respectively) satisfy*

$$\lim_{k\to\infty} (k! \left|\lambda_k(q) - \Lambda_q\right|)^{1/k} = \frac{B}{2} (\mathrm{Cap}\,(\Gamma))^2.$$

In particular, the eigenvalues of A_α accumulate to Λ_q from above (from below, respectively) for all $q \in \mathbb{N}_0$.

Proof. We will show the proof for $\alpha \geq 0$, since the case $\alpha \leq 0$ works analogously. As in the proof of of Theorem 6.8 we will set

$$\begin{aligned}
&W = W_{\lambda_0}, \quad T = (\mathsf{A}_0 - \lambda_0)^{-1}, \quad \Lambda = \frac{1}{\Lambda_q - \lambda_0}, \\
&P_\Lambda = P_q, \quad \epsilon = \frac{1}{2}, \quad \tau_{\pm} = \pm\frac{1}{2} \left[\frac{1}{\Lambda_q \mp B - \lambda_0} - \frac{1}{\Lambda_q - \lambda_0} \right].
\end{aligned} \tag{6.7}$$

Since $\operatorname{rank}(P_q W P_q) = \infty$ by Theorem 6.4 we can apply Proposition 2.30 for (6.7) and arrive at the inequality

$$\frac{1}{2} s_{k+l}(P_q W P_q) \le \left| \frac{1}{\lambda_k(q) - \lambda} - \frac{1}{\Lambda_q - \lambda} \right| \le \frac{3}{2} s_{k-l}(P_q W P_q)$$

for some constant $l = l(q) \in \mathbb{N}$ and all $k \in \mathbb{N}$ sufficiently large. Using Lemma 2.22 in conjuction with Corollary 6.4 we see that

$$\lim_{k\to\infty} s_{k\pm l}(P_q W P_q) = \frac{B}{2} (\operatorname{Cap}(\Gamma))^2,$$

from which we can conclude the asymptotic result on the eigenvalues of A_α in the same way as in the proof of Theorem 6.8(ii). □

Mimicking the proof of the above theorem we are able to obtain a lower bound on the spectral clustering, under the slightly weaker assumption that α is uniformly positive on a subarc of positive measure of Γ.

Proposition 6.10. *Let $\alpha \in L^\infty(\Sigma; \mathbb{R})$ with $\Gamma = \operatorname{supp} \alpha$ and $|\Gamma| > 0$. Furthermore, suppose that α is non-negatve (non-positive) on Γ and uniformly positive (uniformly-negative) on a closed subarc $\Gamma' \subset \Gamma$ with $|\Gamma'| > 0$. Then the eigenvalues $\{\lambda_k(q)\}_{k\in\mathbb{N}_0}$ of A_α lying in the interval I_q^+ (I_q^-, respectively) satisfy*

$$\liminf_{k\to\infty} (k!\,|\lambda_k(q) - \Lambda_q|)^{1/k} \ge \frac{B}{2} (\operatorname{Cap}(\Gamma'))^2.$$

In particular, the eigenvalues of A_α accumulate to Λ_q from above (from below, respectively) for all $q \in \mathbb{N}_0$.

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